Zero–Hopf bifurcation in a hyperchaotic Lorenz system

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Abstract We characterize the zero–Hopf bifurcation at the singular points of a parameter co-dimension four hyperchaotic Lorenz system. Using averaging theory, we find sufficient conditions so that at the bifurcation points two periodic solutions emerge and describe the stability of these orbits.

1 Introduction

In 1963 Edward Lorenz [19] introduced a system of ordinary differential equations in $\mathbb{R}^3$ soon to became famous for exhibiting chaotic solutions for certain parameter values and initial conditions. More precisely, the Lorenz system displays a set of chaotic solutions which, when plotted, looks as a butterfly or figure eight (the Lorenz attractor). The origins of this system lies in atmospheric modeling. However, the Lorenz equations also appear in the modeling of lasers see [11], and dynamos see [15].

Recently, a so-called *hyperchaotic Lorentz* system was introduced; see for instance [1, 6, 10, 13, 14, 25, 27–33] and the references therein. (MathSciNet presently lists 24 papers on *hyperchaotic Lorenz* systems.) We remark that not all these hyperchaotic Lorenz systems coincide, as they can vary in one or two terms. A precise definition of a hyperchaotic system comprises 1) an autonomous differential equations system with a phase space of dimension at least four, 2) a dissipative

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structure, and 3) at least two unstable directions, out of which at least one is due to a nonlinearity. Since it generates multiple positive Lyapunov exponents, the dynamics of hyperchaotic systems is hard to predict and control. Consequently, such systems are of use in secure communication and thus received a great deal of attention mainly in engineering (circuit and communications systems; see, for instance, [24] and references therein).

In this paper, we approach a hyperchaotic system from a dynamical systems point of view. More precisely, we investigate a *4-dimensional zero–Hopf equilibrium* (that is, an isolated equilibrium with a double zero eigenvalue and a pair of purely imaginary eigenvalues), and the birth of periodic solutions as parameters vary.

There are several works studying the unfolding of the 3–*dimensional zero–Hopf equilibria*. Recall that a 3–dimensional zero–Hopf bifurcation is a two–parameter unfolding (or family) of a 3–dimensional autonomous differential system with a zero–Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. The zero–Hopf bifurcation has been studied by Guckenheimer [8], Guckenheimer and Holmes [9], Han [12], Kuznetsov [16], Llibre [18], Marsden and Scheurle [23]. It has been shown that, under certain conditions, some complicated invariant sets of the unfolding could be bifurcated from the isolated zero–Hopf equilibrium and hence, in some cases, a zero–Hopf equilibrium could signal a local birth of "chaos" (see [5,23]). Also, recently Li and Wang [17] published a paper on a Hopf bifurcation in a (three-dimensional) Lorentz-type system. Due to the complexity related to the high dimensionality, there is very little work done on the n–dimensional zero–Hopf bifurcation with $n > 3$.

Here we study the following hyperchaotic Lorenz system (as given in [6,14])

$$
\begin{aligned}
\dot{x} &= a(y - x) + w, \\
\dot{y} &= cx - y - xz, \\
\dot{z} &= -bz + xy, \\
\dot{w} &= dw - xz,
\end{aligned}
\tag{1}
$$

which displays for a zero–Hopf equilibrium for an appropriate choice of the parameters a, b, c and d. Using the method of averaging and a blow-up of both the variables and the parameters at the zero–Hopf equilibrium, we show that two periodic solutions emerge as parameters vary. Further, we characterize the stability of these periodic solutions.

To our knowledge zero–Hopf equilibria and bifurcations in systems (which have a central manifold) of dimension $n \geq 4$ has not been studied yet. While we focus on the case of a hyperchaotic Lorentz system, we mention that the method used here is amenable to a larger class of nonlinear differential systems in $\mathbb{R}^n$. We plan to develop this in future work.

2 Statements of the main results

One may verify that for any choice of the parameters the origin of coordinates of $\mathbb{R}^4$ is always an equilibrium point for the hyperchaotic Lorenz system (1). Moreover,

if

$$abd \neq 1 \quad \text{and} \quad abd(1-c)/(c-ad) > 0,$$

then system (1) has two additional equilibria $p_\pm$, namely

$$\left(\pm \frac{\sqrt{abd(1-c)}}{\sqrt{c-ad}}, \pm \frac{\sqrt{abd(1-c)(c-ad)}}{1-ad}, \frac{ad(1-c)}{1-ad}, \pm \frac{a(1-c)\sqrt{abd(1-c)}}{(1-ad)\sqrt{c-ad}} \right).$$

In the next proposition we characterize when the equilibrium point localized at the origin of coordinates of the hyperchaotic Lorenz system (1) is a zero–Hopf equilibrium point.

Proposition 1 *There is an one–parameter family of hyperchaotic Lorenz system* (1) *for which the origin of coordinates is a zero–Hopf equilibrium point. Namely $a = -1$, $b = d = 0$ and $c > 1$. Moreover, the eigenvalues at the origin for this one–parameter family are 0, 0 and $\pm i\sqrt{c-1}$.*

We shall study when the hyperchaotic Lorenz system (1) having a zero–Hopf equilibrium point at the origin of coordinates has a zero–Hopf bifurcation producing some periodic orbit.

Theorem 1 *For*

$$b \neq 0, \quad d \neq 0 \tag{2}$$

and

$$\frac{1}{2}d^2 - (a+1) > 0 \quad \text{if} \ (a+1)\, d > 0 \tag{3}$$

$$\frac{1}{2}d^2 - (a+1) < 0 \quad \text{if} \ (a+1)\, d < 0 \tag{4}$$

the hyperchaotic Lorenz system has a zero–Hopf bifurcation at the equilibrium point localized at the origin of coordinates. Two periodic solutions born at this equilibrium and they are stable if $(a+1) > 0$, $b > 0$ and $d > 0$.

3 The averaging theory for periodic orbits

The averaging theory is a classical and matured tool for studying the behavior of the dynamics of nonlinear smooth dynamical systems, and in particular of their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure is due to Fatou [7] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [3] in the 1930s and Bogoliubov [2] in 1945. The averaging theory of first order for studying periodic orbits can be found in [26], see also [9]. It can be summarized as follows.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{5}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{6}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (6) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{7}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (7).

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic, where $\mathbf{x}(t, \mathbf{z}, 0)$ denotes the solution of the unperturbed system (6) with $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (5); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the bifurcation problem of T–periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 2 [Perturbations of an isochronous set] *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T} \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}))dt. \tag{8}$$

Then the following statements holds:

(a) *If there exists $\mathbf{a} \in V$ with $\mathcal{F}(\mathbf{a}) = 0$ and $\det\left((\partial \mathcal{F}/\partial \mathbf{z})(\mathbf{a})\right) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (5) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{a}$ as $\varepsilon \to 0$.*
(b) *The type of stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $\left((\partial \mathcal{F}/\partial \mathbf{z})(\mathbf{a})\right)$.*

For an easy proof of Theorem 2(a) see Corollary 1 of [4]. In fact the result of Theorem 2 is a classical result due to Malkin [20] and Roseau [21].

For additional information on averaging theory see the book [22].

4 Proofs

Proof of Proposition 1 The Jacobian matrix evaluated at $(x, y, z, w) = (0, 0, 0, 0)$ is

$$A = \begin{pmatrix} -a & a & 0 & 1 \\ c & -1 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

with characteristic equation $(-b - \lambda)(d - \lambda)[\lambda^2 + (a + 1)\lambda + a(1 - c)] = 0$. Solving this equation we find that the eigenvalues of A are $\lambda_1 = -b$, $\lambda_2 = d$ and $\lambda_{3,4} = (-(a+1) \pm \sqrt{(a+1)^2 - 4a(1-c)})/2$. To satisfy the conditions for the origin to be a zero-Hopf equilibrium, it must be satisfied that $b = d = 0$, and for the other two eigenvalues to be purely imaginary we must have $a = -1$ and $4(c - 1) < 0$, this is $c > 1$. Note that when this conditions are satisfied, the only equilibrium of the system is the origin.

Therefore, when $a = -1$, $b = d = 0$ and $c > 1$, we obtain a one parameter family of hyperchaotic Lorenz systems for which the origin is a zero-Hopf equilibrium. Moreover, the eigenvalues for this one parameter family are $\lambda_1 = \lambda_2 = 0$ and $\lambda_{3,4} = \pm i\sqrt{c - 1}$. $\square$

Proof of Theorem 1 Let $(a, b, d) = (-1 + \varepsilon a_1, \varepsilon b_1, \varepsilon d_1)$ where $\varepsilon > 0$ is a sufficiently small parameter and a_1, b_1 and d_1 are real non-zero numbers. With these substitutions, the Lorenz system becomes

$$\begin{aligned}
\dot{x} &= (-1 + \varepsilon a_1)(y - x) + w, \\
\dot{y} &= cx - y - xz, \\
\dot{z} &= -\varepsilon b_1 z + xy, \\
\dot{w} &= \varepsilon d_1 w - xz.
\end{aligned} \tag{9}$$

Next we rescale the variables. Let $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, and by denoting again the variables (X, Y, Z) by (x, y, z) the system (9) is written as follows

$$\begin{aligned}
\dot{x} &= (-1 + \varepsilon a_1)(y - x) + w, \\
\dot{y} &= cx - y - \varepsilon xz, \\
\dot{z} &= -\varepsilon b_1 z + \varepsilon xy, \\
\dot{w} &= \varepsilon d_1 w - \varepsilon xz.
\end{aligned} \tag{10}$$

To describe the the behavior of the system (10) we will apply the averaging method described in Theorem 2. Using the same notation as in section 3, we have $\mathbf{x} = (x, y, z, w)$,

$$F_0(t, \mathbf{x}) = \begin{pmatrix} -y + x + w \\ cx - y \\ 0 \\ 0 \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} a_1(y - x) \\ -xz \\ -b_1 z + xy \\ d_1 w - xz \end{pmatrix},$$

and $F_2(t, \mathbf{x}, \varepsilon) = 0$. Let us consider the next initial value problem given by the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \quad \mathbf{x}(0, \varepsilon) = (x_0, y_0, z_0, w_0) = \mathbf{z}. \tag{11}$$

The solution to this unperturbed system is $\mathbf{x}(t, \mathbf{z}, \varepsilon) = (x(t), y(t), z(t), w(t))$ where

$$x(t) = \frac{w_0 + (cx_0 - x_0 - w_0)\cos\left(\sqrt{c-1}\,t\right) + \sqrt{c-1}(w_0 + x_0 - y_0)\sin\left(\sqrt{c-1}\,t\right)}{c - 1},$$

$$y(t) = \frac{-cw_0 + (cw_0 - cy_0 + y_0)\cos\left(\sqrt{c-1}\,t\right) + \sqrt{c-1}(y_0 - cx_0)\sin\left(\sqrt{c-1}\,t\right)}{c - 1},$$

$$z(t) = z_0,$$

$$w(t) = w_0.$$

Note that all the solutions $x(t, \mathbf{z}, \varepsilon)$ with $\mathbf{z} \neq 0$ are periodic, and have the same period $T = \dfrac{2\pi}{\sqrt{c-1}}$. We know write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{12}$$

The fundamental matrix $M_{\mathbf{z}}(t)$ is given by

$$M_{\mathbf{z}}(t) = \begin{pmatrix} \dfrac{r\cos(rt) + \sin(rt)}{r} & -\dfrac{\sin(rt)}{r} & 0 & \dfrac{1 - \cos(rt) + r\sin(rt)}{r^2} \\ \dfrac{c\sin(rt)}{r} & \dfrac{r\cos(rt) - \sin(rt)}{r} & 0 & \dfrac{c(1 - \cos(rt))}{r^2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$M_{\mathbf{z}}^{-1}(t) = \begin{pmatrix} \dfrac{r\cos(rt) - \sin(rt)}{r} & \dfrac{\sin(rt)}{r} & 0 & \dfrac{r(1 - \cos(rt) - \sin(rt))}{r^3} \\ \dfrac{c\sin(rt)}{r} & \dfrac{r\cos(rt) - \sin(rt)}{r} & 0 & \dfrac{c(1 - \cos(rt))}{r^2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where $r := \sqrt{c-1}$. Computing the integral established in Theorem 2

$$\mathcal{F}(\mathbf{z}) = \frac{r}{2\pi} \int_0^{\frac{2\pi}{r}} M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z})) dt = (\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z}), \mathcal{F}_3(\mathbf{z}), \mathcal{F}_4(\mathbf{z})),$$

where the components of $\mathcal{F}(\mathbf{z})$ are given as follows

$$\mathcal{F}_1(\mathbf{z}) = \frac{d_1(4w_0 - cw_0 + 2x_0 - y_0 - 2cx_0 + cy_0)}{2(c-1)} + \frac{a_1(-cw_0 + 2w_0 + 2x_0 - y_0 - 2cx_0 + cy_0)}{2(c-1)}$$
$$+ \frac{(c-1)x_0 z_0 - 3w_0 z_0}{2(c-1)^2},$$

$$\mathcal{F}_2(\mathbf{z}) = \frac{cd_1(3w_0 - cx_0 + x_0) + ca_1(cx_0 - x_0 - w_0)}{2(c-1)} + \frac{3cw_0 z_0 + y_0 z_0 - cy_0 z_0}{2(c-1)^2},$$

$$\mathcal{F}_3(\mathbf{z}) = \frac{3cw_0^2 + c(c-1)x_0^2}{2(c-1)^2} - \frac{y_0(w_0 + x_0)}{(c-1)} + \frac{y_0^2}{2(c-1)} - b_1 z_0,$$

$$\mathcal{F}_4(\mathbf{z}) = \frac{w_0 d_1(c-1) - w_0 z_0}{c-1},$$

and solving the nonlinear system given by $\mathcal{F}(\mathbf{z}) = 0$ we can conclude that the system has the next three solutions

$$\mathbf{s}_0 = (0, 0, 0, 0),$$

$$\mathbf{s}_1 = \left(\frac{-\sqrt{b_1 d_1(c-1)}}{\sqrt{c}}, -\sqrt{cb_1 d_1(c-1)}, d_1(c-1), \frac{-\sqrt{b_1 d_1(c-1)^3}}{\sqrt{c}} \right),$$

$$\mathbf{s}_2 = \left(\frac{\sqrt{b_1 d_1(c-1)}}{\sqrt{c}}, \sqrt{cb_1 d_1(c-1)}, d_1(c-1), \frac{\sqrt{b_1 d_1(c-1)^{3/2}}}{\sqrt{c}} \right).$$

The first solution corresponds to the equilibrium point localized at the origin. For the other two solutions, we note that

$$\mathcal{F}(\mathbf{s}_1) = \mathcal{F}(\mathbf{s}_2) = 0$$

$$\det\left((\partial\mathcal{F}/\partial\mathbf{z})\,(\mathbf{s}_1)\right) = \det\left((\partial\mathcal{F}/\partial\mathbf{z})\,(\mathbf{s}_2)\right) = \frac{1}{2}b_1 d_1^3 + a_1 b_1 d_1 (d_1 c - 1)$$

Conditions (2) ensure that

$$\frac{1}{2}b_1 d_1^3 + a_1 b_1 d_1 (d_1 c - 1) \neq 0$$

and so by Theorem 2(a) there exists two T-periodic solution $\mathbf{x}_1(t, \varepsilon)$ and $\mathbf{x}_2(t, \varepsilon)$ of system (10) such that $\mathbf{x}_1(0, \varepsilon) \to \mathbf{s}_1$ and $\mathbf{x}_2(0, \varepsilon) \to \mathbf{s}_2$ as $\varepsilon \to 0$, where its period is $T = \dfrac{2\pi}{\sqrt{c-1}}$.

To determine the type of stability of the two periodic solutions, we look at the eigenvalues of the Jacobian matrices $\Gamma_1 = ((\partial\mathcal{F}/\partial\mathbf{z})\,(\mathbf{s}_1))$ and $\Gamma_2 = ((\partial\mathcal{F}/\partial\mathbf{z})\,(\mathbf{s}_2))$. We have that the eigenvalues are the same for both matrices and given as follows

$$\lambda_{1,2} = \frac{-b_1 \pm \sqrt{b_1(b_1 - 8d_1)}}{2}$$

$$\lambda_{3,4} = \frac{-a_1 \pm i\sqrt{(a_1 + d_1)^2(c - 1)}}{2}$$

The stability follows by imposing a negative real part to all eigenvalues and substituting a_1, b_1 and d_1 in terms if a, b and d.

Finally, the periodic solutions $\mathbf{x_1} = (x_1(t, \varepsilon), y_1(t, \varepsilon), z_1(t, \varepsilon), w_1(t, \varepsilon))$ and $\mathbf{x_2} = (x_2(t, \varepsilon), y_2(t, \varepsilon), z_2(t, \varepsilon), w_2(t, \varepsilon))$ of system (10), provide the periodic solutions

$$\varepsilon\mathbf{x_1} = (\varepsilon x_1(t, \varepsilon), \varepsilon y_1(t, \varepsilon), \varepsilon z_1(t, \varepsilon), \varepsilon w_1(t, \varepsilon))$$

and

$$\varepsilon\mathbf{x_2} = (\varepsilon x_1(t, \varepsilon), \varepsilon y_2(t, \varepsilon), \varepsilon z_2(t, \varepsilon), \varepsilon w_2(t, \varepsilon))$$

of system (9). Since each of the later periodic solutions tends to the equilibrium point $(0, 0, 0, 0)$ when $\varepsilon \to 0$, they correspond to a zero–Hopf bifurcation of the zero–Hopf equilibrium point. $\square$

References

1. R. BARBOZA, *Dynamics of a hyperchaotic Lorenz system*, Int. J. of Bifurcation and Chaos **17**, 4285–4294, (2007).
2. N. N. BOGOLIUBOV, *On some statistical methods in mathematical physics*, Izv. Akad. Nauk Ukr. SSR, Kiev, (1945).
3. N. N. BOGOLIUBOV AND N. KRYLOV, *The application of methods of nonlinear mechanics in the theory of stationary oscillations*, Publ. 8 of the Ukrainian Acad. Sci. Kiev, (1934).

4. A. BUICĂ, J. P. FRANÇOISE AND J. LLIBRE, *Periodic solutions of nonlinear periodic differ-ential systems with a small parameter*, Comm. on Pure and Appl. Anal. **6**, 103–111, (2007).
5. A.R. CHAMPNEYS AND V. KIRK, *The entwined wiggling of homoclinic curves emerging from saddle-node/Hopf instabilities*, Physica D **195**, 77–105, (2004).
6. M.M. EL-DESSOKY AND E. SALEH, *Generalized projective synchronization for different hyperchaotic dynamical systems*, Discrete Dynamics in Nature and Society Volume **2011**, Article ID 437156, 19 pp., (2011).
7. P. FATOU, *Sur le mouvement d'un systàme soumis à des forces à courte période*, Bull. Soc. Math. France **56**, 98–139, (1928).
8. J. GUCKENHEIMER, *On a codimension two bifurcation*, Lecture Notes in Math. **898**, 99–142, (1980).
9. J. GUCKENHEIMER AND P. HOLMES, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, revised and corrected reprint of the 1983 original. Applied Mathematical Sciences **42**, Springer–Verlag, New York, (1990).
10. Fu, Guiyuan ROBUST ADAPTIVE MODIFIED FUNCTION PROJECTIVE SYNCHRONIZATION OF DIFFERENT HYPERCHAOTIC SYSTEMS SUBJECT TO EXTERNAL DISTURBANCE, Commun. Nonlinear Sci. Numer. Simul. **17** , no. 6, 26022608, (2012).
11. H. HAKEN, *Analogy between higher instabilities in fluids and lasers*, Physics Letters A **53**, 77–78, (1975).
12. M. HAN, *Existence of periodic orbits and invariant tori in codimension two bifurcations of three dimensional systems*, J. Sys. Sci & Math. Scis. **18**, 403–409, (1998).
13. J. HUANG, *Chaos synchronization between two novel different hyperchaotic systems with unknown parameters*, Nonlinear Analysis **69**, 4174–4181, (2008).
14. Q. JIA, *Hyperchaos generated from the Lorenz chaotic system and its control*, Physics Letters A **366**, 217–222, (2007).
15. E. KNOBLOCH, *Chaos in the segmented disc dynamo*, Physics Letters A **82**, 439–440, (1981)
16. Y. A. KUZNETSOV, *Elements of Applied Bifurcation Theory*, Springer–Verlag, 3rd Edition, (2004).
17. H. LI AND M. WANG *Hopf bifurcation analysis in a Lorenz-type system*, Nonlinear Dyn. **71**, 235240, (2013)
18. J. LLIBRE, *Zero–Hopf bifurcation in the Rössler system*, preprint, (2012).
19. E.N. LORENZ, E. N. *Deterministic nonperiodic flow*, J. Atmos. Sci. **20**, 130–141, (1963),
20. I.G. MALKIN, *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, (1956).
21. M. ROSEAU, *Vibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol.**8** Springer–Verlag, Berlin–New York, (1966).
22. J. SANDERS, F. VERHULST AND J. MURDOCK, *Averaging method in nonlinear dynamical systems*, Second edition, Applied Mathematical Sciences **59**, Springer, New York, (2007).
23. J. SCHEURLE AND J. MARSDEN, *Bifurcation to quasi-periodic tori in the interaction of steady state and Hopf bifurcations*, SIAM. J. Math. Anal. **15**, 1055–1074, (1984).
24. S. YU, J. LU, X. YU, XINGHUO AND G. CHEN *Design and implementation of grid multiwing hyperchaotic Lorenz system family via switching control and constructing super-heteroclinic loops*, IEEE Trans. Circuits Syst. I. Regul. Pap. **59**, No. 5, 10151028, (2012).
25. K.S. SUDHEER AND M. SABIR, *Adaptive modified function projective synchronization between hyperchaotic Lorenz system and hyperchaotic Lu system with uncertain parameters*, Physics Letters A **373**, 3743–3748, (2009).
26. F. VERHULST, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer, (1991).
27. X. WANG AND M. WANG, *A hyperchaos generated from Lorenz system*, Physica A **387**, 3751–3758, (2008).
28. X. SHI, W. WANG AND Q. LIU *Synchronization of noise perturbed hyperchaotic Lorenz time-delay system via a single controller with one variable*, Int. J. Nonlinear Sci. **14**, No. 1, 31–37, (2012).
29. X. SHI AND Z. WANG *The alternating between complete synchronization and hybrid synchronization of hyperchaotic Lorenz system with time delay*, Nonlinear Dynam. **69**, No. 3, 1177–1190, (2012).
30. X. SHI AND Z. WANG *A single adaptive controller with one variable for synchronizing two identical time delay hyperchaotic Lorenz systems with mismatched parameters*, Nonlinear Dynam. **69**, No. 1-2, 117–125, (2012).
31. C. YANG, C.H. TAOA AND P. WANG, *Comparison of feedback control methods for a hyperchaotic Lorenz system*, Physics Letters A **374**, 729–732, (2010).

32. Y. CHAI, C. YI AND W. LIPING *Ranchao Inverse projective synchronization between two different hyperchaotic systems with fractional order*, J. Appl. Math. **2012** , Art. ID 762807, 18pp., (2012).

33. C. ZHU, *Controlling hyperchaos in hyperchaotic Lorenz system using feedback controllers*, Applied Math. and Computation **216**, 3126–3132, (2010).