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ON THE ANALYTIC INTEGRABILITY OF THE CORED GALACTIC HAMILTONIAN

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ABSTRACT. We provide a complete characterization of the analytic first integrals of the cored galactic Hamiltonian.

1. Introduction and statement of the main results

We consider the planar cored galactic Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2/q) + \sqrt{1 + x^2 + \frac{y^2}{q^2}},$$

where $q > 0$. Its associated Hamiltonian system is

$$\begin{aligned} x' &= p_x, \\ y' &= \frac{p_y}{q}, \\ p_x' &= -\frac{x}{\sqrt{1 + x^2 + y^2/q^2}}, \\ p_y' &= -\frac{y}{q\sqrt{1 + x^2 + y^2/q^2}}, \end{aligned} \tag{1}$$

where the prime denotes derivative with respect to time t. Note that this Hamiltonian system has two degrees of freedom.

The potential

$$\sqrt{1 + x^2 + \frac{y^2}{q^2}}$$

has an absolute minimum and a reflection symmetry with respect the two axis x and y. The motivation for the choice of these symmetries comes from the interest of this potential in galactic dynamics, see for instance [2, 3, 4, 9, 10, 11]. The parameter q gives the ellipticity of the potential, which ranges in the interval $0.6 \le q \le 1$. Lower values of q have no physical meaning and greater values of q are equivalent to reverse the role of the coordinate axes.

The main aim of this paper is to study the existence of a second analytic first integral F of the cored galactic Hamiltonian system (1) *independent* of H, i.e. the gradients of F and H are linearly independent at any point of the phase space except

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perhaps in a zero Lebesgue measure set. The existence of a such second independent first integral allows to describe completely the dynamics of a Hamiltonian system with two degrees of freedom, see for instance [1].

The Hamiltonian vector field X associated to system (1) is

$$X = p_x \frac{\partial}{\partial x} + \frac{p_y}{q} \frac{\partial}{\partial y} - \frac{x}{\sqrt{1 + x^2 + y^2/q^2}} \frac{\partial}{\partial p_x} - \frac{y}{q\sqrt{1 + x^2 + y^2/q^2}} \frac{\partial}{\partial p_y}.$$

Let U be an open and dense set in $\mathbb{R}^4$. We say that the non–constant function $F\colon U \to \mathbb{R}$ is a first integral of the vector field X on U, if $F(x(t), y(t), p_x(t), p_y(t)) = \text{constant}$ for all values of t for which the solution $(x(t), y(t), p_x(t), p_y(t))$ of X is defined on U. Clearly F is a *first integral* of X on U if and only if $XF = 0$ on U. An *analytic* first integral is a first integral F being F an analytic function. Of course the Hamiltonian H is an analytic first integral of system (1). By definition a Hamiltonian system with 2 degrees of freedom having 2 independent first integrals is *completely integrable*, see for more details [1].

Proposition 1. *When $q = 1$ the cored galactic Hamiltonian system* (1) *is completely integrable with the first integrals H and $F = yp_x - xp_y$.*

Since $XF = 0$ when $q = 1$, and clearly H and F are independent the proposition follows. Hence, from now on we will restrict to the case $q \neq 1$.

Note that the cored galactic Hamiltonian system (1) is not a polynomial differential system, and consequently the Darboux theory of integrability (see for instance [8]), which is very useful for finding first integrals, cannot be applied to system (1).

Theorem 2. *The unique analytic first integrals of the cored galactic Hamiltonian system* (1) *with $q \neq 1$ are analytic functions in the variable H.*

The proof of Theorem 2 is given in section 2. As a Corollary we have the following result.

Corollary 3. *The cored galactic Hamiltonian system* (1) *with $q \neq 1$ is not completely integrable with analytic first integrals.*

2. Proof of Theorem 2

Doing the change of the independent variable from t to s given by

$$dt = \sqrt{1 + x^2 + y^2/q^2}\, ds,$$

the differential system (1) becomes

$$\begin{aligned} \dot{x} &= p_x \sqrt{1 + x^2 + y^2/q^2}, \\ \dot{y} &= \frac{p_y}{q} \sqrt{1 + x^2 + y^2/q^2}, \\ \dot{p}_x &= -x, \\ \dot{p}_y &= -\frac{y}{q}, \end{aligned}$$

where the dot denotes derivative with respect to the variable s. Taking the notation $Y = y/q$, $Q = 1/q^2 > 0$ we get

$$\begin{aligned}
\dot{x} &= p_x\sqrt{1+x^2+Y^2},\\
\dot{Y} &= Qp_y\sqrt{1+x^2+Y^2},\\
\dot{p}_x &= -x,\\
\dot{p}_y &= -Y.
\end{aligned}$$

We denote again Y by y, and the previous differential system writes as

$$\begin{aligned}
\dot{x} &= p_x\sqrt{1+x^2+y^2},\\
\dot{y} &= Qp_y\sqrt{1+x^2+y^2},\\
\dot{p}_x &= -x,\\
\dot{p}_y &= -y.
\end{aligned} \tag{2}$$

Note that system (2) has the first integral

$$H_0 = \frac{1}{2}(p_x^2 + Qp_y^2) + \sqrt{1+x^2+y^2}.$$

Now we observe that since $\dot{H}_0 = 0$ we can rewrite system (2) as the following system in the five variables (x, y, p_x, p_y, H_0):

$$\begin{aligned}
\dot{x} &= p_x\Big(H_0 - \frac{1}{2}(p_x^2 + Qp_y^2)\Big) = P_1(x, y, p_x, p_y, H_0),\\
\dot{y} &= Qp_y\Big(H_0 - \frac{1}{2}(p_x^2 + Qp_y^2)\Big) = P_2(x, y, p_x, p_y, H_0),\\
\dot{p}_x &= -x = P_3(x, y, p_x, p_y, H_0),\\
\dot{p}_y &= -y = P_4(x, y, p_x, p_y, H_0),\\
\dot{H}_0 &= 0 = P_5(x, y, p_x, p_y, H_0).
\end{aligned} \tag{3}$$

Note that F is an analytic first integral of system (1) if and only if it is an analytic first integral of system (3). So now we will look for analytic first integrals of system (3).

Now the differential system (3) is a polynomial differential system in the variables (x, y, p_x, p_y, H_0). Let $\mathbb{N}$ denote the set of positive integers. We recall that the polynomial differential system (3) is *quasi-homogeneous* if there exist $s_1, s_2, s_3, s_4, s_5, d \in \mathbb{N}$ such that for arbitrary $\lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R}, \lambda > 0\}$ we have

$$P_i(\lambda^{s_1}x, \lambda^{s_2}y, \lambda^{s_3}p_x, \lambda^{s_4}p_y, \lambda^{s_5}H_0) = \lambda^{s_i-1+d}P_i(x, y, p_x, p_y, H_0)$$

for $i = 1, \dots, 5$. We call $(s_1, \dots, s_5)$ the *weight exponents* of system (3) and d the *weight degree* with respect to the weight exponents $(s_1, \dots, s_5)$. For the weigh homogeneous polynomial differential systems we have the following well known result (see for instance [7] for a proof).

Lemma 4. *Let F be an analytic function and let $F = \sum_i F_i$ be its decomposition into weight homogeneous polynomials of degree i. Then F is an analytic first integral of a weight homogeneous polynomial differential system if and only if F_i is a weight homogeneous polynomial first integral of such polynomial differential system for all i.*

Note that the differential system (3) is a weight–homogeneous polynomial differential system with weight exponents $(2, 2, 1, 1, 2)$ and weight degree $d = 2$.

In view of Lemma 4 for studying the existence of analytic first integrals of the weight–homogeneous polynomial system (4) it is enough to study the existence of polynomial first integrals.

We restrict the differential system (3) to $H_0 = 0$ (which obviously gives a restriction to x, y, p_x, p_y). Then system (3) restricted to $H_0 = 0$ becomes

$$
\begin{aligned}
\dot{x} &= -\frac{1}{2}p_x(p_x^2 + Qp_y^2),\\
\dot{y} &= -\frac{1}{2}Qp_y(p_x^2 + Qp_y^2),\\
\dot{p}_x &= -x,\\
\dot{p}_y &= -y
\end{aligned}
\tag{4}
$$

Note that system (4) is a weight–homogeneous polynomial differential system with weight exponents $(2, 2, 1, 1)$ and weight degree $d = 2$.

The differential system (4) is a Hamiltonian system with Hamiltonian

$$
H_1 = \frac{1}{2}(x^2 + y^2) - \frac{1}{8}(p_x^2 + Qp_y^2)^2. \tag{5}
$$

Setting $p_1 = x$, $p_2 = y$, $q_1 = p_x$, $q_2 = p_y$, system (4) is a particular case of the Hamiltonian systems associated to the following family of Hamiltonians

$$
H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \tag{6}
$$

with V a homogeneous polynomial of degree four, i.e.

$$
V(q_1, q_2) = -\frac{1}{8}(p_x^2 + Qp_y^2)^2. \tag{7}
$$

In [5, 6] the authors characterized all Hamiltonian systems with a Hamiltonian (6) that are completely integrable with polynomial first integrals. This is the contents of the following result. First we note that we can always do a rotation so that a homogeneous polynomial potential of degree 4 can be chosen to be equal to

$$
V = V(q_1, q_2) = \frac{a_1}{4}q_1^4 + \frac{a_2}{3}q_1^3 q_2 + \frac{a_3}{2}q_1^2 q_2^2 + \frac{a_4}{4}q_2^4, \tag{8}
$$

$a_1, a_2, a_3, a_4 \in \mathbb{C}$, that is we can take the coefficient of $q_1 q_2^3$ equal to zero.

Theorem 5. *The Hamiltonian systems with Hamiltonian* (6) *where V is a homogenous polynomial of degree* 4 *as in* (8) *are completely integrable with polynomial first* *integrals if and only if* V *is one of the potentials* $V_0, V_1, \ldots, V_9$ *given by*

$$\begin{aligned} V_k &= a(q_2 - iq_1)^k (q_2 + iq_1)^{4-k} \quad \textit{for } k = 0, \ldots, 4, \\ V_5 &= a q_2^4, \\ V_6 &= \frac{1}{4} a q_1^4 + q_2^4, \\ V_7 &= 4q_1^4 + 3q_1^2 q_2^2 + \frac{1}{4} q_2^4, \\ V_8 &= 2q_1^4 + \frac{3}{2} q_1^2 q_2^2 + \frac{1}{4} q_2^4, \\ V_9 &= -q_1^2 (q_1 + iq_2)^2 + \frac{1}{4} (q_1^2 + q_2^2)^2. \end{aligned}$$

In view of Theorem 5 in order that the Hamiltonian system associated to the Hamiltonian (5) has a second independent polynomial first integral the potential V given in (7) must be one of the potentials given in Theorem 5. This is not possible because $Q \neq 0, 1$. Hence the unique polynomial first integral of system (4) is a polynomial function in the variable H_1. But system (4) is the restriction of system (3) to $H_0 = 0$. When $H_0 = 0$ we have that

$$\sqrt{1 + x^2 + y^2} = -\frac{1}{2}(p_x^2 + Q p_y^2),$$

and then

$$H_1 = \frac{1}{2}(x^2 + y^2) - \frac{1}{8} 4(1 + x^2 + y^2) = -\frac{1}{2}.$$

Hence the unique polynomial first integrals of system (3) are polynomials in the variable H_0. Thus, the unique analytic first integrals of system (3) are analytic first integrals of H_0 which implies that the unique analytic first integrals of system (1) are analytic functions in the variable H. This concludes the proof of Theorem 2.

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