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THE SYMMETRIC CENTRAL CONFIGURATIONS OF THE 4–BODY PROBLEM WITH MASSES $m_1 = m_2 \neq m_3 = m_4$

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Abstract. We characterize the planar central configurations of the 4-body problem with masses $m_1 = m_2 \neq m_3 = m_4$ which have an axis of symmetry.

It is known that this problem has exactly two classes of convex central configurations, one with the shape of a rhombus and the other with the shape of an isosceles trapezoid.

We show that this 4-body problem also has exactly two classes of concave central configurations with the shape of a kite, this proof is assisted by computer.

1. Introduction and statement of the main results

The main problem of the classical Celestial Mechanics is the *n-body problem*; i.e. the description of the motion of n particles of positive masses under their mutual Newtonian gravitational forces. This problem is completely solved only when $n = 2$, and for $n > 2$ there are only few partial results. In this paper all the results commented or proved will be on positive masses, and we do not mention this again.

Central configurations are initial positions of the n bodies where the position and the acceleration vector of each particle with respect to the center of mass are proportional, with the same constant of proportionality for all the n particles. Central configurations started to be studied in the second part of the 18th century, there is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in the books of Wintner [28] and Hagihara [10]. For a modern background see, for instance, the papers of Saari [24], McCord [18], Palmore [22], Schmidt [25], Xia [29], ...

One of the reasons why central configurations are important is that they allow to obtain the unique explicit solutions in function of the time of the n–body problem known until now, the *homographic solutions* for which the

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ratios of the mutual distances between the bodies remain constant. They are also important because the total collision or the total parabolic escape at infinity in the n-body problem is asymptotic to central configurations, see for more details Dziobek [7] and Saari [24]. Also if we fix the total energy h and the angular momentum c of the n–body problem, then some of the bifurcation points (h, c) for the topology of the level sets with energy h and angular momentum c are related with the central configurations, see Meyer [19] and Smale [27] for a full background on these topics.

We note that if $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n) \in (\mathbb{R}^2)^n$ is a planar central configuration for the n–body problem, then $k\mathbf{q}$ and $A\mathbf{q} = (A\mathbf{q}_1, \cdots, A\mathbf{q}_n)$ are also central configurations for any constant $k > 0$ and any rotation $A \in SO(2)$. Therefore we count the planar central configurations modulo rotations and scalar changes, and we call them *classes* of central configurations.

In 1910 Moulton [20] classified the collinear central configurations by showing that there exist exactly $n!/2$ classes of collinear central configurations of the n-body problem for a given set of masses, one for each possible ordering of the particles.

For an arbitrary given set of masses the number of classes of planar non-collinear central configurations of the n-body problem has been only solved for $n = 3$. In this case they are the three collinear and the two equilateral triangle central configurations, due to Euler [8] and Lagrange [12] respectively. Recently, Hampton and Moeckel [11] proved that for any choice of four masses there exist a finite number of classes of central configurations. For five or more masses this result is unproved, but recently an important contribution to the case of five masses has been made by Albouy and Kaloshin [4].

Under the assumption that every central configuration of the 4-body problem has an axis of symmetry when the four masses are equal, the central configurations were characterized studying the intersection points of two planar curves in [15]. Later on in [1, 2] Albouy provided a complete proof for the classes of central configurations of the 4–body problem with equal masses.

We say that a planar non–collinear central configuration of the 4-body problem has a *kite shape* or simply is a *kite central configuration* if it has an axis of symmetry passing through two of the masses.

Bernat, Llibre and Pérez–Chavela [5] complet the characterization of the kite planar non–collinear classes of central configurations with three equal masses, started by Leandro in [14].

The goal in this paper is to complete the characterization of the classes of central configurations having an axis of symmetry for the planar 4–body problem with masses $m_1 = m_2 \neq m_3 = m_4$.

For the 4-body problem a central configuration is *convex* if none of the bodies is located in the interior of the convex hull of the other three, otherwise and if the configuration is not collinear we say that the central configuration is *concave*.

MacMillan and Bartky [17] stated that, for any four masses and any assigned order, there is a convex planar central configuration of the 4-body problem, also they shown that there is a unique isosceles trapezoid central configuration of the planar Newtonian 4–body problem with $m_1 = m_2 \neq m_3 = m_4$ when two pairs of equal masses are located at adjacent vertices of the trapezoid. Xia in [30] gave a simpler proof of the first mentioned result of [17]. Recently the isosceles trapezoid central configuration has been revisited by Xie [31].

Also for the 4-body problem with masses $m_1 = m_2 > m_3 = m_4$ Long and Sun [16] proved that any convex central configuration such that the diagonal determined by the masses m_1 is not shorter than that determined by the masses m_3, must possess an axis of symmetry and this central configuration has the shape of a rhombus.

All these results about the central configurations with an axis of symmetry for the 4–body problem with masses $m_1 = m_2 \neq m_3 = m_4$ can be summarized as follows.

Theorem 1. *There are exactly two classes of convex central configurations having an axis of symmetry for the planar* 4*–body problem with masses* $m_1 = m_2 \neq m_3 = m_4$*. One has the shape of a rhombus when the diagonal opposite masses are equal, and the other has the shape of an isosceles trapezoid when the equal masses are adjacent.*

Related with Theorem 1 the readers can look the results of the papers of Albouy, Fu, and Sun[3] and of Perez–Chavela and Santoprete [23].

The rest of the paper is dedicated to show the following result.

Theorem 2. *There are exactly two classes of concave kite central configurations for the planar* 4*–body problem with masses* $m_1 = m_2 \neq m_3 = m_4$*.*

The proof of Theorem 2 is analytical modulo the computations of the roots of polynomials of one variable which are computed numerically with the help of the algebraic manipulator mathematica.

We must mention that the results of these two theorems are compatible with the numerical results obtained by Simó in [26], and by Grebenikov, Ikhsanov and Prokopenya [9].

The paper is organized as follows. In section 2 we present the system of equations for the kite concave central configurations of the 4–body problem here studied. Finally, in section 3 we prove Theorem 2.

2. Equations of the concave kite central configurations

Newton's planar n–body problem is about the motion of n point particles with masses m_i and position vectors $\mathbf{q}_i \in \mathbb{R}^2$ for $i = 1, \ldots, n$, under their mutual attractions due to Newtonian gravitation. Its equations of motion are

$$m_i\,\ddot{\mathbf{q}}_i = -\sum_{j=1,\,j\neq i}^{n} m_i\,m_j\,\frac{\mathbf{q}_i-\mathbf{q}_j}{\|\mathbf{q}_i-\mathbf{q}_j\|^3}\;, \qquad i=1,\ldots,n\;,$$

where the units have been taken in such a way that the gravitation constant be equal to the unity. Here $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^2$.

Assume that the center of mass of the n particles is at the origin of coordinates, i.e.

$$\sum_{j=1}^{n} m_i\mathbf{q}_i = 0.$$

We say that the n bodies form a *central configuration* if there exists a constant λ such that

$$\lambda\,\mathbf{q}_i = -\sum_{j=1,\,j\neq i}^{n} m_j\,\frac{\mathbf{q}_i-\mathbf{q}_j}{\|\mathbf{q}_i-\mathbf{q}_j\|^3}\;, \qquad i=1,\ldots,n\;. \tag{1}$$

In the case of $n = 4$ bodies, Dziobek [7] reduces the equations of the central configurations (1) to the following system of 12 equations and 12 unknowns:

$$\begin{aligned}\frac{1}{r_{ij}^3} &= c_1 + c_2\frac{\Delta_i\Delta_j}{m_im_j},\\ t_i - t_j &= 0,\end{aligned} \tag{2}$$

for $1 \le i < j \le 4$, with

$$\begin{aligned} r_{ij} &= \|\mathbf{q}_i-\mathbf{q}_j\|,\\ t_i &= \sum_{j=1,\,j\neq i}^{4} \Delta_j\, r_{ij}^2,\end{aligned}$$

where Δ_1, Δ_2, Δ_3 and Δ_4 denote the oriented area of the triangle of vertices (m_2, m_3, m_4), (m_4, m_3, m_1), (m_1, m_2, m_4) and (m_3, m_2, m_1), respectively. More precisely,

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1\\ x_3 & y_3 & 1\\ x_4 & y_4 & 1\end{vmatrix},$$

where (x_i, y_i) is the position vector of the mass m_i. Similarly, for Δ_2, Δ_3 and Δ_4. The 12 unknowns in equations (2) are the 6 mutual distances r_{ij}, the 4 oriented areas Δ_i, and the two constants c_k.

Now we shall obtain the equations for the concave kite central configurations of the 4–body problem with masses $m_1 = m_2 \neq m_3 = m_4$.

By convenience, but without loss of generality, we can assume that the bodies have masses $m_1 = m_2 = 1 > m_3 = m_4 = m$. Moreover, due to the fact that we are interested in the classes of central configurations, again without loss of generality we can suppose that the positions of the four particles are $\mathbf{q}_1 = (-1/2, 0)$, $\mathbf{q}_2 = (1/2, 0)$, $\mathbf{q}_3 = (0, a)$ and $\mathbf{q}_4 = (0, b)$ with $0 < a < b$. Of course, the axis of symmetry here is the y–axis and the configuration is kite because the masses m_3 and m_4 are on this axis. Then, easy computations shown that the 12 equations (2) reduce to the following two equations

(3)

$$\begin{aligned} f_1 &= 8(a-b)^2(m^2a^2 + m^2b^2 - 2m^2ab + 4ab) + (4b^2+1)^{3/2}(2ma^4 - 4a^3b \\ &\quad -6ma^3b + 8a^2b^2 + 6ma^2b^2 - 4ab^3 + m^2a - 2mab^3 + 2ma - m^2b) \\ &= A + B, \\ f_2 &= 8(a-b)^2(m^2a^2 + m^2b^2 - 2m^2ab + 4ab) - (4a^2+1)^{3/2}(-2mb^4 + 4ab^3 \\ &\quad +6mab^3 - 8a^2b^2 - 6ma^2b^2 + 4a^3b + m^2b + 2ma^3b + 2mb - m^2a) \\ &= A - C, \end{aligned}$$

where $f_i = f_i(a, b)$ for $i = 1, 2$.

For a fixed $m \in (0, 1)$ the solutions (a, b) with $0 < a < b$ of system $f_1 = f_2 = 0$, provide the classes of concave kite central configurations of the 4–body problem with masses $m_1 = m_2 = 1 > m_3 = m_4 = m$.

3. Proof of Theorem 2

For solving system $f_1(a, b) = f_2(a, b) = 0$ we define the polynomials

$$g_1(a, b) = A^2 - B^2 \quad \text{and} \quad g_2(a, b) = A^2 - C^2.$$

Clearly if (a, b) is a solution of the system $f_1(a, b) = f_2(a, b) = 0$, then (a, b) is a solution of the polynomial system $g_1(a, b) = g_2(a, b) = 0$. Of course, the converse is not true. We shall work with g_1 and g_2 because they are polynomials in the variables a and b, and for studying the solutions of the polynomial system $g_1(a, b) = g_2(a, b) = 0$ we can use resultants, see [13, 21] for definitions and results on the resultants.

Now we consider the resultant $h(a)$ of the polynomials $g_1(a, b)$ with $g_2(a, b)$ with respect to the variable b. Therefore $h(a)$ is a polynomial in the variable a with coefficients polynomials in the variable m. More precisely, $h(a)$ is

$$\begin{aligned} &268435456m^{12}(1+m)^4a^4(1+4a^2)^9(4a+4a^2+m^2)^4(-4032a^6+768a^8+ \\ &3072a^{10}+4096a^{12}+16ma^3+192ma^5+8192ma^6+768ma^7+1024ma^9+ \\ &m^2+12m^2a^2+48m^2a^4-4032m^2a^6)(16a^2-16a^3+16a^4-4m^2a- \\ &4m^2a^2+m^4)^4h_1(a), \end{aligned}$$

where $h_1(a)$ is a polynomial in the variable a of degree 90 with coefficients polynomials in the variable m of degree at most 34 that we do not write here because is too much long. By the properties of the resultant note that if (a, b) is a solution of the polynomial system $g_1(a, b) = g_2(a, b) = 0$, then a is a root of the polynomial $h(a)$. Again the converse is not true.

We define $\bar{h}(a)$ as the following polynomial

$$(-4032a^6 + 768a^8 + 3072a^{10} + 4096a^{12} + 16ma^3 + 192ma^5 + 8192ma^6 + 768ma^7 + 1024ma^9 + m^2 + 12m^2a^2 + 48m^2a^4 - 4032m^2a^6)(16a^2 - 16a^3 + 16a^4 - 4m^2a - 4m^2a^2 + m^4)h_1(a).$$

Note that the polynomials $h(a)$ and $\bar{h}(a)$ have the same real positive roots. These are the roots of the polynomial $h(a)$ which we need to study.

Now we compute the resultant $r(m)$ of the polynomials $\bar{h}(a)$ and $d\bar{h}(a)/da$ with respect to the variable a. Thus $r(m)$ is a polynomial in the variable m. Since the common roots of $\bar{h}(a)$ and $d\bar{h}(a)/da$ are the roots of the polynomial $\bar{h}(a)$ with multiplicity larger than one. By the properties of the resultant the real roots $m \in (0, 1)$ of the polynomial $r(m)$ provide the possible values of m where the number of classes of concave kite central configurations of the 4–body problem with masses $m_1 = m_2 = 1 > m_3 = m_4 = m$ can change.

The 21 roots $m \in (0, 1)$ of the polynomial $r(m)$ are

0.0766875918991752976	0.4209299414821865653
0.1059682673291668694	0.4695378062378132584
0.1607304170827556946	0.4836525776747220757
0.2394770604609108232	0.4916016592961264249
0.2500000000000000000	0.5543951429506980495
0.2515934674257706168	0.5836310277625886914
0.3491312860079313915	0.5946353542558213906
0.3500392425656423452	0.7333650742201544896
0.3574351108259872461	0.8144614240009864465
0.4072033461399394284	0.8291393975330882794
0.4139942466981757925	

Now we choose 22 values of m in the interval $(0, 1)$ separating the 21 previous roots, and we will compute the solutions (a, b) with $0 < a < b$ of system $f_1(a, b) = f_2(a, b) = 0$ for every one of these 22 values of m. Using resultants the computation of these solutions is analytic with the exception of the roots of a polynomial in one variable. More precisely, by the properties of the resultant if (a^*, b^*) is a solution of the system $f_1(a, b) = f_2(a, b) = 0$, then a^* is a root of the polynomial $h(a)$, and b^* is a root of the polynomial $i(b)$, where this polynomial is the resultant of the polynomials $g_1(a, b)$ with $g_2(a, b)$ with respect to the variable a. Therefore, given a value of m we can compute all the roots of the polynomials $h(a)$ and $i(b)$, and check which pairs of these roots are solutions of the system $f_1(a, b) = f_2(a, b) = 0$. In this way we compute all the solutions of the system $f_1(a, b) = f_2(a, b) = 0$, only computing numerically the roots of polynomials of one variable.

In short, we obtain that for every value m of these 22 values described in Table 1, there are exactly two solutions (a, b) satisfying $0 < a < b$. Hence, the number of kite concave central configurations is constant and equal to two for all $m \in (0, 1)$, and consequently Theorem 2 is proved.

m	a	b
0.0755	0.0035279616758150515	0.7900771163967845
	0.7147850192650812	1.03417855000925
0.09	0.004441954024599277	0.7787882094296523
	0.7060019036050289	1.0438844559856253
0.135	0.007775225074186264	0.7487832179510557
	0.6833789836115126	1.0672672711484206
0.18	0.011866067055554435	0.7250808571727829
	0.6648260814917821	1.0839750871480232
0.245	0.019115712686377427	0.6995511906542532
	0.6417643498270774	1.1005808387742313
0.25075	0.019833062485639572	0.6977131904357448
	0.6398615024672192	1.1017168361652598
0.30036	0.02653232580677746	0.6843003904109473
	0.6240559396301911	1.1096706193316743
0.34958	0.03407707966214664	0.6748652570504416
	0.6091407677058371	1.1146855833194436
0.351	0.03430795950783352	0.6746447867500059
	0.6087181752812121	1.1147921039179474
0.4	0.04272794868139528	0.6686650390177226
	0.5943059037979875	1.1172619449723382
0.41	0.04455456411806872	0.6678152168702742
	0.591393537473783	1.1174898328710987
0.42	0.046417967023960775	0.6670846706971966
	0.5884868015506847	1.1176279758001728
0.46	0.054241355381497006	0.6653092022875151
	0.5768835718417511	1.1173085360386747
0.4765	0.05764233692960265	0.6650905027128892
	0.5720923676000066	1.1167808549456162
0.486	0.059647067866711305	0.6650959384734173
	0.569328491469086	1.116374289208752
0.5	0.06266403664343717	0.6652756726893431
	0.5652452037426297	1.1156395115984925
0.57	0.07890214700031103	0.6691408852540138
	0.544516240507697	1.1095658817324812
0.589	0.0836559924036139	0.6710211897239738
	0.5387579029545071	1.107226090713527
0.6	0.08647950174898113	0.6722698996426629
	0.5353900418852396	1.1057346294920007
0.8	0.14927214021151938	0.7169152728637981
	0.466392164652557	1.0586892090164706
0.82175	0.15791851398180692	0.7247768403083018
	0.4573924814606644	1.050704688313241
0.91455	0.2028555259533622	0.7695992886389915
	0.4114255661938648	1.0054497508935787

Table 1

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