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CUBIC HOMOGENEOUS POLYNOMIAL CENTERS

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Abstract. First, doing a combination of analytical and algebraic computations, we determine by first time an explicit normal form depending only on three parameters for all cubic homogeneous polynomial differential systems having a center.

After using the averaging method of first order we show that we can obtain at most 1 limit cycle bifurcating from the periodic orbits of the mentioned centers when they are perturbed inside the class of all cubic polynomial differential systems. Moreover, there are examples with 1 limit cycles.

1. Introduction and statement of the main results

We consider *polynomial differential systems* in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y. We say that this system has *degree* m, if m is the maximum of the degrees of P and Q.

In his address to the International Congress of Mathematicians in Paris in 1900, Hilbert [7] asked for the maximum number of limit cycles which real polynomial differential systems of degree m could have, this problem is known as the 16th *Hilbert problem*, for more details about it see the surveys [8, 9].

Since the 16th Hilbert problem is too much difficult Arnold, in [1], stated the *weakened* 16*th Hilbert problem*, one version of this problem is to determine an upper bound for the number of limit cycles which can bifurcate from the periodic orbits of a polynomial Hamiltonian center when it is perturbed inside a class of polynomial differential systems, see for instante [3, 4, 6].

We recall that a singular point $p \in \mathbb{R}^2$ of a differential system (1) is a *center* if there is a neighborhood U of p such that $U \setminus \{p\}$ is filled of periodic orbits of (1). The *period annulus* of a center is the region fulfilled by all the periodic orbits surrounding the center. We say that a center located at the origin is *global* if its period annulus is $\mathbb{R}^2 \setminus \{0\}$.

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In this paper we shall study the weakened 16th Hilbert's problem but *perturbing mainly non–Hamiltonian centers* using the technique of the averaging method of first order. More precisely, our first objective will be to characterize all planar cubic homogeneous polynomial differential system having a center, and after to analyze how many limit cycles can bifurcate from the periodic orbits of these cubic homogeneous polynomial centers when we perturb them inside the class of all cubic polynomial differential systems.

The next result characterize all planar cubic homogeneous polynomial differential system having a center.

Theorem 1. *Any planar cubic homogeneous polynomial differential system having a center after a rescaling of its independent variable and a linear change of variables can be written into the form*

$$\begin{aligned} \dot{x} &= ax^3 + (b - 3\alpha\mu)x^2y - axy^2 - \alpha y^3, \\ \dot{y} &= \alpha x^3 + ax^2y + (b + 3\alpha\mu)xy^2 - ay^3, \end{aligned} \tag{2}$$

where $\alpha = \pm 1$, $a, b, \mu \in \mathbb{R}$ *and* $\mu > -1/3$.

Theorem 1 will be proved in section 2.

Our objective now is to study how many limit cycles can bifurcate from the periodic orbits of the cubic homogeneous polynomial centers (2) when we perturb them inside the class of all cubic polynomial differential systems. We consider for ε sufficiently small the following cubic polynomial differential systems

$$\begin{aligned} \dot{x} &= ax^3 + (b - 3\alpha\mu)x^2y - axy^2 - \alpha y^3 + \varepsilon p(x, y), \\ \dot{y} &= \alpha x^3 + ax^2y + (b + 3\alpha\mu)xy^2 - ay^3 + \varepsilon q(x, y), \end{aligned} \tag{3}$$

where p and q are arbitrary polynomials of degree 3, i.e.

$$\begin{aligned} p(x, y) = \;& a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 \\ & +a_7x^2y + a_8xy^2 + a_9y^3, \\ q(x, y) = \;& b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 \\ & +b_7x^2y + b_8xy^2 + b_9y^3. \end{aligned} \tag{4}$$

Clearly from (2) it is easy to check that the cubic homogeneous polynomial centers in general are non–Hamiltonian, this justifies what we have said that mainly in (3) we are perturbing non–Hamiltonian centers.

In [10] the authors provide upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of any homogeneous and quasi–homogeneous center, which can be obtained using the Abelian integral method of first order. They also show that these bounds are the best possible using the Abelian integral method of first order. In particular, applying statement (c) of Theorem A of [10] with $p = q = 1$ and $n = 3$ it follows that with the Abelian integral method of first order it can be obtained for systems (3) at most 1 limit cycle. For more details on the Abelian integral method see the second part of the book [4].

What it is not provide in [10] is an explicit formula for computing the limit cycle which can bifurcate from the periodic orbits of any homogeneous and quasi–homogeneous center. We shall provide here such a formula for the cubic homogeneous polynomial centers (2). For doing that we shall use the averaging method of first order described in section 3.

We must mention that for planar polynomial differential systems since the Abelian integral method of first order as the averaging method of first order correspond to the coefficient of first order in the small parameter of the perturbation of the displacement function, both methods provide the same information.

We need to define the following three functions:

$$
\begin{aligned}
f(\theta) &= a\cos^4\theta + (b+\alpha-3\alpha\mu)\cos^3\theta\sin\theta \\
&\quad +(b-\alpha+3\alpha\mu)\cos\theta\sin^3\theta - a\sin^4\theta, \\
g(\theta) &= \alpha\left(\cos^4\theta + 6\,\mu\cos^2\theta\sin^2\theta + \sin^4\theta\right), \\
k(\theta) &= \exp\left(\int_0^\theta \frac{f(s)}{g(s)}ds\right).
\end{aligned}
\tag{5}
$$

Let (r,θ) be the polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$. The following result is well known, but since its proof is almost immediate we shall give it in section 4.

Proposition 2. *In polar coordinates the periodic solution $r(\theta,z)$ of the cubic homogeneous polynomial center* (2) *such that $r(0,z)=z$ is $r(\theta,z)=k(\theta)z$.*

For studying the limit cycle that can bifurcate from the periodic orbits of a cubic homogeneous polynomial center perturbed inside the class of all cubic polynomial differential equations we need to define the following two functions and two numbers:

$$
\begin{aligned}
a(\theta) &= f(\theta)\sin\theta - g(\theta)\cos\theta, \\
b(\theta) &= f(\theta)\cos\theta + g(\theta)\sin\theta, \\
c_0 &= \int_0^{2\pi} \frac{a(\theta)(a_1\cos\theta + a_2\sin\theta) - b(\theta)(b_1\cos\theta + b_2\sin\theta)}{g(\theta)^2 k(\theta)^2} d\theta, \\
c_2 &= \int_0^{2\pi} \Big[\frac{a(\theta)(a_6\cos^3\theta + a_7\sin\theta\cos^2\theta + a_8\sin^2\theta\cos\theta + a_9\sin^3\theta)}{g(\theta)^2} \\
&\quad - \frac{b(\theta)(b_6\cos^3\theta + b_7\sin\theta\cos^2\theta + b_8\sin^2\theta\cos\theta + b_9\sin^3\theta)}{g(\theta)^2}\Big] d\theta.
\end{aligned}
\tag{6}
$$

Theorem 3. *The following two statements hold.*

(a) *Consider the polynomial $h(z) = c_0 + c_2z^2$. Clearly it can have at most one real positive root. Assume that z_0 is a positive root of $h(z)$. Then, for $\varepsilon \neq 0$ sufficiently small the perturbed cubic polynomial differential system* (3) *has a limit cycle such that in polar* *coordinates tends to the periodic orbit $k(\theta)z_0$ of the cubic homogeneous polynomial center* (3) *for* $\varepsilon = 0$.

(b) *There are cubic homogeneous polynomial centers for which the polynomial $h(z)$ has a real positive root.*

Theorem 3 will be proved in section 4.

2. Proof of Theorem 1

From Proposition 4.2 of [5] and its proof we have the following result.

Proposition 4. *Assume that the polynomials P and Q of the differential system* (1) *are homogeneous of degree n. If*

$$\begin{aligned} f(\theta) &= \cos\theta P(\cos\theta, \sin\theta) + \sin\theta Q(\cos\theta, \sin\theta),\\ g(\theta) &= \cos\theta Q(\cos\theta, \sin\theta) - \sin\theta P(\cos\theta, \sin\theta),\end{aligned}$$

we define

$$I = \int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta.$$

Then the origin $(0,0)$ *of system* (1) *is a global center if and only if the polynomial $xQ(x,y) - yP(x,y)$ has no real linear factors and $I = 0$.*

In Corollary 3.3 of [5] there are ten normal forms for all the cubic homogeneous polynomial differential systems. Only the normal formals (8) and (9) of that corollary satisfy that the polynomial $xQ(x,y) - yP(x,y)$ has no real linear factors, but these two normal forms can be joined in the following unique normal form

$$\begin{aligned} \dot{x} &= ax^3 + (b - 3\alpha\mu)x^2y + cxy^2 - \alpha y^3,\\ \dot{y} &= \alpha x^3 + ax^2y + (b + 3\alpha\mu)xy^2 + cy^3,\end{aligned} \tag{7}$$

where $\alpha = \pm 1$, $a, b, c, \mu \in \mathbb{R}$ and $\mu > -1/3$.

In short, in order that the cubic homogeneous polynomial differential system (7) has a center at the origin only remains that its corresponding integral I be zero. For system (7) we have

$$\begin{aligned} f(\theta) = &\ a\cos^4\theta + (b + \alpha - 3\alpha\mu)\cos^3\theta\sin\theta + (a+c)\cos^2\theta\sin^2\theta\\ &+(b - \alpha + 3\alpha\mu)\cos\theta\sin^3\theta + c\sin^4\theta\\ = &\ a\left(\cos^4\theta - \sin^4\theta\right) + (b + \alpha - 3\alpha\mu)\cos^3\theta\sin\theta\\ &+(b - \alpha + 3\alpha\mu)\cos\theta\sin^3\theta + (a+c)\sin^2\theta,\\ g(\theta) = &\ \alpha\left(\cos^4\theta + 6\,\mu\cos^2\theta\sin^2\theta + \sin^4\theta\right).\end{aligned} \tag{8}$$

Note that when $\mu > -1/3$ we have

$$g_1(\theta) = g(\theta)/\alpha = \cos^4\theta + 6\,\mu\cos^2\theta\sin^2\theta + \sin^4\theta > 0.$$

By using the symmetry of g_1 with respect to $\cos\theta$ and $\sin\theta$, and using the change of variable $\theta = \frac{\pi}{2} - \phi$, we find

$$\int_0^{2\pi} \frac{\cos^4\theta}{g_1(\theta)} d\theta = 4\int_0^{\frac{\pi}{2}} \frac{\cos^4\theta}{g_1(\theta)} d\theta = 4\int_0^{\frac{\pi}{2}} \frac{\sin^4\phi}{g_1(\phi)} d\phi = \int_0^{2\pi} \frac{\sin^4\theta}{g_1(\theta)} d\theta.$$

On the other hand, splitting the integration interval $[0,\pi]$ to $[0,\pi/2]$ and $[\pi/2,\pi]$ and making the change $\theta=\pi-\phi$ in $[\pi/2,\pi]$, we have

$$\int_0^{2\pi}\frac{\cos^3\theta\sin\theta}{g_1(\theta)}d\theta=2\int_0^{\pi}\frac{\cos^3\theta\sin\theta}{g_1(\theta)}d\theta=0.$$

Similarly,

$$\int_0^{2\pi}\frac{\cos\theta\sin^3\theta}{g_1(\theta)}d\theta=2\int_0^{\pi}\frac{\cos\theta\sin^3\theta}{g_1(\theta)}d\theta=0.$$

Hence, from (8) we obtain

$$I=\frac{(a+c)}{\alpha}h(\mu), \tag{9}$$

where

$$h(\mu)=\int_0^{2\pi}\frac{\sin^2\theta}{g_1(\theta)}d\theta>0.$$

Consequently, from (9) we obtain that $I=0$ if and only if $c=-a$, and Theorem 1 follows from (7) with $c=-a$.

3. Averaging theory of first order

One of the averaging theories considers the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}}=F_0(t,\mathbf{x})+\varepsilon F_1(t,\mathbf{x})+\varepsilon^2F_2(t,\mathbf{x},\varepsilon), \tag{10}$$

with $\varepsilon=0$ to $\varepsilon\neq 0$ sufficiently small. Here the functions $F_0,F_1:\mathbb{R}\times\Omega\rightarrow\mathbb{R}^n$ and $F_2:\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\rightarrow\mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}}=F_0(t,\mathbf{x}), \tag{11}$$

has a submanifold of periodic solutions.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of system (11) such that $\mathbf{x}(0,\mathbf{z},\varepsilon)=\mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}}=D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{12}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (12).

We assume that there exists an open set V with $\mathrm{Cl}(V)\subset\Omega$ such that for each $\mathbf{z}\in\mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (11) with $\mathbf{x}(0,\mathbf{z},0)=\mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (10); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the bifurcation problem of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 5. [Perturbations of an isochronous set] *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T} \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}, 0)) dt. \tag{13}$$

Then the following statements holds:

(a) *If there exists $\mathbf{a} \in V$ with $\mathcal{F}(\mathbf{a}) = 0$ and $\det\left((\partial \mathcal{F}/\partial \mathbf{z})(\mathbf{a})\right) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (10) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{a}$ as $\varepsilon \to 0$.*

(b) *The type of stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $\left((\partial \mathcal{F}/\partial \mathbf{z})(\mathbf{a})\right)$.*

For an easy proof of Theorem 5(a) see Corollary 1 of [2]. In fact the result of Theorem 5 is a classical result due to Malkin [11] and Roseau [12].

For additional information on averaging theory see the book [13].

4. Proof of Theorem 3

We write the polynomial differential system (3) in polar coordinates (r, θ) through $x = r \cos\theta$ and $y = r \sin\theta$. In the new coordinates the system becomes

$$\begin{aligned} \dot{r} = {} & f(\theta) r^3 + \varepsilon\big(\cos\theta p(r\cos\theta, r\sin\theta) + \sin\theta q(r\cos\theta, r\sin\theta)\big), \\ \dot{\theta} = {} & g(\theta) r^2 + \varepsilon \frac{1}{r}\big(\cos\theta q(r\cos\theta, r\sin\theta) - \sin\theta p(r\cos\theta, r\sin\theta)\big). \end{aligned} \tag{14}$$

Here the functions $f(\theta)$ and $g(\theta)$ are the ones given in (8).

In the differential system (14) we take as new independent variable the variable θ, then system (14) can be written as

$$\frac{dr}{d\theta} = F_0(\theta, r) + \varepsilon F_1(\theta, r) + O(\varepsilon^2), \tag{15}$$

where

$$\begin{aligned} F_0(\theta, r) = {} & \frac{f(\theta)}{g(\theta)} r, \\ F_1(\theta, r) = {} & \frac{\cos\theta P(r\cos\theta, r\sin\theta) + \sin\theta Q(r\cos\theta, r\sin\theta)}{g(\theta) r^2} \\ & - \frac{f(\theta)(\cos\theta Q(r\cos\theta, r\sin\theta) - \sin\theta P(r\cos\theta, r\sin\theta))}{g(\theta)^2 r^2}. \end{aligned}$$

Now the differential equation (15) is into the normal form (10) taking

$$\mathbf{x} = r, \quad t = \theta, \quad T = 2\pi, \quad \Omega = \mathbb{R}, \quad \mathbf{y} = y, \quad \mathbf{z} = z, \quad V = (-\rho, \rho)$$

with $\rho > 0$ arbitrarily large. Note that the differential equation (15) satisfies the assumptions of Theorem 5.

Using the notation of section 3 the periodic solution $\mathbf{x}(t,\mathbf{z},0)=r(\theta,z,0)$ of the differential equation (11) for our differential system is

$$r(\theta,z,0)=k(\theta)z=e^{k_1(\theta)}k_2(\theta)z,$$

where

$$\begin{aligned}
k_1(\theta) &= -\frac{a}{2R\alpha}\left(\frac{(R-3\mu-1)\arctan\left(\frac{\tan\theta}{\sqrt{3\mu-R}}\right)}{\sqrt{3\mu-R}}+\frac{(R+3\mu+1)\arctan\left(\frac{\tan\theta}{\sqrt{R+3\mu}}\right)}{\sqrt{R+3\mu}}\right),\\
k_2(\theta) &= (R-3\mu)^{\frac{\alpha-\frac{b}{R}}{4\alpha}}(R+3\mu)^{\frac{\frac{b}{R}+\alpha}{4\alpha}}|\sec\theta|\left(R-\tan^2\theta-3\mu\right)^{\frac{1}{4}\left(\frac{b}{R\alpha}-1\right)}\\
&\quad\left(\tan^2\theta+R+3\mu\right)^{-\frac{\frac{b}{R}+\alpha}{4\alpha}},
\end{aligned}$$

with $R=\sqrt{9\mu^2-1}$.

Since our differential system (15) has dimension 1 we have that the solution $M_{\mathbf{z}}(t)=\mathbf{y}(t)=y(\theta)$ of the variational equation (12) for our differential system is

$$y(\theta)=\frac{y_1(\theta)}{y_2(\theta)},$$

where

$$\begin{aligned}
y_1(\theta) &= \sqrt{3\mu-R}\Big(\sqrt{3\mu+R}\big(4\alpha R+2\alpha R\log(\sec^2\theta)\\
&\qquad -\alpha R\log(R-\tan^2\theta-3\mu)-\alpha R\log(R+\tan^2\theta+3\mu)\\
&\qquad +(\alpha R-b)\log(R-3\mu)+(\alpha R+b)\log(R+3\mu)\\
&\qquad +b\log(R-\tan^2\theta-3\mu)-b\log(R+\tan^2\theta+3\mu)\big)\\
&\qquad -2a(R+3\mu+1)\arctan\left(\frac{\tan\theta}{\sqrt{3\mu+R}}\right)\Big)\\
&\quad -2a(R-3\mu-1)\sqrt{3\mu+R}\arctan\left(\frac{\tan\theta}{\sqrt{3\mu-R}}\right),\\
y_2(\theta) &= 4\alpha\sqrt{3\mu-R}\sqrt{3\mu+R}\sqrt{9\mu^2-1}.
\end{aligned}$$

Note that $M_{\mathbf{z}}(t)$ does not depend on z.

The integrant

$$M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))=\frac{F_1(\theta,r(\theta,z,0))}{y(\theta)}=\frac{F_1(\theta,k(\theta)z)}{y(\theta)}$$

of the integral (13) for our differential system is

$$\begin{aligned}
&\frac{f(\theta)(\sin\theta p(zk(\theta)\cos\theta,zk(\theta)\sin\theta))-\cos\theta q(zk(\theta)\cos\theta,zk(\theta)\sin\theta)}{z^2g(\theta)^2k(\theta)^2y(\theta)}\\
&-\frac{\cos\theta p(zk(\theta)\cos\theta,zk(\theta)\sin\theta)+\sin\theta q(zk(\theta)\cos\theta,zk(\theta))\sin\theta}{z^2g(\theta)k(\theta)^2y(\theta)}.
\end{aligned}\tag{16}$$

Using the expressions of the polynomials p and q given in (4) the function (16) can be written as

$$\frac{A(\theta)}{z^2}+\frac{B(\theta)}{z}+C(\theta)+D(\theta)z=\frac{A(\theta)+B(\theta)z+C(\theta)z^2+D(\theta)z^3}{z^2},\tag{17}$$

we do not provide the huge expressions for the functions $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$, which are easy to obtain with an algebraic manipulator as mathematica.

According to Theorem 5 we must compute the integral

$$\mathcal{F}(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{A(\theta)+B(\theta)z+C(\theta)z^2+D(\theta)z^3}{z^2}\right)d\theta.$$

With the help of an algebraic manipulator (for instance mathematica) is not difficult to see that

$$\int_0^{2\pi} A(\theta)d\theta = 0, \qquad \int_0^{2\pi} C(\theta)d\theta = 0.$$

Therefore

$$\mathcal{F}(z) = \frac{1}{2\pi z}\left(\int_0^{2\pi} B(\theta)d\theta + z^2\int_0^{2\pi} D(\theta)d\theta\right) = \frac{c_0+c_2z^2}{2\pi z} = \frac{h(z)}{2\pi z}. \tag{18}$$

Clearly the numerator of $\mathcal{F}(z)$ can have at most 1 simple positive zero in the interval $(0,\rho)$. Recall that z is the initial condition for the periodic solution $r(\theta,z,0)$ such that $r(0,z,0)=z$. So we are only interested in the positive zeros of the function $\mathcal{F}(z)$. Moreover, the condition $\mathcal{F}(a)=0$ of Theorem 5 applied to our function $\mathcal{F}(z)$ implies that the positive zero of $\mathcal{F}(z)$ must be simple, i.e. in the zero the derivative $\mathcal{F}'(z)$ must be non–zero. The sign of this derivative provides the stability or inestability of these periodic orbits (see statement (b) of Theorem 5), consequently these periodic orbits are limit cycles.

Finally, from statement (a) of Theorem 5 we know that if z_0 is the real positive zero of $\mathcal{F}(z)=0$, then its corresponding limit cycle in polar coordinates tends to the periodic orbit $k(\theta)z_0$ of the cubic homogeneous polynomial center (3) for $\varepsilon=0$. This completes the proof of statement (a) of Theorem 3.

Now we shall provide a cubic polynomial differential system (3) such that for $\varepsilon\neq 0$ sufficiently small will have 1 limit cycle computed using the averaging method of first order. Therefore, statement (b) of Theorem 3 will be proved.

Consider the cubic homogeneous center

$$\dot{x} = -y^3, \qquad \dot{y} = x^3,$$

and the following perturbation of it inside the cubic polynomial differential systems

$$\dot{x} = -y^3 + \varepsilon(x-x^3), \qquad \dot{y} = x^3.$$

For this system we have

$$\begin{aligned} k(\theta) &= \frac{\sqrt{2}}{\sqrt[4]{\cos(4\theta)+3}}, \\ y(\theta) &= \frac{1}{4}(4+\log 4 - \log(\cos(4\theta)+3)), \end{aligned}$$

and the integrant (16) for this particular system writes

$$\frac{B(\theta)+D(\theta)z^2}{z},$$

where

$$B(\theta) = \frac{4\big(\cos^4\theta - 6\cos^2\theta\sin^2\theta + 4\cos^2\theta + \sin^4\theta - 4\sin^2\theta + 3\big)}{2L(\theta)^2 k(\theta) y(\theta)},$$

$$D(\theta) = -\frac{d(\theta)}{2L(\theta)^2 k(\theta) y(\theta) z}$$

with

$$\begin{aligned} d(\theta) = \;& k(\theta)^2\big(\cos^6\theta - 15\cos^4\theta\sin^2\theta + 6\cos^4\theta + 15\cos^2\theta\sin^4\theta \\ & -36\cos^2\theta\sin^2\theta + 15\cos^2\theta - \sin^6\theta + 6\sin^4\theta - 15\sin^2\theta + 10\big), \\ L(\theta) = \;& \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + 3. \end{aligned}$$

Then, computing numerically the integral (18) we get

$$\mathcal{F}(z) = \frac{0.5932194545078... - 0.5327637396520...z^2}{z}.$$

So the function $\mathcal{F}(z)$ has a unique positive simple zero. This completes the proof of statement (b) of Theorem 3.

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