A NOTE ON THE PERIODIC ORBITS
OF A KIND OF DUFFING EQUATIONS

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ABSTRACT. We study the periodic orbits of the modified Duffing differential equation $\ddot{y} + ay - \varepsilon y^3 = \varepsilon h(y, \dot{y})$, with $a > 0$, ε a small parameter and h a $\mathcal{C}^2$ function in its variables.

1. INTRODUCTION

In a paper published in 1922, Hamel [7] provided the first general results for the existence of periodic solutions of the periodically forced pendulum equation

(1) $$\ddot{y} + a \sin y = b \sin t.$$

This equation was the main subject of a monograph published four years earlier by Duffing [5], who had restricted his study to the approximate determination of the periodic solutions for the following approximation of equation (1):

$$\ddot{y} + ay - cy^3 = b \sin t,$$

which now is known as the *Duffing differential equation.* For more details on the history of these differential equations see the paper of Mawhin [9]. Many of the 190 references quoted in this last paper are on the periodic orbits of different kind of Duffing equations, and from its publication many new papers working on these type of periodic orbits also have been published, see for instance the papers [3, 4, 12] and the quoted references in there.

Here we consider the following *modified Duffing differential equation*

(2) $$\ddot{y} + ay - \varepsilon y^3 = \varepsilon h(y, \dot{y}),$$

with $a > 0$, ε a small parameter and h a $\mathcal{C}^2$ function in its variables.

We recall that $\bar{k}$ is a *simple zero* of a real function $f(k)$ if $f(\bar{k}) = 0$ and $(df/dk)(\bar{k}) \neq 0$.

Our main result is the following.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small and for every simple positive zero $\bar{k}$ of the function*

$$f(k) = \int_0^{2\pi} h\left(k \cos \theta, \frac{k \sin \theta}{\sqrt{a}}\right) \cos \theta \, d\theta,$$

the modified Duffing differential equation (2) has a periodic orbit $y(t)$ such that when $\varepsilon \to 0$ we have that $(\dot{y}(t), y(t))$ tends to the periodic orbit given by the ellipse

(3) $$\frac{x^2}{k^2} + \frac{y^2}{k^2/a} = 1,$$

of the differential system $\dot{x} = -ay$, $\dot{y} = x$.

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Theorem 1 will be proved in section 3 using the averaging theory, see section 2 where is described the result on averaging theory that we shall need to use here. In general the main difficulty for studying the existence of periodic orbits using the averaging theory is to find a change of variables which write the studied differential system into the normal form of the averaging theory (see (4)).

Now we shall do some applications of Theorem 1.

Corollary 2. *Let $h(x,y) = \sin x$. Then for every positive integer n there exists an $\varepsilon_0 > 0$ sufficiently small such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the modified Duffing differential equation (2) has at least n periodic orbits.*

Corollary 3. *Let $h(x,y) = \displaystyle\sum_{i+j=0}^{3} a_{ij} x^i y^j$ be an arbitrary polynomial of degree 3. If $a_{10}(a_{12} + 3\, a\, a_{30}) < 0$, then the modified Duffing differential equation (2) has at least one periodic orbit $y(t)$ such that when $\varepsilon \to 0$ we have that $(\dot{y}(t), y(t))$ tends to the periodic orbit of the differential system $\dot{x} = -ay$, $\dot{y} = x$, given by the ellipse (3) with*

$$k = 2\sqrt{-\frac{a\, a_{10}}{a_{12} + 3\, a\, a_{30}}}.$$

Corollaries 2 and 3 are proved in section 4.

2. BASIC RESULTS

In this section we present the basic result from the averaging theory that we shall need for proving the main result of this paper. For a general introduction to the averaging theory and related topics see the books [2, 6, 8, 10, 11].

We consider the differential system

$$(4) \qquad\qquad \dot{\mathbf{x}} = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon),$$

with $\mathbf{x} \in U \subset \mathbb{R}^n$, U a bounded domain and $t \geq 0$. Moreover, we assume that $F(t, \mathbf{x})$ and $R(t, \mathbf{x}, \varepsilon)$ are T–periodic in t.

The averaged system associated to system (4) is defined by

$$(5) \qquad\qquad \dot{\mathbf{y}} = \varepsilon f(\mathbf{y}),$$

where

$$(6) \qquad\qquad f(\mathbf{y}) = \frac{1}{T} \int_0^T F(s, \mathbf{y}) ds.$$

The next theorem says us under which conditions the singular points of the averaged system (5) provide T–periodic orbits of system (4). For a proof see Theorem 2.6.1 of [10], Theorems 11.5 and 11.6 of [11], and Theorem 4.1.1 of [6].

Theorem 4. *We consider system (4) and assume that the vector functions F, R, $D_{\mathbf{x}}F$, $D_{\mathbf{x}}^2 F$ and $D_{\mathbf{x}}R$ are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times U$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover we suppose that F and R are T–periodic in t, with T independent of ε.*

    (a) *If $a \in U$ is a singular point of the averaged system (5) such that $\det(D_{\mathbf{x}}f(a)) \neq 0$ then, for $|\varepsilon| > 0$ sufficiently small there exists a unique T–periodic solution $\mathbf{x}_\varepsilon(t)$ of system (4) such that $\mathbf{x}_\varepsilon(0) \to a$ as $\varepsilon \to 0$.*

(b) *If the singular point a of the averaged system* (5) *is hyperbolic then, for* $|\varepsilon| > 0$ *sufficiently small the corresponding periodic solution* $\mathbf{x}_\varepsilon(t)$ *of system* (4) *is hyperbolic and of the same stability type than a.*

3. Proof of Theorem 1

We write the second order differential equation (2) as the system of two differential equations of first order

$$(7) \qquad \begin{aligned} \dot{x} &= -ay + \varepsilon\left(y^3 + h(x, y)\right), \\ \dot{y} &= x. \end{aligned}$$

Now we shall study the solution of the differential system (7) with $\varepsilon = 0$.

Lemma 5. *With the exception of the singular point located at the origin of coordinates, the differential system* (7) *with* $\varepsilon = 0$ *has all its orbits periodic leaving on the ellipses*

$$\frac{x^2}{k^2} + \frac{y^2}{k^2/a} = 1,$$

with k real and positive.

Proof. Assuming $a > 0$, the general solution of the system (7) with $\varepsilon = 0$ and with initial conditions $x(0) = x_0, \ y(0) = y_0$ is

$$x(t) = x_0 \cos(\sqrt{a}t) - \sqrt{a}y_0 \sin(\sqrt{a}t),$$

$$(8)$$

$$y(t) = \frac{\sqrt{a}y_0 \cos(\sqrt{a}t) + x_0 \sin(\sqrt{a}t)}{\sqrt{a}}.$$

All the orbits of the system (7) with $\varepsilon = 0$ with exception of the origin are periodic of period $2\pi/\sqrt{a}$.

We solve (8) with respect to $\cos(\sqrt{a}t)$ and $\sin(\sqrt{a}t)$ and we get

$$\cos(\sqrt{a}t) = \frac{-xx_0 - ayy_0}{x_0^2 + ay_0^2},$$

$$(9)$$

$$\sin(\sqrt{a}t) = \frac{-\sqrt{a}x_0y + \sqrt{a}xy_0}{x_0^2 + ay_0^2}.$$

using the trigonometric relation $\sin^2(x) + \cos^2(x) = 1$ we get after simplifying

$$\frac{x^2 + ay^2}{x_0^2 + ay_0^2} = 1.$$

Thus, we get the ellipsis in the statement of Lemma 5 with $\overline{k}^2 = x_0^2 + ay_0^2$. $\qquad \square$

Proof of Theorem 1. Recall Lemma 5. The solution of system (7) with $\varepsilon = 0$ is the ellipsis

$$\frac{x^2}{x_0^2 + ay_0^2} + \frac{y^2}{(x_0^2/a + y_0^2)} = 1,$$

centered at the origin with semi-major axis on the x-axis and semi-minor axis on the y-axis. The semi-major axis is equal to $\sqrt{x_0^2 + ay_0^2}$ and the semi-minor axis is equal

to $\sqrt{(x_0^2 + ay_0^2)/a}$. So we write the original system of two differential equations of first order (7) in elliptic coordinates. We do the change of coordinates

$$x = \sqrt{x_0^2 + ay_0^2}\, r \cos\theta, \ y = \sqrt{x_0^2/a + y_0^2}\, r \sin\theta.$$

In these new coordinates, system (7) becomes

(10)
$$\dot{r} = \frac{\varepsilon \cos\theta}{k}\left[h\left(rk\cos\theta, r\frac{k}{\sqrt{a}}\sin\theta\right) + r^3 k^3 \sin^3\theta\right],$$

$$\dot{\theta} = \sqrt{a} - \frac{\varepsilon \sin\theta}{rk}\left[h\left(rk\cos\theta, r\frac{k}{\sqrt{a}}\sin\theta\right) + r^3 k^3 \sin^3\theta\right].$$

where $k = \sqrt{x_0^2 + ay_0^2}$.

Now applying Theorem 4 of the averaging theory described in Section 2 to system (10) and since

$$\int_0^{2\pi} \frac{\cos\theta}{k/r}\left[k^3 \sin^3\theta\right]d\theta = 0,$$

it follows immediately Theorem 1. $\square$

4. APPLICATIONS

In these section we shall prove the corollaries.

Proof of Corollary 2. We apply Theorem 1 to system (7) with

$$h(x, y) = b \sin x.$$

We write the second order differential equation as the system of two differential equations of first order

(11)
$$\begin{aligned}\dot{x} &= -ay + \varepsilon(y^3 + b\sin x),\\ \dot{y} &= x.\end{aligned}$$

We write system (11) in elliptic coordinates

$$x = r\cos(\theta), \ y = r\sin\theta/\sqrt{a}$$

and we get the system in the new coordinates (r, θ) given by

(12)
$$\dot{r} = \varepsilon \cos\theta\left(\frac{r^3 \sin^3\theta}{a^{3/2}} + b\sin t\right),$$

$$\dot{\theta} = \sqrt{a} - \frac{\varepsilon\left(\dfrac{r^3 \sin^4\theta}{a^{3/2}} + b\sin\theta\sin t\right)}{r}.$$

The system of equations (12) with $\varepsilon = 0$ and initial conditions $r(0) = r_0$, $\theta(0) = \theta_0$ has the solutions

$$r(t) = r_0, \ \theta(t) = \sqrt{a}t + \theta_0,$$

this is, all orbits with $r_0 > 0$ are periodic of period $2\pi/\sqrt{a}$.

We obtain after some tedious computations that the function (f_1, f_2) are given by

$$f_1 = b\left(-2\cos\theta_0 \sin^2\left(\tfrac{\pi}{\sqrt{a}}\right) + \sqrt{a}\sin\left(\tfrac{\pi}{\sqrt{a}}\right)\sin\theta_0\right),$$
$$f_2 = 3(-1 + a)\pi r_0^3 - 4a^{5/2}b\cos\theta_0 \sin\left(\tfrac{2\pi}{\sqrt{a}}\right) - 8a^2 b\sin^2\left(\tfrac{\pi}{\sqrt{a}}\right)\sin\theta_0.$$

Now solving $f_1 = 0$ with respect to θ_0 we get

$$\theta_0^{\pm} = \pm \arccos \left[\frac{\sqrt{a} \sin\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}} \right].$$

Substituting θ_0^- in $f_2 = 0$ and simplifying we get

$$(13) \qquad f_2 = 3(a-1)\pi r_0^3 + A,$$

where

$$A = -4a^2 b \left[\frac{a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}} \right] - 2\sqrt{\frac{1}{a + (1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)}} \left| \sin^3\left(\frac{\pi}{\sqrt{a}}\right) \right|.$$

Thus if $A(a-1) < 0$, we have a positive solution for r_0.

Now substituting θ_0^+ in $f_2 = 0$ and simplifying we get

$$(14) \qquad f_2 = 3(a-1)\pi r_0^3 + B,$$

where

$$B = -4a^2 b \left[\frac{a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}} \right] + 2\sqrt{\frac{1}{a + (1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)}} \left| \sin^3\left(\frac{\pi}{\sqrt{a}}\right) \right|.$$

Thus if $B(a-1) < 0$, we have a positive solution for r_0. Now A and B have the same sign (and consequently there are two periodic orbits) if the following expression is positive:
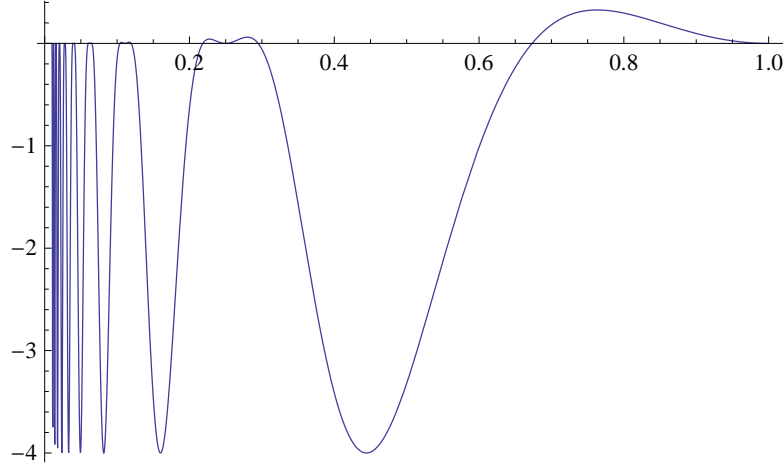
$$(15) \qquad -\frac{4\sin^6\left(\frac{\pi}{\sqrt{a}}\right)}{a - (-1+a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)} + \frac{a^2\sin^4\left(\frac{2\pi}{\sqrt{a}}\right)}{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}.$$

We analyze in Figure 1 the sign of this expression. Note that it takes positive and negative values.

In fact, we must check that both solutions when $AB > 0$ do not correspond to different initial conditions of the same periodic orbit.

We solve equation (14) with respect to r_0 and we get two different solutions

$$(16) \qquad r = \frac{B^{1/3}}{(3\pi)^{1/3}(1-a)^{1/3}}, \quad \theta = \arccos \left[\frac{\sqrt{a}\sin\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}} \right]$$

FIGURE 1. Plot of expression (15) as a function of a.

and

$$(17) \quad r = \frac{A^{1/3}}{(3\pi)^{1/3}(1-a)^{1/3}}, \ \theta = -\arccos\left[\frac{\sqrt{a}\sin\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}}\right].$$

Substituting the values for r and θ we got in (16) in the expression for x_0 and y_0

$$(18) \qquad\qquad\qquad x_0 = r\cos\theta, \ y_0 = r\sin\theta/\sqrt{a}$$

holds

$$
x_0 = \frac{\sqrt{a}B^{1/3}\sin\left(\frac{2\pi}{\sqrt{a}}\right)}{(1-a)^{1/3}(3\pi)^{1/3}\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right) + a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}},
$$

(19)

$$
y_0 = \frac{B^{1/3}\sqrt{\dfrac{1}{1-a+a\csc^2\left(\frac{\pi}{\sqrt{a}}\right)}}}{(1-a)^{1/3}\sqrt{a}(3\pi)^{1/3}}.
$$

Substituting now the values for r and θ we got in (17) in the expression for x_0 and y_0 given by (18) we get

(20)
$$x_0 = \frac{\sqrt{a}A^{1/3}\sin\left(\frac{2\pi}{\sqrt{a}}\right)}{(1-a)^{1/3}(3\pi)^{1/3}\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right)+a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}},$$

$$y_0 = -\frac{A^{1/3}\sqrt{\dfrac{1}{1-a+a\csc^2\left(\frac{\pi}{\sqrt{a}}\right)}}}{(1-a)^{1/3}\sqrt{a}(3\pi)^{1/3}}.$$

Consider the semi-axis

$$x_0^2 + ay_0^2$$

of the ellipsis. Restricted to the solution we found in (19) holds

$$\frac{2}{3^{2/3}(\pi-a\pi)^{2/3}}\left(-\frac{a^2b\left[4\dfrac{\left|\sin^3\left(\frac{\pi}{\sqrt{a}}\right)\right|}{\sqrt{\frac{1}{1+a+(-1+a)\cos\left(\frac{2\pi}{\sqrt{a}}\right)}}}+2a\cos^2\left(\frac{\pi}{\sqrt{a}}\right)\xi(a)\right]}{1+a+(-1+a)\cos\left(\frac{2\pi}{\sqrt{a}}\right)}\right)^{2/3},$$

where

$$\xi(a) = \sqrt{3+a-4\cos\left(\frac{2\pi}{\sqrt{a}}\right)-(-1+a)\cos\left(\frac{4\pi}{\sqrt{a}}\right)},$$

and restricted to the solution given by (20) we get

$$\frac{2}{3^{2/3}(\pi-a\pi)^{2/3}}\left(-\frac{a^2b\left[-4\dfrac{\left|\sin^3\left(\frac{\pi}{\sqrt{a}}\right)\right|}{\sqrt{\frac{1}{1+a+(-1+a)\cos\left(\frac{2\pi}{\sqrt{a}}\right)}}}+2a\cos^2\left(\frac{\pi}{\sqrt{a}}\right)\xi(a)\right]}{1+a+(-1+a)\cos\left(\frac{2\pi}{\sqrt{a}}\right)}\right)^{2/3}.$$

Clearly there are two different periodic orbits when both exist and for $a \to 1$ they tend to the same orbit, note that

$$\lim_{a\to 1}\frac{a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}{\sqrt{4\sin^4\left(\frac{\pi}{\sqrt{a}}\right)+a\sin^2\left(\frac{2\pi}{\sqrt{a}}\right)}} = 0.$$

Moreover, we have $a+(1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)>0$ if $a>0$. Indeed, we plot in figures 2, 3 and 4 the function $a+(1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)$ for different ranges of a.

Assume now that $a=1$. We get after a simple computation that f_1 and f_2 are given by

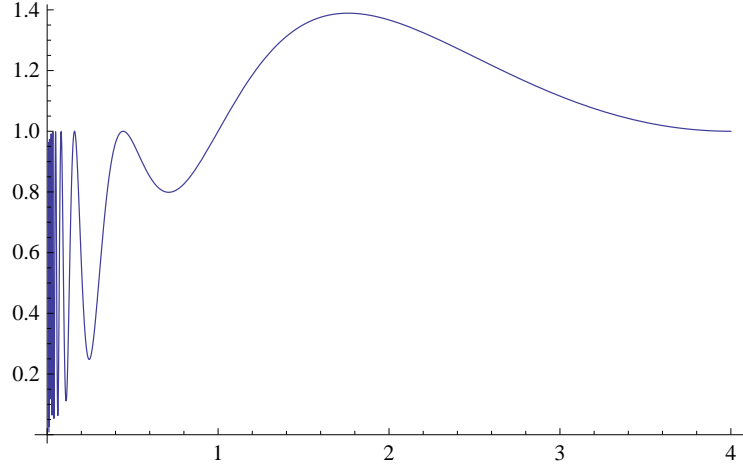$$f_1 = -\frac{1}{2}b\sin\theta_0, \ \ f_2 = -\frac{3r_0^3+4b\cos\theta_0}{8r_0}.$$

FIGURE 2. Plot of expression $a + (1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)$ as a function of a for $a \in [0.01, 4]$.
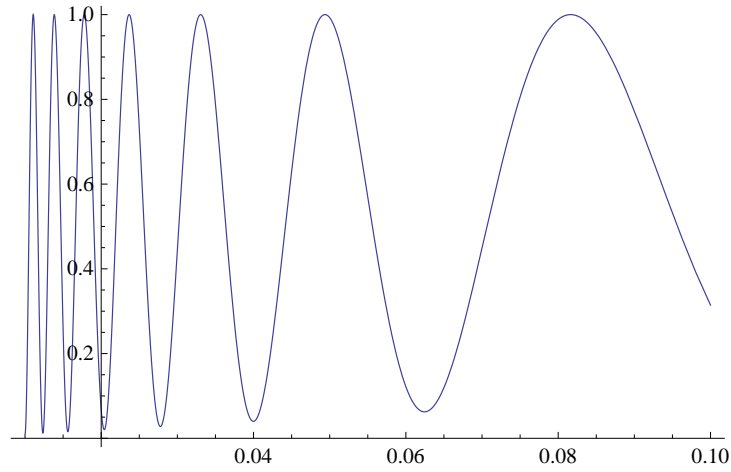

FIGURE 3. Plot of expression $a + (1-a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)$ as a function of a for $a \in [0.01, 0.1]$.

Thus, we have one positive solution if $b < 0$ and one positive solution if $b > 0$. Now substituting $x_0 = r\cos\theta$, $y_0 = r\sin\theta/\sqrt{a}$ with $r = (4|b|/3)^{1/3}$ and $\theta = 0$ or $\theta = \pi$ in $x_0^2 + ay_0^2$ holds $\frac{2^{4/3}|b|^{2/3}}{3^{2/3}}$. This finishes the proof of Corollary 2. $\qquad\square$

Proof of Corollary 3. We apply Theorem 1 to system (7) with

$$h(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3.$$
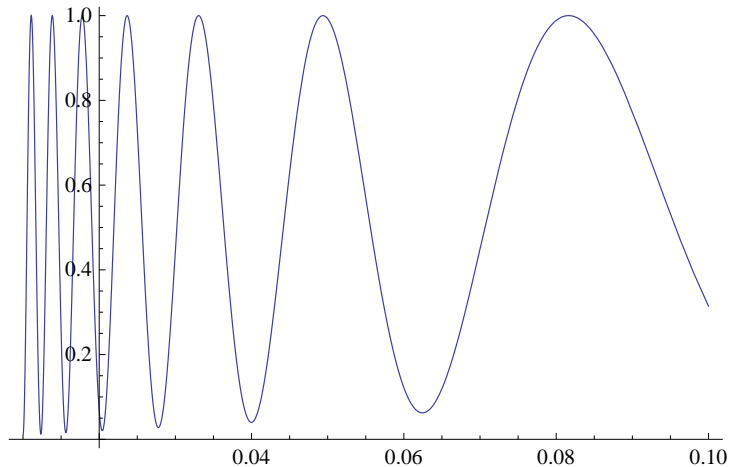
FIGURE 4. Plot of expression $a + (1 - a)\sin^2\left(\frac{\pi}{\sqrt{a}}\right)$ as a function of a for $a \in [1, 10]$.

After some simple computation we obtain that the function $f(k)$ is given by

$$f(k) = \frac{1}{8}k\left(4a_{10} + \frac{(a_{12} + 3aa_{30})k^2}{a}\right).$$

Solving $f(k) = 0$ with respect to k holds

$$k = 0, \ k = -2\sqrt{-\frac{aa_{10}}{a_{12} + 3aa_{30}}}, \ k = 2\sqrt{-\frac{aa_{10}}{a_{12} + 3aa_{30}}},$$

and this finishes the proof. $\qquad\square$

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