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ON THE NUMBER OF INVARIANT CONICS FOR THE POLYNOMIAL VECTOR FIELDS DEFINED ON QUADRICS

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Abstract. The quadrics here considered are the nine real quadrics: parabolic cylinder, elliptic cylinder, hyperbolic cylinder, cone, hyperboloid of one sheet, hyperbolic paraboloid, elliptic paraboloid, ellipsoid and hyperboloid of two sheets. Let $\mathcal{Q}$ be one of these quadrics. We consider a polynomial vector field $\mathcal{X} = (P, Q, R)$ in $\mathbb{R}^3$ whose flow leaves $\mathcal{Q}$ invariant. If $m_1 = \text{degree } P$, $m_2 = \text{degree } Q$ and $m_3 = \text{degree } R$, we say that $\mathbf{m} = (m_1, m_2, m_3)$ is the degree of $\mathcal{X}$. In function of these degrees we find a bound for the maximum number of invariant conics of $\mathcal{X}$ that result from the intersection of invariant planes of $\mathcal{X}$ with $\mathcal{Q}$. The conics obtained can be degenerate or not. Since the first six quadrics mentioned are ruled surfaces, the degenerate conics obtained are formed by a point, a double straight line, two parallel straight lines, or two intersecting straight lines; thus for the vector fields defined on these quadrics we get a bound for the maximum number of invariant straight lines contained in invariant planes of $\mathcal{X}$. In the same way, if the conic is non degenerate, it can be a parabola, an ellipse or a hyperbola and we provide a bound for the maximum number of invariant non degenerate conics of the vector field $\mathcal{X}$ depending on each quadric $\mathcal{Q}$ and of the degrees m_1, m_2 and m_3 of $\mathcal{X}$.

1. Introduction

In this paper we deal with polynomial vector fields

$$\mathcal{X} = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z},$$

where P, Q and R are polynomials of degrees m_1, m_2 and m_3 respectively in the variables x, y, z with coefficients in $\mathbb{R}$. We say that $\mathbf{m} = (m_1, m_2, m_3)$ is the *degree* of the polynomial vector field. We assume that $\mathcal{X}$ has an invariant quadric $\mathcal{Q}$, then we say that $\mathcal{X}$ is a polynomial vector field defined on the quadric $\mathcal{Q}$.

An invariant plane of $\mathcal{X}$ intersects the invariant quadric $\mathcal{Q}$ in an invariant conic. Our main objective is to study the maximum number of invariant conics of this kind that the polynomial vector field $\mathcal{X}$ can have in function of its degrees m_1, m_2 and m_3. If the conic is non degenerate, then it is an ellipse, a parabola or a hyperbola. If it is degenerate, then it is formed by a point, two straight lines that intersect at a point, or two parallel straight lines, or a double straight line.

The study of the maximum number of invariant classes of algebraic curves in $\mathbb{R}^2$, of invariant classes of surfaces in $\mathbb{R}^3$, and of invariant classes of hypersurfaces in $\mathbb{R}^n$ have been studied recently for several authors. Thus, the maximum number of

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straight lines that a polynomial vector field in $\mathbb{R}^2$ can have in function of its degree has been studied in [1, 14, 17]. The maximum number of algebraic limit cycles that a polynomial vector field in $\mathbb{R}^2$ can have has been studied in [9, 10, 18]. The maximum number of invariant meridians and parallels for polynomial vector fields on a 2–dimensional torus have been considered in [8, 11]. The maximum number of invariant hyperplanes (respectively $\mathbb{S}^{n-1}$ spheres) that polynomial vector fields can have in $\mathbb{R}^n$ have been determined in [7] (respectively [2]).

The paper is organized as follows. In section 2 we provide some basic definitions that will need later on. Next we introduce the quadrics and theirs canonical forms, and in the following sections we give the bounds for the maximum number of the invariant conics living on invariant planes that a polynomial vector field defined on a quadric can have.

2. Basic definitions and results

We recall that the *polynomial differential system* in $\mathbb{R}^3$ of degree $\mathbf{m}$ associated with the vector field $\mathcal{X}$ is

$$\frac{dx}{dt} = P(x,y,z), \quad \frac{dy}{dt} = Q(x,y,z), \quad \frac{dz}{dt} = R(x,y,z).$$

Let $\mathcal{Q}$ be a quadric given by $\mathcal{Q} = \{(x,y,z) \in \mathbb{R}^3 : G(x,y,z) = 0\}$, where $G : \mathbb{R}^3 \to \mathbb{R}$ is a polynomial of degree 2. We say that $\mathcal{X}$ defines a *polynomial vector field on the quadric* $\mathcal{Q}$ if $\mathcal{X}G = (P,Q,R)\cdot\nabla G = 0$ on all points of $\mathcal{Q}$. Here, as usual ∇G denotes the gradient of the function G.

We denote by $\mathbb{R}[x,y,z]$ the ring of all polynomials in the variables x, y, z with coefficients in $\mathbb{R}$. Let $f(x,y,z) \in \mathbb{R}[x,y,z] \setminus \mathbb{R}$. We say that $\{f = 0\} \cap \mathcal{Q} \subset \mathbb{R}^3$ is an *invariant algebraic curve of the vector field* $\mathcal{X}$ *on* $\mathcal{Q}$ (or simply an *invariant algebraic curve of* $\mathcal{Q}$) if it satisfies:

(i) There exists a polynomial $k \in \mathbb{R}[x,y,z]$ such that

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = kf \quad \text{on } \mathcal{Q},$$

the polynomial k is the *cofactor* of $f = 0$ on $\mathcal{Q}$; and

(ii) the two surfaces $f = 0$ and $\mathcal{Q}$ have transversal intersection; i.e. $\nabla G \wedge \nabla f \neq 0$ on the curve $\{f = 0\} \cap \mathcal{Q}$, where $\wedge$ denotes the wedge product of two vectors of $\mathbb{R}^3$.

By the Darboux theory of integrability we know that the existence of a sufficiently large number of invariant algebraic surfaces of $\mathcal{X}$ guarantees the existence of a first integral for the polynomial vector field $\mathcal{X}$ which can be calculated explicitly, see for instance [4, 13]. Note that if the vector field $\mathcal{X}$ has degree $\mathbf{m}$, then any cofactor has degree at most $\max\{m_1, m_2, m_3\} - 1$.

One of the best tools in order to look for invariant algebraic surfaces is the following. Let W be a vector subspace of the space of polynomials $\mathbb{R}[x,y,z]$ generated by the independent polynomials $v_1, \ldots, v_l$ in the $\mathbb{R}$–vector space $\mathbb{R}[x,y,z]$, which is denoted by $W = \langle v_1, \ldots, v_l \rangle$. The *extactic polynomial of* $\mathcal{X}$ *associated with* W is the polynomial

$$\varepsilon_W(\mathcal{X}) = \begin{vmatrix} v_1 & \cdots & v_l \\ \mathcal{X}(v_1) & \cdots & \mathcal{X}(v_l) \\ \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \cdots & \mathcal{X}^{l-1}(v_l) \end{vmatrix},$$

where $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$ for all i, j. From the properties of the determinants and of the derivation we know that the definition of the extactic polynomial is independent of the chosen basis of W.

We note that $\varepsilon_W(\mathcal{X})$ is the determinant of an $l \times l$ matrix.

The notion of extactic polynomial $\varepsilon_W(\mathcal{X})$ is important here for two reasons. First it allows to detect when an algebraic surface $f = 0$ with $f \in W$ is invariant by the polynomial vector field $\mathcal{X}$, see the next proposition. Second $\varepsilon_W(\mathcal{X})$ is also important because it allows to define and compute easily the multiplicity of an invariant algebraic surface. For more details see [3, 13].

Proposition 1. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ and let W be a finitely generated $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$ with $\dim(W) > 1$. Then, every invariant algebraic surface $f = 0$ for the vector field $\mathcal{X}$, with $f \in W$, is a factor of the polynomial $\varepsilon_W(\mathcal{X})$.*

The *multiplicity of an invariant algebraic surface* $f = 0$ with $f \in W$ is the largest positive integer k such that f^k divides the polynomial $\varepsilon_W(\mathcal{X})$ when $\varepsilon_W(\mathcal{X}) \neq 0$, otherwise the multiplicity is infinite. For more details on the multiplicity see [3, 13].

3. Quadrics and conics

It is well known that the quadrics can be classified into seventeen types. Since here we only consider real polynomial vector fields defined on real quadrics, we omit the five imaginary quadrics. Also we omit the three types of quadrics that are formed by planes since the study of the Darboux theory of integrability of the vector fields on these planes is reduced to the classic study of the Darboux theory of integrability of planar polynomial vector fields. Thus we only work with the remainder nine type of quadrics: parabolic cylinder, elliptic cylinder, hyperbolic cylinder, cone, hyperboloid of one sheet, hyperbolic paraboloid, elliptic paraboloid, ellipsoid or sphere, and hyperboloid of two sheets.

Recall that the reduction of each quadric to its canonical form is done through a linear isomorphism, and that this isomorphism and its inverse are linear diffeomorphism. Therefore, it is not restrictive to study the Darboux theory of integrability on the corresponding canonical forms of the quadrics (for more details see [12]). So, in the rest of this paper we will work with the nine canonical forms: $z^2 - x = 0$, $x^2 + z^2 - 1 = 0$, $x^2 - z^2 - 1 = 0$, $x^2 + y^2 - z^2 = 0$, $x^2 + y^2 - z^2 - 1 = 0$, $y^2 - z^2 - x = 0$, $y^2 + z^2 - x = 0$, $x^2 + y^2 + z^2 - 1 = 0$ and $x^2 + y^2 - z^2 + 1 = 0$.

Some of the quadrics are ruled surfaces and this will be used in the following sections. A surface is *ruled* [5, 6] if through every point of the surface there is a straight line that lies on it. A surface is *doubly ruled* if through every one of its points there are two different straight lines that lie on the surface. Of the nine real quadrics that we consider six of them are ruled, these are the parabolic cylinder, elliptic cylinder, hyperbolic cylinder, cone, hyperboloid of one sheet and hyperbolic paraboloid. Furthermore, the hyperboloid of one sheet and the hyperbolic paraboloid are doubly ruled surfaces.

Since we are interested in studying the number of invariant conics obtained from the intersection of invariant planes with a quadric, we recall in the following result (see [15]) the classification of conics.

Proposition 2. *Let $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with $A^2 + B^2 + C^2 \neq 0$ be the equation corresponding to a conic and consider*

$$d_1 = \begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix}.$$

The conic is non degenerate if $d_1 \neq 0$. In this case it is a hyperbola if and only if $d_2 < 0$, a parabola if and only if $d_2 = 0$, or an ellipse if and only if $d_2 > 0$. If the conic is degenerate ($d_1 = 0$), it is a pair of intersecting straight lines if and only if $d_2 < 0$, a pair of parallel straight lines if and only if $d_2 = 0$, or a point if and only if $d_2 > 0$. In the case of parallel straight lines ($d_2 = 0$), these are distinct and real if $E^2 - 4CF > 0$, double if $E^2 - 4CF = 0$, and distinct and imaginary if $E^2 - 4CF < 0$.

4. Results on the parabolic cylinder

We are going to work with the following canonical form $z^2 - x = 0$ of a parabolic cylinder $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ defined on $\mathcal{Q}$. We will see that the intersection of a parabolic cylinder with a plane are straight lines or parabolas.

In the following subsections we will find an upper bound on the maximum number of these invariant straight lines and parabolas of $\mathcal{X}$ defined on $\mathcal{Q}$, depending on the degrees m_1, m_2 and m_3 of the vector field $\mathcal{X}$.

4.1. **Invariant straight lines.** As mentioned above the parabolic cylinder is a ruled surface.

Lemma 3. *All straight lines contained in the parabolic cylinder $z^2 - x = 0$ are parallel to the y axis.*

Proof. We are going to see that any straight line that is non parallel to the y axis cannot be contained in the parabolic cylinder.

Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ be a point of the parabolic cylinder and consider a straight line in $\mathbb{R}^3$ non parallel to the y axis that goes through the point (x_0, y_0, z_0). Let (a, b, c) with $a^2 + c^2 \neq 0$ be the direction vector of the straight line. Its equation is $(x, y, z) = (x_0, y_0, z_0) + \lambda(a, b, c)$, $\lambda \in \mathbb{R}$. We will prove that this straight line is not contained in the parabolic cylinder.

If the points (x, y, z) of the straight line are on the parabolic cylinder, then it must satisfy $(z_0 + \lambda c)^2 - (x_0 + \lambda a) = 0$ for all λ. From this it follows that $z_0^2 + 2\lambda c z_0 + \lambda^2 c^2 - x_0 - \lambda a = 0$. Since $z_0^2 - x_0 = 0$, we obtain

$$2z_0 c + \lambda c^2 - a = 0. \tag{1}$$

If $c \neq 0$, equality (1) is satisfied only for $\lambda = (a - 2z_0 c)/c^2$. Since $a^2 + c^2 \neq 0$, if $c = 0$, then $a \neq 0$, which contradicts equality (1). □

Lemma 4. *Any plane non parallel to the y axis intersects the parabolic cylinder $z^2 - x = 0$ in a parabola.*

Proof. Let $ax + by + cz + d = 0$ with $b \neq 0$ be the equation of a plane non parallel to the y axis. If $c \neq 0$, then $z = -(ax + by + d)/c$. Replacing it in $z^2 - x = 0$ we obtain $a^2x^2 + 2abxy + b^2y^2 + (2ad - c^2)x + 2bdy + d^2 = 0$, that is the equation of the intersection conic. Applying Proposition 2, $d_1 = -b^2c^4/4 \neq 0$ and $d_2 = 0$. So the conic is a parabola.

Now let $c = 0$. If $a \neq 0$, then $x = -(b\,y + d)/a$. Replacing it in $z^2 - x = 0$ we get the equation of the intersection conic $az^2 + by + d = 0$. In this case $d_1 = -a\,b^2/4 \neq 0$ and $d_2 = 0$. So the conic is a parabola. If $c = a = 0$, the equation of the plane takes the form $y = -d/b$. The intersection of the parabolic cylinder with this plane is a parabola. □

Proposition 5. *The intersection conic of a parabolic cylinder $\mathcal{Q}$ with a plane is either a double straight line, or a pair of parallel straight lines, or a parabola.*

Proof. By Lemma 3 if we have a plane parallel to the y axis that intersects the parabolic cylinder $\mathcal{Q}$, its intersection is a double straight line or two parallel straight lines. If the plane is non parallel to the y axis, from Lemma 4, it intersects $\mathcal{Q}$ in a parabola. □

Denote by $\mathcal{X} = (P, Q, R)$ the polynomial vector field defined on the parabolic cylinder $z^2 - x = 0$. This implies that $P = 2zR$ and $m_1 = m_3 + 1$.

Proposition 6. *There are polynomial vector fields $\mathcal{X}$ defined on a parabolic cylinder $\mathcal{Q}$ that have infinitely many invariant straight lines, taking into account their multiplicities.*

Proof. Let $W = \langle 1, x, z\rangle$. If $f \in W$, then $f = 0$ is a parallel plane to the y axis. If these planes are invariant by $\mathcal{X}$, their intersections with $\mathcal{Q}$ are straight lines also invariant. By Proposition 1 and Lemma 3 it follows that a necessary condition in order that $\mathcal{X}$ has infinitely many invariant straight lines contained in $\mathcal{Q}$ is $\varepsilon_W(\mathcal{X}) = 0$. We know that

$$\varepsilon_W(\mathcal{X}) = \begin{vmatrix} 1 & x & z \\ 0 & \mathcal{X}(x) & \mathcal{X}(z) \\ 0 & \mathcal{X}^2(x) & \mathcal{X}^2(z) \end{vmatrix}. \tag{2}$$

Since $\mathcal{X}(x) = 2zR$, $\mathcal{X}(z) = R$ and $P = 2zR$ we obtain

$$\begin{aligned} \mathcal{X}^2(x) &= \mathcal{X}(2zR) = 2\mathcal{X}(z)R + 2z\mathcal{X}(R) \\ &= 2R^2 + 4z^2R_xR + 2zR_yQ + 2zR_zR, \\ \mathcal{X}^2(z) &= \mathcal{X}(R) = 2zRR_x + QR_y + RR_z. \end{aligned}$$

So,

$$\varepsilon_W(\mathcal{X}) = \mathcal{X}(x)\mathcal{X}^2(z) - \mathcal{X}(z)\mathcal{X}^2(x) = 2R^3. \tag{3}$$

Thus, $\varepsilon_W(\mathcal{X}) = 0$ if and only if $R = 0$. Then, we can define $\mathcal{X}$ as the vector field whose associated differential system is $\dot{x} = 0$, $\dot{y} = Q$, $\dot{z} = 0$. As $\dot{x} = 0$, the planes $f = x + c = 0$ with $c \in \mathbb{R}$ are invariant by $\mathcal{X}$. Since they are parallel to the y axis, by Proposition 5, for all $c < 0$ they intersect $\mathcal{Q}$ in a pair of parallel straight lines, or in a double straight line if $x = 0$. Therefore the vector field $\mathcal{X}$ has infinitely many invariant straight lines. □

Theorem 7. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the parabolic cylinder $z^2 - x = 0$ has finitely many invariant straight lines, taking into account their multiplicities. Then, the maximum number of invariant straight lines of $\mathcal{X}$ contained in different invariant planes parallel to the y axis is at most*

(a) $2m_3$ if $\mathcal{X}$ has no invariant double straight lines,

(b) m_3 if all invariant straight lines are double, and

(c) is between m_3 and $2m_3 - 1$ if there are invariant double straight lines.

Proof. Let $W = \langle 1, x, z\rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (3). According with Proposition 1 any invariant plane parallel to the y axis must divide R^3, and so R. Since the degree of R is m_3, R has at most m_3 different linear factors, so $\mathcal{X}$ can have at most m_3 different invariant planes. Therefore an upper bound for the maximum number of invariant straight lines parallel to the y axis contained in such invariant planes is $2m_3$ if all straight lines are not double, and m_3 if all are double. $\square$

Proposition 8. *The hypotheses of Theorem 7 are satisfied if $m_3 \geq 1$ and the bounds can be reached.*

Proof. By equation (3), if $\mathcal{X} = (P, Q, R)$ with $R = k \in \mathbb{R}\backslash\{0\}$ is a polynomial vector field defined on the parabolic cylinder $\mathcal{Q}$, then $\varepsilon_W(\mathcal{X}) = 2k^3$ with $W = \langle 1, x, z\rangle$. Since any invariant plane parallel to the y axis, $ax + bz + c = 0$, must satisfy that $ax + bz + c$ divides to the extactic polynomial $2k^3$ (see Proposition 1), there are no such invariant planes. Then, $m_3 \geq 1$.

Now we prove that if $m_3 \geq 1$, the bounds can be reached. Let $\mathcal{X}$ be a polynomial vector field on $\mathcal{Q}$ whose associated differential system is $\dot{x} = 2z\prod_{l=1}^{m_3} f_l$, $\dot{y} = q$, $\dot{z} = \prod_{l=1}^{m_3} f_l$, where $f_l = a_l x + b_l z + c_l$ with $a_l, b_l, c_l, q \in \mathbb{R}$ for all $l = 1, \ldots, m_3$ and the planes $f_l = 0$ are distinct. The plane $f_l = 0$ is invariant by $\mathcal{X}$ with cofactor $k_l = (2a_l z + b_l) \prod_{\substack{j=1 \\ j \neq l}}^{m_3} f_l$, $l = 1, \ldots, m_3$. So, this system has at most $2m_3$ invariant parallel straight lines contained in parallel planes to the y axis o m_3 invariant double straight lines of $\mathcal{X}$ contained in planes parallel to the y axis, which coincides with the bounds given by Theorem 7. $\square$

4.2. **Invariant parabolas.** According with Lemma 4, if the intersection of the parabolic cylinder $z^2 - x = 0$ with a plane non parallel to the y axis is not empty, then it is a parabola.

Proposition 9. *There are polynomial vector fields $\mathcal{X}$ defined on the parabolic cylinder $\mathcal{Q}$ that have infinitely many invariant parabolas, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be the polynomial vector field on $\mathcal{Q}$ whose associated differential system is $\dot{x} = 2z^2$, $\dot{y} = -z(1 + 2z)$, $\dot{z} = z$.

A simple calculation verifies that $f = x + y + z + d = 0$ is an invariant plane of $\mathcal{X}$ with cofactor $k = 0$ for all $d \in \mathbb{R}$. In other words $x + y + z$ is a first integral of the vector field, i.e. a function constant on the solutions of the vector field. The planes $f = 0$ are non parallel to the y axis, so they intersect $\mathcal{Q}$ in a parabola. Therefore, the vector field $\mathcal{X}$ have infinitely many invariant parabolas. $\square$

Theorem 10. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a parabolic cylinder $z^2 - x = 0$ has finitely many invariant parabolas, taking into account their multiplicities. Then, the maximum number of invariant parabolas of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 3m_2+3m_3-2 & \textit{if } m_2\geq m_3+1,\\ m_2+5m_3 & \textit{if } m_2\leq m_3+1. \end{array}$$

Proof. Let $ax+by+cz+d=0$ be a plane that intersects $\mathcal{Q}$ and consider $W=\langle 1,x,y,z\rangle$, the extactic polynomial of $\mathcal{X}$ associated to W is

$$\varepsilon_W(\mathcal{X})=\begin{vmatrix} 1 & x & y & z\\ 0 & \mathcal{X}(x) & \mathcal{X}(y) & \mathcal{X}(z)\\ 0 & \mathcal{X}^2(x) & \mathcal{X}^2(y) & \mathcal{X}^2(z)\\ 0 & \mathcal{X}^3(x) & \mathcal{X}^3(y) & \mathcal{X}^3(z)\end{vmatrix}. \tag{4}$$

From its expression we obtain that the degree of the monomials of the extactic polynomial are m_2+5m_3-2, $2m_2+4m_3-2$, $3m_2+3m_3-2$, m_2+5m_3-1, $2m_2+4m_3-1$ and m_2+5m_3. The degree of the polynomial $\varepsilon_W(\mathcal{X})$ is the greatest of these values. Note that $3m_2+3m_3-2\geq 2m_2+4m_3-1$ if $(3m_2+3m_3-2)-(2m_2+4m_3-1)\geq 0$, or equivalently if $m_2\geq m_3+1$.

Similarly if $m_2\geq m_3+1$, it follows that $2m_2+4m_3-1\geq m_2+5m_3$. Therefore, the degree of $\varepsilon_W(\mathcal{X})$ is $3m_2+3m_3-2$ if $m_2\geq m_3+1$, and in a similar way we obtain that the degree of $\varepsilon_W(\mathcal{X})$ is $5m_3+m_2$ if $m_2\leq m_3+1$. Consequently, the maximum number of invariant planes of $\mathcal{X}$ (counting their multiplicities) that intersect $\mathcal{Q}$ is $3m_2+3m_3-2$ if $m_2\geq m_3+1$, or $5m_3+m_2$ if $m_2\leq m_3+1$, and so the statement of theorem holds. □

Proposition 11. *The bound provided by Theorem 10 can be reached.*

Proof. Consider a vector field $\mathcal{X}$ on $x=z^2$ whose associated differential system is $\dot{x}=2z$, $\dot{y}=(y+1)^m$, $\dot{z}=1$. The extactic polynomial (4) for this case is $\varepsilon_W(\mathcal{X})=2m(2m-1)(y+1)^{3m-2}$. Let $f=y+1$, then $f=0$ intersects $\mathcal{Q}$ in a parabola and it is invariant by $\mathcal{X}$ with cofactor $k=(y+1)^{m-1}$. As $\mathcal{X}$ cannot have more invariant planes than the degree of $\varepsilon_W(\mathcal{X})$ and f has multiplicity $3m-2$, the bound of Theorem 10 is obtained. □

5. Results on the elliptic cylinder

We will work with the following canonical form $x^2+z^2-1=0$ of an elliptic cylinder $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field of $\mathbb{R}^3$ defined on $\mathcal{Q}$. We will see that the conics obtained by intersecting the elliptic cylinder $\mathcal{Q}$ with a plane are straight lines or ellipses.

5.1. **Invariant straight lines.** The elliptic cylinder is a ruled surface. Below we determine which are the straight lines that contains.

Lemma 12. *All straight lines contained in the elliptic cylinder $x^2+z^2-1=0$ are parallels to the y axis.*

Proof. The proof follows the same steps than the proof of Lemma 3, and instead of (1) we get $\lambda(a^2+c^2)+2(ax_0+cz_0)=0$. From here we arrive to a contradiction as in Lemma 3. □

Proposition 13. *The intersection conic of an elliptic cylinder $\mathcal{Q}$ with a plane is either a double straight line, or a pair of parallel straight lines, or an ellipse.*

Proof. Consider a plane that intersects $\mathcal{Q}$, if the plane is parallel to the y axis, the intersection conic is a double straight line or two parallel straight lines. By Lemma 12, $\mathcal{Q}$ only contains straight lines parallel to the y axis, so any other plane non parallel to the y axis intersects the elliptic cylinder in an ellipse, a parabola or a hyperbola. Since the intersection is a closed curve, it could not be a parabola or a hyperbola. Therefore, a plane non parallel to the y axis only can cut this cylinder in an ellipse. $\square$

Let $\mathcal{X} = (P, Q, R)$ be the polynomial vector field defined on the elliptic cylinder $x^2 + z^2 - 1 = 0$. Then, $xP = -zR$ and $m_1 = m_3$.

Proposition 14. *There are polynomial vector fields $\mathcal{X}$ defined on an elliptic cylinder $\mathcal{Q}$ that have infinitely many invariant straight lines, taking into account their multiplicities.*

Proof. Let $W = \langle 1, x, z\rangle$. Then, the extactic polynomial $\varepsilon_W(\mathcal{X})$ is given by equation (2). Since $xP = -zR$ is follows that $xP_x = -zR_x - P$, $xP_y = -zR_y$ and $xP_z = -R - zR_z$, moreover taking into account that $\mathcal{X}(x) = P$ and $\mathcal{X}(z) = R$, we obtain

$$\varepsilon_W(\mathcal{X}) = (x^2 + z^2)\left(\frac{R}{x}\right)^3. \tag{5}$$

So, $\varepsilon_W(\mathcal{X}) = 0$ if and only if $R = 0$.

We define $\mathcal{X}$ as the vector field whose associated differential system is $\dot{x} = 0$, $\dot{y} = Q$, $\dot{z} = 0$. As $\dot{x} = 0$, the planes $f = x + c = 0$ with $c \in \mathbb{R}$ are invariant by $\mathcal{X}$. Since these planes are parallel to the y axis, when they intersect $\mathcal{Q}$, they provide a pair of parallel straight lines, or a double straight line (see Proposition 13). Therefore, the vector field $\mathcal{X}$ has infinitely many invariant straight lines. $\square$

Theorem 15. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the elliptic cylinder $x^2 + z^2 - 1 = 0$ has finitely many invariant straight lines, taking into account their multiplicities. Then, the maximum number of invariant straight lines of $\mathcal{X}$ contained in different invariant planes parallel to the y axis is at most*

(a) $2m_3 - 2$ if $\mathcal{X}$ has no invariant double straight lines,
(b) $m_3 - 1$ if all invariant straight lines are double, and
(c) is between $m_3 - 1$ and $2m_3 - 1$ if there are invariant double straight lines.

Proof. Let $W = \langle 1, x, z\rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (5). Since $\varepsilon_W(\mathcal{X})$ is a polynomial, x must divide R. So, there exists a polynomial R_1 of degree m_3-1 such that $R = xR_1$, and therefore the degree of $\varepsilon_W(\mathcal{X})$ is $3(m_3 - 1) + 2$. Thus the maximum number of real different invariant planes of $\mathcal{X}$, is at most $m_3 - 1$. Note that there are no real planes dividing to $x^2 + z^2$. Therefore the maximum number of invariant straight lines contained in these invariant planes is $m_3 - 1$ if all straight lines are double, and $2m_3 - 2$ if $\mathcal{X}$ has no invariant double straight lines. $\square$

Proposition 16. *The bounds provided by Theorem 15 can be reached.*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on $\mathcal{Q}$ whose associated differential system is $\dot{x} = -z \prod_{l=1}^{m_3-1} f_l$, $\dot{y} = q$, $\dot{z} = x \prod_{l=1}^{m_3-1} f_l$, where $f_l = a_l x + b_l z + c_l$ with a_l, b_l, c_l and q are real constants and $a_l x + b_l z + c_l = 0$ distinct planes for all

$l = 1, \dots, m_3 - 1$. The planes $f_l = 0$ with $l = 1, \dots, m_3 - 1$ are invariant by $\mathcal{X}$ with cofactor $k_l = -(a_l z + b_l x) \prod_{\substack{j=1 \\ j \neq l}}^{m_3-1} (a_j x + b_j z + c_j)$. So this system has at most $2m_3 - 2$ invariant parallel straight lines contained in planes parallel to the y axis, or $m_3 - 1$ invariant double straight lines contained in planes parallel to the y axis, which coincides with the bounds given by Theorem 15. □

5.2. **Invariant ellipses.** According with the proof of Proposition 13 the intersection of the elliptic cylinder $x^2 + z^2 - 1 = 0$ with a plane non parallel to the y axis is not empty and it is an ellipse.

Proposition 17. *There are polynomial vector fields $\mathcal{X}$ defined on the elliptic cylinder $\mathcal{Q}$ that have infinitely many invariant ellipses, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be the polynomial vector field on $\mathcal{Q}$ whose associated differential system is $\dot{x} = -z$, $\dot{y} = z - x$, $\dot{z} = x$. The planes $f = x + y + z + d = 0$ with $d \in \mathbb{R}$ are invariant by $\mathcal{X}$ with cofactor $k = 0$ for all $d \in \mathbb{R}$. Since they are non parallel to the y axis, these planes intersect $\mathcal{Q}$ in ellipses. Therefore, we have obtained infinitely many invariant ellipses for the vector field $\mathcal{X}$ on $\mathcal{Q}$. □

Theorem 18. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on an elliptic cylinder $x^2 + z^2 - 1 = 0$ has finitely many invariant ellipses, taking into account their multiplicities. Then, the maximum number of invariant ellipses of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 3m_2 + 3m_3 - 5 & \text{if } m_2 \geq m_3, \\ m_2 + 5m_3 - 5 & \text{if } m_2 \leq m_3. \end{array}$$

Proof. Let $W = \langle 1, x, y, z \rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (4). The degrees of the monomials of $\varepsilon_W(\mathcal{X})$ are $m_2 + 5m_3 - 3$, $2m_2 + 4m_3 - 3$ and $3m_2 + 3m_3 - 3$. If $m_2 \geq m_3$ the following inequalities are satisfied $3m_2 + 3m_3 - 3 \geq 2m_2 + 4m_3 - 3 \geq m_2 + 5m_3 - 3$. So, the degree of $\varepsilon_W(\mathcal{X})$ is $3m_2 + 3m_3 - 3$ if $m_2 \geq m_3$, and in a similar way we obtain that the degree of $\varepsilon_W(\mathcal{X})$ is $m_2 + 5m_3 - 3$ if $m_2 \leq m_3$. Therefore, the maximum number of real invariant planes of $\mathcal{X}$ (counting their multiplicities) that intersect $\mathcal{Q}$ is $3m_2 + 3m_3 - 5$ if $m_2 \geq m_3$, or $m_2 + 5m_3 - 5$ if $m_2 \leq m_3$. □

Proposition 19. *The bound provided by Theorem 18 can be reached.*

Proof. Consider a vector field $\mathcal{X}$ on $x^2 + z^2 - 1 = 0$ whose associated differential system is $\dot{x} = -z(y+1)^{m-1}$, $\dot{y} = (y+1)^m$, $\dot{z} = x(y+1)^{m-1}$. The extactic polynomial (4) for this case is $\varepsilon_W(\mathcal{X}) = -2(x^2 + z^2)(y + 1)^{6m-5}$. Let $f = y + 1$, then $f = 0$ intersects $\mathcal{Q}$ in an ellipse, and moreover it is an invariant plane of $\mathcal{X}$ with cofactor $k = (y+1)^{m-1}$. As $\mathcal{X}$ on $\mathcal{Q}$ cannot have more invariant planes and the multiplicity of f is $6m - 5$, the bound of Theorem 18 is reached. □

6. Results on the hyperbolic cylinder

We will work with the following canonical form $x^2 - z^2 - 1 = 0$ of a hyperbolic cylinder $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ defined on $\mathcal{Q}$. We will see that the intersection of a hyperbolic cylinder with a plane are straight lines or hyperbolas.

6.1. **Invariant straight lines.** As in the previous two quadrics the hyperbolic cylinder $\mathcal{Q}$ is a ruled surface and its straight lines are parallel the y axis.

Lemma 20. *All straight lines contained in the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ are parallel to the y axis.*

Proof. The proof follows the same steps than the proof of Lemma 3, and instead of (1) we get $\lambda(a^2 - c^2) + 2(ax_0 - cz_0) = 0$. From here we arrive to a contradiction as in Lemma 3. □

Lemma 21. *Any plane non parallel to the y axis intersects the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ in a hyperbola.*

Proof. Consider the plane $ax + by + cz + d = 0$ with $b \neq 0$. If $c \neq 0$, then $z = -(ax + by + d)/c$. Replacing it in $x^2 - z^2 - 1 = 0$ we obtain that the equation of the intersection conic is $(c^2 - a^2)x^2 - 2abxy - b^2y^2 - 2adx - 2bdy - c^2 - d^2 = 0$. Applying Proposition 2 we obtain $d_1 = b^2c^4 \neq 0$ and $d_2 = -b^2c^2 < 0$. So, the conic is a hyperbola.

Now let $c = 0$. If $a \neq 0$, we have $x = -(by+d)/a$. Replacing it in $z^2 - x^2 - 1 = 0$, it follows that the equation of the intersection conic is $b^2y^2 - a^2z^2 + 2bdy + d^2 - a^2 = 0$. As $d_1 = b^2a^2 \neq 0$ and $d_2 = -b^2 < 0$ the conic is a hyperbola. If $c = a = 0$, the equation of the plane takes the form $y = -d/b$. The intersection of the hyperbolic cylinder with this plane is clearly a hyperbola. □

Proposition 22. *The intersection conic of a hyperbolic cylinder $\mathcal{Q}$ with a plane is either a double straight line, or a pair of parallel straight lines, or a hyperbola.*

Proof. By Lemma 20, given a plane parallel to the y axis that intersects $\mathcal{Q}$, its intersection is a double straight line, or two parallel straight lines. If this plane is non parallel to the y axis, by Lemma 21, it intersects $\mathcal{Q}$ in a hyperbola. □

If $\mathcal{X} = (P, Q, R)$ is a polynomial vector field defined on the hyperbolic cylinder $\mathcal{Q}$, we have $xP = zR$ which implies $m_1 = m_3$.

Proposition 23. *There are polynomial vector fields $\mathcal{X}$ defined on the hyperbolic cylinder $\mathcal{Q}$ that have infinitely many invariant straight lines, taking into account their multiplicities.*

Proof. Consider $W = \langle 1, x, z\rangle$. Then, the extactic polynomial $\varepsilon_W(\mathcal{X})$ is given by equation (2). Since $xP = zR$, taking into account that $\mathcal{X}(x) = P$ and $\mathcal{X}(z) = R$ we obtain

$$\varepsilon_W(\mathcal{X}) = -(x^2 - z^2)\left(\frac{R}{x}\right)^3. \tag{6}$$

It follows that $\varepsilon_W(\mathcal{X}) = 0$ if and only if $R = 0$. Therefore we can define $\mathcal{X}$ as the vector field whose associated differential system is $\dot{x} = 0$, $\dot{y} = Q$, $\dot{z} = 0$. The planes $f = x + c = 0$ with $c \in \mathbb{R}$ are invariant by $\mathcal{X}$, moreover when they intersect $\mathcal{Q}$, we obtain a pair of parallel straight lines, or a double straight line (see Proposition 22). So, $\mathcal{X}$ on $\mathcal{Q}$ has infinitely many invariant straight lines. □

Theorem 24. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ has finitely many invariant straight lines, taking into account their multiplicities. Then, the maximum number of invariant straight lines of $\mathcal{X}$ contained in different invariant planes parallel to the y axis is at most*

(a) $2m_3 - 2$ if $\mathcal{X}$ has no invariant double straight lines,

(b) $m_3 - 1$ if all invariant straight lines are double, and

(c) is between $m_3 - 1$ and $2m_3 - 1$ if there are invariant double straight lines.

Proof. By equation (6) we have that x must divide R, i.e. $R = xR_1$ for some polynomial R_1 of degree $m_3 - 1$, and hence the degree of $\varepsilon_W(\mathcal{X})$ is $3(m_3 - 1) + 2$. Since the planes $x - z = 0$ and $x + z = 0$ do not intersect the hyperbolic cylinder, the maximum number of different invariant planes of $\mathcal{X}$, is at most $m_3 - 1$. So, the maximum number of invariant straight lines contained in these invariant planes is $m_3 - 1$ if all straight lines are double, and $2m_3 - 2$ if the straight lines are not double. □

Proposition 25. *The bounds provided by Theorem 24 can be reached.*

Proof. Let $\mathcal{X}$ be a polynomial vector field defined on the hyperbolic cylinder $\mathcal{Q}$ whose associated differential system is $\dot{x} = z \prod_{l=1}^{m_3-1} f_l$, $\dot{y} = q$, $\dot{z} = x \prod_{l=1}^{m_3-1} f_l$, where $f_l = a_l x + b_l z + c_l$ with $a_l, b_l, c_l, q \in \mathbb{R}$ and the planes $f_l = 0$ distinct for all $l = 1, \ldots, m_3 - 1$. Moreover $f_l = 0$ is invariant by $\mathcal{X}$ with cofactor $k_l = (a_l z + b_l x) \prod_{\substack{j=1 \\ j \neq l}}^{m_3-1} (a_j x + b_j z + c_j)$. So, this implies that the bounds given by Theorem 24 is reached. □

6.2. **Invariant hyperbolas.** According with the proof of Proposition 22 the intersection of the hyperbolic cylinder $\mathcal{Q}$ with a plane non parallel to the y axis is a hyperbola.

Proposition 26. *There are polynomial vector fields $\mathcal{X}$ defined on $\mathcal{Q}$ that have infinitely many invariant hyperbolas, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be the polynomial vector field on $\mathcal{Q}$ whose associated differential system is $\dot{x} = z$, $\dot{y} = x + z$, $\dot{z} = x$. The planes $-x + y - z + d = 0$ are invariant by $\mathcal{X}$ with cofactor $k = 0$ for all $d \in \mathbb{R}$. These planes are not parallels the y axis, so $\mathcal{X}$ has infinitely many invariant hyperbolas. □

Theorem 27. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a hyperbolic cylinder $x^2 - z^2 - 1 = 0$ has finitely many invariant hyperbolas, taking into account their multiplicities. Then, the maximum number of invariant hyperbolas of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 3m_2 + 3m_3 - 5 & \text{if } m_2 \geq m_3, \\ m_2 + 5m_3 - 5 & \text{if } m_2 \leq m_3. \end{array}$$

Proof. Let $W = \langle 1, x, y, z \rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (4). From its expression we obtain that the degree of the monomials of $\varepsilon_W(\mathcal{X})$ are $m_2 + 5m_3 - 3$, $2m_2 + 4m_3 - 3$ and $3m_2 + 3m_3 - 3$. If $m_2 \geq m_3$, then $3m_2 + 3m_3 - 3 \geq 2m_2 + 4m_3 - 3 \geq m_2 + 5m_3 - 3$. Therefore, the degree of $\varepsilon_W(\mathcal{X})$ is $3m_2 + 3m_3 - 3$ if $m_2 \geq m_3$, and in a similar way we prove that the degree of $\varepsilon_W(\mathcal{X})$ is $m_2 + 5m_3 - 3$ if $m_2 \leq m_3$. Thus, the maximum number of invariant planes (counting their multiplicities) that intersect $\mathcal{Q}$ is $3m_2 + 3m_3 - 5$ if $m_2 \geq m_3$ and $m_2 + 5m_3 - 5$ if $m_2 \leq m_3$ as stated in the theorem. □

Proposition 28. *The bound that provides Theorem 27 can be reached.*

Proof. Let $\mathcal{X}$ be a polynomial vector field defined on $x^2 - z^2 - 1 = 0$ whose associated differential system is $\dot{x} = z(2y+1)^{m-1}$, $\dot{y} = (2y+1)^m$, $\dot{z} = x(2y+1)^{m-1}$. The extactic polynomial (4) for this case is $\varepsilon_W(\mathcal{X}) = 3(x-z)(x+z)(2y+1)^{6m-5}$. Consider $f = 2y+1$, then $f = 0$ intersects $\mathcal{Q}$ in a hyperbola. Moreover $f = 0$ is an invariant plane of $\mathcal{X}$ with cofactor $k = 2(2y+1)^{m-1}$. Since f has multiplicity $6m-5$ and $\mathcal{X}$ cannot have more invariant planes, $\mathcal{X}$ reaches the bound stated in Theorem 27. □

7. Results on the cone

We are going to work with the following canonical form $x^2 + y^2 - z^2 = 0$ of a cone $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ defined on $\mathcal{Q}$. It is known that the intersection of a plane and $\mathcal{Q}$ is a conic that can be degenerate: one point, two intersecting straight lines, a double straight line, or non degenerate: a parabola, an ellipse or a hyperbola.

7.1. **Invariant degenerate conics.**

The cone is also a ruled surface. Indeed, the intersection of the cone with a plane passing through its vertex can be only a point, a double straight line, or a pair of intersecting straight lines.

If $\mathcal{X} = (P, Q, R)$ is defined on $\mathcal{Q}$, then this implies that $zR = xP + yQ$ and $\mathcal{X} = (P, Q, (xP + yQ)/z)$.

Proposition 29. *There are polynomial vector fields $\mathcal{X}$ defined on the cone $\mathcal{Q}$ that have infinitely many invariant degenerate conics, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be a vector field on $\mathcal{Q}$ whose associated differential system is $\dot{x} = zx$, $\dot{y} = zy$, $\dot{z} = x^2 + y^2$. Consider $f = ax + by$, with $a, b \in \mathbb{R}$. The planes $f = 0$ are invariant by $\mathcal{X}$ with cofactor $k = z$, moreover they contain the z axis, so they cut $\mathcal{Q}$ in a pair of intersecting straight lines. Therefore, $\mathcal{X}$ has infinitely many invariant degenerate conics. □

Theorem 30. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a cone $x^2 + y^2 - z^2 = 0$ has finitely many invariant degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant degenerate conics of $\mathcal{X}$ contained in invariant planes is at most*

$$\begin{array}{ll} 3m_1 & \text{if } m_1 \geq m_2, \\ 3m_2 & \text{if } m_1 \leq m_2. \end{array}$$

Proof. As we study the invariant degenerate conics we need only consider the planes passing through the vertex of the cone $x^2 + y^2 - z^2 = 0$, otherwise we obtain non degenerate conics. Let $W = \langle x, y, z\rangle$, the extactic polynomial associated to W is

$$\varepsilon_W(\mathcal{X}) = \begin{vmatrix} x & y & z \\ \mathcal{X}(x) & \mathcal{X}(y) & \mathcal{X}(z) \\ \mathcal{X}^2(x) & \mathcal{X}^2(y) & \mathcal{X}^2(z) \end{vmatrix}. \tag{7}$$

From its expression, the degrees of the monomials of $\varepsilon_W(\mathcal{X})$ are $3m_1$, $2m_1 + m_2$, $m_1 + 2m_2$ and $3m_2$. We can see that $3m_1 \geq 2m_1 + m_2 \geq m_1 + 2m_2 \geq 3m_2$ if $m_1 \geq m_2$. So, the degree of $\varepsilon_W(\mathcal{X})$ is $3m_1$ if $m_1 \geq m_2$, and in a similar way we obtain that the degree of $\varepsilon_W(\mathcal{X})$ is $3m_2$ if $m_1 \leq m_2$. Therefore, we get the maximum number of invariant degenerate conics contained in these planes passing through the origin and intersecting $\mathcal{Q}$ described in the statement of the theorem. □

Proposition 31. *The bound provided by Theorem 30 can be reached.*

Proof. Let $\mathcal{X}$ be a polynomial vector field defined on the cone $x^2 + y^2 - z^2 = 0$ such that its associated differential system is $\dot{x} = 0$, $\dot{y} = z(y - z/2)^{m-1}$, $\dot{z} = y(y - z/2)^{m-1}$. The bound provided by Theorem 30 for the maximum number of invariant degenerate conics is $3m$. The extactic polynomial (7) in this case is $\varepsilon_W(\mathcal{X}) = 4x(y - z)(y + z)(y - z/2)^{3m-3}$. The plane $x = 0$ is invariant by $\mathcal{X}$ with cofactor $k = 0$. Also $y \pm z = 0$ are invariant planes with cofactor $k = (y - z/2)^{m-1}$. The plane $y - z/2 = 0$ is invariant with $k = -(y - 2z)(y - z/2)^{m-2}/2$ and its multiplicity is $3(m - 1)$. These planes pass through the origin, so its intersection with the cone $\mathcal{Q}$ are degenerate conics. Therefore we obtain $3m$ invariant degenerate conics. □

7.2. **Invariant non degenerate conics.** The intersection of the cone with a plane not passing through its vertex is a non degenerate conic.

Proposition 32. *There are polynomial vector fields $\mathcal{X}$ defined on $\mathcal{Q}$ that have infinitely many invariant non degenerate conics, taking into account their multiplicities.*

Proof. The system of the proof of Proposition 31 has infinitely many non degenerate conics obtained intersecting the invariant planes $x =$ constant with the cone. □

Theorem 33. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the cone $x^2 + y^2 - z^2 = 0$ has finitely many invariant non degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant non degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1 + m_2 - 4 & \text{if } m_1 \geq m_2, \\ m_1 + 5m_2 - 4 & \text{if } m_1 \leq m_2. \end{array}$$

Proof. Let $W = \langle 1, x, y, z\rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (4). From its expression we obtain that the degree of the monomials of the extactic polynomial $\varepsilon_W(\mathcal{X})$ are $5m_1 + m_2 - 3$, $4m_1 + 2m_2 - 3$, $3m_1 + 3m_2 - 3$, $2m_1 + 4m_2 - 3$ and $m_1 + 5m_2 - 3$. We observe that if $m_1 \geq m_2$, then $5m_1 + m_2 - 3 \geq 4m_1 + 2m_2 - 3 \geq 3m_1 + 3m_2 - 3 \geq 2m_1 + 4m_2 - 3 \geq m_1 + 5m_2 - 3$. So the degree of $\varepsilon_W(\mathcal{X})$ is $5m_1 + m_2 - 3$ if $m_1 \geq m_2$, and in a similar way we obtain that the degree of $\varepsilon_W(\mathcal{X})$ is $m_1 + 5m_2 - 3$ if $m_1 \leq m_2$.

Now in order to get the bound provided by the theorem, we see that the extactic polynomial (4) vanishes at the origin. The polynomials P and Q can be written as polynomials in the variable z as follows $P(x,y,z)=\sum_{i=0}^{m_1} P_i(x,y)z^i$, $Q(x,y,z)=\sum_{j=1}^{m_2} Q_j(x,y)z^j$, where P_i and Q_j are polynomials in the variables x,y for all $i=1,\ldots,m_1$ and $j=1,\ldots,m_2$; and hence P and Q can be rewritten as $P(x,y,z)=P_0(x,y)+zh_1(x,y,z)$ and $Q(x,y,z)=Q_0(x,y)+zh_2(x,y,z)$, where h_1 and h_2 are polynomials in the variables x,y,z. Since $zR=xP+yQ$, then $zR=xP_0(x,y)+xzh_1(x,y,z)+yQ_0(x,y)+yzh_2(x,y,z)$, so $xP_0(x,y)+yQ_0(x,y)=0$, this is $xP_0=-yQ_0$. Therefore, there are polynomials $\overline{P}_0$ and $\overline{Q}_0$ such that $P_0(x,y)=y\overline{P}_0(x,y)$ and $Q_0(x,y)=x\overline{Q}_0(x,y)$, thus $\overline{P}_0=-\overline{Q}_0$. So, if $h_3(x,y)=\overline{P}_0(x,y)=-\overline{Q}_0(x,y)$, the associated differential system to the vector field $\mathcal{X}$ can be written as $\dot{x}=yh_3(x,y)+zh_1(x,y,z)$, $\dot{y}=-xh_3(x,y)+zh_2(x,y,z)$, $\dot{z}=xh_1(x,y,z)+yh_2(x,y,z)$.

Taking into account that $\mathcal{X}(x)=yh_3+zh_1$, $\mathcal{X}(y)=-xh_3+zh_2$ and $\mathcal{X}(z)=xh_1+yh_2$, the second row of the extactic polynomial (4) is zero, therefore it vanishes at the origin. So, the factorization of $\varepsilon_W(\mathcal{X})$ into linear factors contains at least one factor without independent term and in consequence, the plane associated to this factor intersects the cone in a degenerate conic, which implies that $\mathcal{X}$ would have at least an invariant degenerated conic. Hence the maximum bound of invariant non degenerate conics contained in invariant planes is at least one degree less than the degree of the $\varepsilon_W(\mathcal{X})$. Thus the theorem is proved. □

Proposition 34. *The bound provided by Theorem 33 decreased in six can be reached.*

Proof. Let $\mathcal{X}$ be a polynomial vector field defined on a cone $\mathcal{Q}$ such that its associated differential system is $\dot{x}=(x+1)(y+1)^{m-2}z$, $\dot{y}=z^2(y+1)^{m-2}$, $\dot{z}=(y+1)^{m-2}(x+x^2+yz)$. According with Theorem 33 the maximum number of invariant non degenerate conics of $\mathcal{X}$ is at most $6m-4$. The extactic polynomial (4) for this case is $\varepsilon_W(\mathcal{X})=(x+1)^2(y+1)^{6m-12}A(x,y,z)$, where A is a polynomial of degree 7. The plane $x+1=0$ is invariant by $\mathcal{X}$ with cofactor $k=z(y+1)^{m-2}$ and its multiplicity is 2. Also, $y+1=0$ is an invariant plane with $k=z^2(1+y)^{m-3}$ and multiplicity $6m-12$. The intersection of these planes with the cone are non degenerate conics. So, we obtain $6m-10$ invariant non degenerate conics, this is, the bound provided by Theorem 33 decreased in 6. □

8. Results on the hyperboloid of one sheet

We will work with the following canonical form $x^2+y^2-z^2=1$ of a hyperboloid of one sheet $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field of $\mathbb{R}^3$ defined on $\mathcal{Q}$. We will see that the conic obtained by intersecting the hyperboloid of one sheet $\mathcal{Q}$ with a plane is formed, when it is degenerate, by a pair of intersecting straight lines or a pair of parallel straight lines; and when it is non degenerate by a parabola, an ellipse, or a hyperbola.

8.1. **Invariant straight lines.** It is known that a hyperboloid of one sheet is a doubly ruled surface. That is, for each of its points pass exactly two straight lines that are completely contained in it [6]. Furthermore, there is no other straight line fully contained in the hyperboloid since the only quadratic surface that contains three different straight lines which pass through each of its points is the plane (see [5, 16]).

Proposition 35. *If the intersection of a hyperboloid of one sheet $\mathcal{Q}$ with a plane is a degenerate conic, then it can be a pair of intersecting straight lines or a pair of parallel straight lines.*

Proof. Consider a plane $ax + by + cz + d = 0$ with $a, b, c, d \in \mathbb{R}$ that intersects $x^2 + y^2 - z^2 = 1$ in a degenerate conic.

Suppose $c \neq 0$, then $z = -(ax + by + d)/c$. Replacing it in the equation of the hyperboloid, we obtain the equation of the intersection conic $(c^2 - a^2)x^2 - 2abxy + (c^2 - b^2)y^2 - 2adx - 2bdy - c^2 - d^2 = 0$. Applying Proposition 2 to this expression results $d_1 = c^4(a^2 + b^2 - c^2 - d^2) = 0$ and $d_2 = c^2(c^2 - a^2 - b^2)$. Note that $d_1 = 0$ because the conic is degenerate and hence $d^2 = a^2 + b^2 - c^2 \geq 0$, this implies $c^2 \leq a^2 + b^2$. If $c^2 < a^2 + b^2$, then $d_2 < 0$ and the degenerate conic is a pair of intersecting straight lines. If $c^2 = a^2 + b^2$, it follows that $d_2 = 0$, so the degenerate conic is a pair of parallel straight lines.

Now consider $c = 0$, the equation of the plane that intersects $\mathcal{Q}$ is $ax + by + d = 0$. If $a \neq 0$, $x = -(by + d)/a$ and replacing it in $x^2 + y^2 - z^2 = 1$ we have $(b^2 + a^2)y^2 - a^2z^2 + 2bdy + d^2 - a^2 = 0$. Once again by Proposition 2 we obtain $d_1 = a^4(a^2 + b^2 - d^2) = 0$ and $d_2 = -a^2(a^2 + b^2) < 0$, so the degenerate conic is a pair of intersecting straight lines. If $c = a = 0$, we obtain the plane $by + d = 0$ with $b \neq 0$. Then, $d_1 = b^2 - d^2 = 0$ and $d_2 = -1 < 0$. In this case the degenerate conic is a pair of intersecting straight lines. □

As we said it is well known that for every point of a hyperboloid of one sheet pass two straight lines. We provide now a proof of this fact because we shall need later on the explicit expression of these two straight lines.

Proposition 36. *For each $p = (x_0, y_0, z_0)$ of a hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ pass two straight lines.*

Proof. Let $\lambda, \mu \in \mathbb{R}$ such that $\mu(x_0 - z_0) = \lambda(1 - y_0)$. If $\mu \neq 0$, replacing this expression in the equation of the hyperboloid $x^2 - z^2 = 1 - y^2$, or what is the same in $(x - z)(x + z) = (1 - y)(1 + y)$ we obtain $\lambda(x_0 + z_0) = \mu(1 + y_0)$ whenever $1 - y_0 \neq 0$. If $1 - y_0 = 0$, we have $x - z = 0$ or $x + z = 0$. Considering $\mu = 0$ we would have $1 - y_0 = 0$ which coincides with the previous case.

Now, let $\mu, \lambda \in \mathbb{R}$ such that $\mu(x_0 - z_0) = \lambda(1 + y_0)$. If $\mu \neq 0$, from the equation of the hyperboloid we obtain $\lambda(x_0 + z_0) = \mu(1 - y_0)$ whenever $1 + y_0 \neq 0$. If $1 + y_0 = 0$, then $x_0 - z_0 = 0$ or $x_0 + z_0 = 0$. If $\mu = 0$, we have $1 + y_0 = 0$ that corresponds to the previous case.

In short, the point p is on the straight line $\mu(x - z) - \lambda(1 - y) = 0$, $\mu(1 + y) - \lambda(x + z) = 0$ with $\lambda = \mu(x_0 - z_0)/(1 - y_0)$ and $1 - y_0 \neq 0$; or on the straight line $\mu(x - z) - \lambda(1 + y) = 0$, $\mu(1 - y) - \lambda(x + z) = 0$ when $\lambda = \mu(x_0 - z_0)/(1 + y_0)$ and $1 + y_0 \neq 0$, or on the straight lines $1 - y = 0$, $x - z = 0$ and $1 - y = 0$, $x + z = 0$ when $1 - y_0 = 0$, or on the straight lines $1 + y = 0$, $x - z = 0$ and $1 + y = 0$, $x + z = 0$ if $1 + y_0 = 0$. □

Remark 37. *From the proof of Proposition 36 we obtain that the hyperboloid of one sheet $x^2 + y^2 - z^2 - 1 = 0$ contains the two families of straight lines*

$$\begin{array}{ccc} \mu(x - z) - \lambda(1 - y) = 0, & & \mu(x - z) - \lambda(1 + y) = 0, \\ \mu(1 + y) - \lambda(x + z) = 0, & \text{and} & \mu(1 - y) - \lambda(x + z) = 0, \\ (a) & & (b) \end{array} \tag{8}$$

for all λ, $\mu \in \mathbb{R}$.

As we study invariant straight lines, from the proof of Proposition 36 we need to consider only the equation of the plane $\mu(x-z)-\lambda(1-y)=0$ or $\mu(x-z)-\lambda(1+y)=0$. The family of planes $\mu(x-z)-\lambda(1-y)=0$ for each $\mu,\lambda\in\mathbb{R}$ intersects the hyperboloid of one sheet in the family of straight lines $(8)(a)$ and in the straight line $1-y=0$, $x-z=0$ that belongs to the family $(8)(b)$. In a similar way, the intersection of the plane $\mu(x-z)-\lambda(1+y)=0$ and the hyperboloid for each $\mu,\lambda\in\mathbb{R}$ is a pair of parallel or intersecting straight lines of the family $(8)(b)$ and the straight line $1+y=0$, $x-z=0$.

If $\mathcal{X}=(P,Q,R)$ is a polynomial vector field defined on $\mathcal{Q}=x^2+y^2-z^2-1=0$, then it satisfies $zR=xP+yQ$ and so $\mathcal{X}=(P,Q,(xP+yQ)/z)$.

Proposition 38. *There are polynomial vector fields $\mathcal{X}$ defined on a hyperboloid of one sheet $\mathcal{Q}$ that have infinitely many invariant straight lines, taking into account their multiplicities.*

Proof. We need to determine $\mathcal{X}$ on $\mathcal{Q}$ such that all the straight lines of the family $(8)(a)$ are invariant by $\mathcal{X}$. Let $W=\langle x-z,1-y\rangle$, the extactic polynomial of $\mathcal{X}$ associated to W is

$$\begin{aligned}\varepsilon_W(\mathcal{X}) &= \begin{vmatrix} x-z & 1-y \\ \mathcal{X}(x-z) & \mathcal{X}(1-y)\end{vmatrix} \\ &= (1-y)(x-z)P+(y-y^2-xz+z^2)Q.\end{aligned} \tag{9}$$

In order that $\mathcal{X}$ has infinitely many invariant straight lines we need that $\varepsilon_W(\mathcal{X})=0$. Then, from (9) there exist polynomials P_1 and Q_1 such that $P=(y-y^2-xz+z^2)P_1$ and $Q=(y-1)(x-z)Q_1$. So $\varepsilon_W(\mathcal{X})=(P_1-Q_1)(y-1)(x-z)(y-y^2-xz+z^2)=0$. Doing $P_1=Q_1=1$ we get the vector field $\mathcal{X}$ with $P=y-y^2-xz+z^2$, $Q=(y-1)(x-z)$, $R=-x^2+y-y^2+xz$.

Now we verify that all the straight lines of the family $(8)(a)$ are invariant by $\mathcal{X}$ for all $\mu,\lambda\in\mathbb{R}$. Let $f=\mu(x-z)+\lambda(1-y)=0$, then $\mathcal{X}f=(x-z)(\mu(x-z)+\lambda(1-y))$, so $f=0$ is an invariant plane with cofactor $k=x-z$ for all $\mu,\lambda\in\mathbb{R}$. Therefore, the polynomial vector field $\mathcal{X}$ has infinitely many invariant straight lines. □

Theorem 39. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m}=(m_1,m_2,m_3)$ defined on a hyperboloid of one sheet $x^2+y^2-z^2-1=0$ has finitely many invariant straight lines, taking into account their multiplicities. Then, the maximum number of straight lines, contained in invariant planes of $\mathcal{X}$, of each of the families of equation* (8) *is at most*

$$\begin{aligned} &m_1+1 \text{ if } m_1\geq m_2,\\ &m_2+1 \text{ if } m_1\leq m_2,\end{aligned}$$

Proof. Consider $W_1=\langle x-z,1-y\rangle$, $W_2=\langle x-z,1+y\rangle$ and their associated extactic polynomials $\varepsilon_{W_1}(\mathcal{X})=\begin{vmatrix} x-z & 1-y \\ \mathcal{X}(x-z) & \mathcal{X}(1-y)\end{vmatrix}$, and $\varepsilon_{W_2}(\mathcal{X})=\begin{vmatrix} x-z & 1+y \\ \mathcal{X}(x-z) & \mathcal{X}(1+y)\end{vmatrix}$. Developing these determinants and since $zR=xP+yQ$, $\mathcal{X}(x-z)=P-R$ and $\mathcal{X}(1\pm y)=\pm Q$ results $\varepsilon_{W_1}(\mathcal{X})=-\left(P(-x+xy+z-yz)+Q(-y+y^2+xz-z^2)\right)/z$, and $\varepsilon_{W_2}(\mathcal{X})=\left(P(x+xy-z-yz)+Q(y+y^2+xz-z^2)\right)/z$. The degree of the monomials of $\varepsilon_{W_1}(\mathcal{X})$ and $\varepsilon_{W_2}(\mathcal{X})$ are m_1, m_1+1, m_2 and m_2+1. So, the degree of both polynomials is m_1+1 if $m_1\geq m_2$, or m_2+1 if $m_1\leq m_2$.

We know from Proposition 1 that the degree of the extactic polynomial ε_{W_1} gives an upper bound for the maximum number of invariant planes $f=0$ of $\mathcal{X}$ with

$f \in W_1$. As mentioned before, from the proof of Proposition 36, the intersection of the planes $\mu(x-z)-\lambda(1-y)=0$ and the hyperboloid of one sheet $x^2+y^2-z^2=1$ is the family of straight lines $(8)(a)$ and the straight line $1-y=0$, $x-z=0$. Since this latter belongs to the family $(8)(b)$ we obtain that the maximum number of invariant straight lines of the form $(8)(a)$ corresponds to the degree of the polynomial $\varepsilon_{W_1}(\mathcal{X})$. Similarly it is proved that the maximum number of invariant straight lines of the form $(8)(b)$ is the degree of the polynomial $\varepsilon_{W_2}(\mathcal{X})$. Therefore, the theorem is shown. □

Proposition 40. *The bound provided by Theorem 39 for the invariant degenerate conics of family* $(8)(a)$ *is reached .*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on the hyperboloid of one sheet $x^2+y^2-z^2=1$ whose associated differential system is $\dot{x} = z\prod_{i=1}^{m-1}(\mu_i(x-z)-\lambda_i(1-y))$, $\dot{y}=0$, $\dot{z} = x\prod_{i=1}^{m-1}(\mu_i(x-z)-\lambda_i(1-y))$, with $m \in \mathbb{N}$. According with Theorem 39 the maximum number of invariant degenerate conics of $\mathcal{X}$ is at the most $m+1$. We prove that $\mathcal{X}$ reaches this bound for the family of straight lines $(8)(a)$. The extactic polynomial $\varepsilon_{W_1}(\mathcal{X})$ with $W_1=\langle x-z, 1-y\rangle$ is $\varepsilon_{W_1}(\mathcal{X}) = (1-y)(x-z)\prod_{i=1}^{m-1}(\mu_i(x-z)-\lambda_i(1-y))$. The planes $1\pm y=0$ are invariant by $\mathcal{X}$ since the function y is a first integral of $\mathcal{X}$, $x-z=0$ is invariant by $\mathcal{X}$ with cofactor $k=-\prod_{i=1}^{m-1}(\mu_i(x-z)-\lambda_i(1-y))$ and the planes $\mu_i(x-z)-\lambda_i(1-y)=0$ with $i=1,\ldots,m-1$ are invariant by $\mathcal{X}$ with cofactors $k=\mu_i(z-x)\prod_{\substack{j=1\\ j\neq i}}^{m-1}(\mu_j(x-z)-\lambda_j(1-y))$ respectively. Thus, by addition the number of invariant planes of $\mathcal{X}$ we obtain $m+1$ invariant straight lines of $\mathcal{X}$. □

In a similar way we can obtain a polynomial vector field which reaches the bound provided for the family $(8)(b)$.

8.2. **Invariant non degenerate conics.** In this subsection we work with the non degenerate conics obtained of the intersection of the hyperboloid of a sheet and a plane.

Proposition 41. *If the intersection of a hyperboloid of one sheet with a plane is a non degenerate conic, then it is a parabola, an ellipse or a hyperbola.*

Proof. Consider $x^2+y^2-z^2=1$ be the equation of the hyperboloid of one sheet. If $ax+by+cz+d=0$ is a plane that intersects the hyperboloid with $c\neq 0$, then $z=-(ax+by+d)/c$. Replacing it in $x^2+y^2-z^2=1$ we obtain the equation of the intersection conic $(c^2-a^2)x^2-2abxy+(c^2-b^2)y^2-2adx-2bdy-c^2-d^2=0$. From Proposition 2, we obtain $d_1=c^4(a^2+b^2-c^2-d^2)$ and $d_2=c^2(c^2-a^2-b^2)$. Note that $d_1\neq 0$ since the conic obtained is non degenerate, then $a^2+b^2\neq c^2+d^2$. Moreover, if $c^2=a^2+b^2$, we have $d_2=0$, so the conic is a parabola; if $c^2>a^2+b^2$, then $d_2 > 0$ and the conic is an ellipse; if $c^2 < a^2 + b^2$, result $d_2 < 0$ and hence the conic is a hyperbola.

Now consider $c = 0$, then the plane is $ax+by+d = 0$. Let $a \neq 0$, then $x = -(by+d)/a$. Replacing it in $x^2+y^2-z^2 = 1$ we obtain $(a^2+b^2)y^2-a^2z^2+2bdy+d^2-a^2 = 0$, in this case $d_1 = a^4(a^2+b^2-d^2) \neq 0$ and $d_2 = -a^2(a^2+b^2) < 0$, so the intersection conic is a hyperbola whenever $d^2 \neq a^2+b^2$. If $c = a = 0$, the equation of the plane is $by + d = 0$. Considering $b \neq 0$, $y = -d/b$ and replacing it in $x^2 + y^2 - z^2 = 1$ we obtain $b^2x^2 - b^2z^2 + d^2 - b^2 = 0$, then $d_1 = b^4(b^2 - d^2) \neq 0$ and $d_2 = -b^4 < 0$. Hence the intersection conic is a hyperbola whenever $b^2 \neq d^2$. □

Proposition 42. *There are polynomial vector fields $\mathcal{X}$ defined on the hyperboloid of one sheet $\mathcal{Q}$ that have infinitely many invariant non degenerate conics, taking into account their multiplicities.*

Proof. Note that the vector field of the proof of Proposition 40 has infinitely many invariant non degenerate conics, due to the fact that the planes $y =$ constant are invariant. □

The proof of the following theorem is essentially the same as the proof of Theorem 33 for vector fields defined on the cone, but in this case we cannot subtract one to the degree of the extactic polynomial, because now any plane that contains the origin does not intersect the hyperboloid in a degenerate conic. So we cannot exclude this plane to establish the bound for non degenerate conics.

Theorem 43. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a hyperboloid of one sheet $x^2 + y^2 - z^2 - 1 = 0$ has finitely many invariant non degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant non degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1 + m_2 - 3 & \textit{if } m_1 \geq m_2, \\ m_1 + 5m_2 - 3 & \textit{if } m_1 \leq m_2. \end{array}$$

Proposition 44. *The bound of Theorem 43 decreased by one can be reached.*

Proof. Let $\mathcal{X}$ be a polynomial vector field defined on the hyperboloid of one sheet $x^2+y^2-z^2 = 1$ whose associated differential system is given by $\dot{x} = z(x^2-y^2)x^{m-3}$, $\dot{y} = -zy\,(x^2 - y^2)\,x^{m-4}$, $\dot{z} = (x^2 - y^2)^2\,x^{m-4}$, con $m \in \mathbb{N}$, $m \geq 4$. According with Theorem 43 the maximum number of invariant non degenerate conics that this vector field can have is at most $6m - 3$. Now we shall verify that $\mathcal{X}$ reaches this bound decreased by one. The extactic polynomial $\varepsilon_W(\mathcal{X})$ with $W = \langle 1, x, y, z\rangle$ is $\varepsilon_W(\mathcal{X}) = 6x^{6m-23}yz(x-y)^7(x+y)^7(x-y-z)(x+y-z)(x-y+z)(x+y+z)$.

The planes $x = 0$ and $y = 0$ are invariant by $\mathcal{X}$ with cofactor $k = \pm x^{m-4}(x-y)(x+y)z$ respectively. Also $x \pm y = 0$ are invariant planes with cofactor $k = x^{m-4}(x \mp y)^2 z$ respectively. The planes $x - y \pm z = 0$ are invariant with $k = \pm x^{m-4}(x-y)(x+y)^2$ respectively and $x + y \pm z = 0$ are invariant with cofactor $k = \pm x^{m-4}(x-y)^2(x+y)$ respectively. Finally, $z = 0$ is not an invariant plane of $\mathcal{X}$. Therefore, we obtain $6m-4$ invariant planes. Since these planes do not have the form of the planes of equation (8) we have that all they intersect the hyperboloid of one sheet in non degenerate conics. □

9. Results on the hyperbolic paraboloid

In this section we will use the following canonical form $x = y^2 - z^2$ of a hyperbolic paraboloid $\mathcal{Q}$. Let $\mathcal{X}$ be a polynomial vector field of $\mathbb{R}^3$ defined on $\mathcal{Q}$. The conic obtained by intersecting the hyperbolic paraboloid $\mathcal{Q}$ with a plane is formed by a pair of intersecting straight lines, or a pair of parallel straight lines, when it is degenerate; and it is a parabola, or a hyperbola when the conic is non degenerate.

9.1. **Invariant straight lines.**

As the hyperboloid of one sheet, the hyperbolic paraboloid is also a doubly ruled surface (see [5, 16]).

Proposition 45. *If the intersection of a hyperbolic paraboloid $\mathcal{Q}$ with a plane is a degenerate conic, then it can be a pair of intersecting straight lines, or a pair of parallel straight lines.*

Proof. Following the same arguments used in the proof of Proposition 35 this proposition is proved. $\square$

As mentioned above it is well known that for every point of a hyperbolic paraboloid pass two straight lines. We provide now a proof of this fact to obtain the explicit expression of the two straight lines.

Proposition 46. *For each point $p = (x_0, y_0, z_0)$ of a hyperbolic paraboloid pass two straight lines.*

Proof. If $x_0 = 0$, p is on the two straight lines $x = 0$, $y - z = 0$ and $x = 0$, $y + z = 0$.

If $x_0 \neq 0$, then there is $\lambda \in \mathbb{R} \setminus \{0\}$ such that $y_0 + z_0 = \lambda x_0$. From the equation of the paraboloid $y_0 - z_0 = 1/\lambda$. So, the point p is on the straight line $y + z = \lambda x$, $y - z = 1/\lambda$ with $\lambda = (y_0 + z_0)/x_0$.

On the other hand, if $x_0 \neq 0$, then there is $\mu \in \mathbb{R} \setminus \{0\}$ such that $y_0 - z_0 = \mu x_0$. From the equation of the paraboloid $y_0 + z_0 = 1/\mu$. So, p is on the straight line $y - z = \mu x$, $y + z = 1/\mu$ with $\mu = (y_0 - z_0)/x_0$. $\square$

Remark 47. *From the proof of Proposition 46 we obtain that the hyperbolic paraboloid $x = y^2 - z^2$ contains the two families of straight lines*

$$\begin{array}{cc} y + z = \lambda x & y + z = 0 \\ y - z = 1/\lambda, & x = 0 \\ (a) & (b) \end{array} \tag{10}$$

and,

$$\begin{array}{cc} y - z = \mu x & y - z = 0 \\ y + z = 1/\mu, & x = 0 \\ (a) & (b) \end{array} \tag{11}$$

with $\lambda \neq 0$ and $\mu \neq 0$.

As we study the invariant straight lines, from the proof of Proposition 46 we need only consider the planes whose equation is $y + z - \lambda x = 0$ or $y - z - \mu x = 0$. The family of planes $y + z - \lambda x = 0$ intersects the hyperbolic paraboloid in the family of straight lines (10)(a) when $\lambda \neq 0$ and $x \neq 0$, and in the straight line (10)(b) when $\lambda = 0$. While the family of planes $y - z - \mu x = 0$ intersects the hyperbolic paraboloid in the family of straight lines (11)(a) whenever $\mu \neq 0$ and $x \neq 0$, and in the straight line (11)(b) when $\mu = 0$.

If $\mathcal{X} = (P, Q, R)$ is the polynomial vector field on $\mathcal{Q}$, then $P = 2(yQ - zR)$.

Proposition 48. *There are polynomial vector fields $\mathcal{X}$ defined on the hyperbolic paraboloid $\mathcal{Q}$ that have infinitely many invariant straight lines, taking into account their multiplicities.*

Proof. We determine $\mathcal{X}$ such that all straight lines of the family (10)(a) are invariant by $\mathcal{X}$. Consider $W_1 = \langle x, y+z\rangle$, if $f = y+z-\lambda x$ with $\lambda \in \mathbb{R}$, then $f \in W_1$. The extactic polynomial of $\mathcal{X}$ associated to W_1 is

$$\begin{aligned} \varepsilon_{W_1}(\mathcal{X}) &= \begin{vmatrix} y+z & x \\ \mathcal{X}(y+z) & \mathcal{X}(x) \end{vmatrix} \\ &= (x-2y^2-2yz)Q + (x+2yz+2z^2)R. \end{aligned} \tag{12}$$

In order that $\mathcal{X}$ has infinitely many invariant straight lines it is necessary that $\varepsilon_{W_1}(\mathcal{X}) = 0$. Then, by (12) there are polynomials Q_1 and R_1 such that $Q = (x+2yz+2z^2)Q_1$ and $R = (x-2y^2-2yz)R_1$. So, $\varepsilon_{W_1}(\mathcal{X}) = (Q_1+R_1)(x-2y^2-2yz)(x+2yz+2z^2)$. Doing $Q_1 = 1$ and $R_1 = -1$ we get the vector field $\mathcal{X}$ with $P = 2z(x-2y^2-2yz)+2y(x+2yz+2z^2)$, $Q = x+2yz+2z^2$, $R = -x+2y^2+2yz$. Let $f = y+z-\lambda x$ with $\lambda \in \mathbb{R}$, then $\mathcal{X}f = 2(y+z)(y+z-\lambda x)$, so $f = 0$ is an invariant plane with cofactor $k = 2(y+z)$ for all $\lambda \in \mathbb{R}$. Therefore all straight lines (10)(a) are invariant by $\mathcal{X}$. $\square$

Theorem 49. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a hyperbolic paraboloid $x = y^2 - z^2$ has finitely many invariant straight lines, taking into account their multiplicities. Then, the maximum number of straight lines contained in invariant planes of $\mathcal{X}$ of each family* (10) *and* (11)*, is at most*

$$\begin{aligned} &2+m_2 \quad \text{if } m_2 \geq m_3, \\ &2+m_3 \quad \text{if } m_2 \leq m_3. \end{aligned}$$

Proof. Let $W_1 = \langle x, y+z\rangle$ and $W_2 = \langle x, y-z\rangle$, we calculate the extactic polynomial $\varepsilon_{W_1}(\mathcal{X})$ and $\varepsilon_{W_2}(\mathcal{X})$ and their corresponding degrees. Since $\varepsilon_{W_1}(\mathcal{X}) = \begin{vmatrix} x & y+z \\ \mathcal{X}(x) & \mathcal{X}(y+z) \end{vmatrix}$, $\varepsilon_{W_2}(\mathcal{X}) = \begin{vmatrix} x & y-z \\ \mathcal{X}(x) & \mathcal{X}(y-z) \end{vmatrix}$ and $P = 2(yQ - zR)$, we develop the two previous determinants obtaining $\varepsilon_{W_1}(\mathcal{X}) = x(Q+R) - 2(y+z)(yQ-zR)$ and $\varepsilon_{W_2}(\mathcal{X}) = x(Q-R) - 2(y-z)(yQ-zR)$. The degrees of the monomials of $\varepsilon_{W_1}(\mathcal{X})$ and $\varepsilon_{W_2}(\mathcal{X})$ are $1+m_2$, $2+m_2$, $1+m_3$ and $2+m_3$. So, the degree of both polynomials is $2+m_2$ if $m_2 \geq m_3$, or $2+m_3$ if $m_2 \leq m_3$. Hence we obtain the maximum number of invariant straight lines described in the statement of the theorem. $\square$

Proposition 50. *The bound provided by Theorem 49 can be reached by the family* (11)*.*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on the hyperbolic paraboloid $x = y^2 - z^2$ whose associated differential system is $\dot{x} = 2(y-z)\prod_{i=1}^{m}(y-z-\mu_i x)$, $\dot{y} = \prod_{i=1}^{m}(y-z-\mu_i x)$, $\dot{z} = \prod_{i=1}^{m}(y-z-\mu_i x)$, with $m \in \mathbb{N}$ and $\mu_i \in \mathbb{R}$ for all $i = 1, \ldots, m$. According with Theorem 49 the maximum number of invariant degenerate conics that $\mathcal{X}$ can have is at the most $m+2$. We consider $W_2 = \langle x, y-z\rangle$, then $\varepsilon_{W_2}(\mathcal{X}) =$

$-2(y-z)^2 \prod_{i=1}^{m} (y-z-\mu_i x)$. The planes $y-z-\mu_i x = 0$ for all $\mu_i \in \mathbb{R}$ are invariant by $\mathcal{X}$ with cofactor $k = -2(y-z)\mu_i \prod_{\substack{j=1 \\ j\neq i}}^{m} (y-z-\mu_j x)$. Furthermore, the plane $y-z=0$ also is invariant by $\mathcal{X}$ since $\mathcal{X}(y-z)=0$ and this plane has multiplicity 2. So, we obtain $m+2$ invariant straight lines of the family (11), i.e. the maximum number provided by Theorem 49. □

A similar result to Proposition 50 could be obtained for the family (10).

9.2. **Invariant non degenerate conics.** In this subsection we study the maximum number of invariant non degenerate conics of polynomial vector fields defined on the hyperbolic paraboloid $x = y^2 - z^2$ living on invariant planes of the vector field.

Proposition 51. *If the intersection of a hyperbolic paraboloid with a plane is a non degenerate conic, then it is a parabola or a hyperbola.*

Proof. It can be obtained in a similar way to the proof of Proposition 41. □

Proposition 52. *There are polynomial vector fields $\mathcal{X}$ defined on the hyperbolic paraboloid $\mathcal{Q}$ that have infinitely many invariant non degenerate conics, taking into account their multiplicities.*

Proof. The system of the proof of Proposition 50 has infinitely many invariant non degenerate conics that result of intersecting the invariant planes $y-z=$ constant with the hyperbolic paraboloid. □

Theorem 53. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the hyperbolic paraboloid $x = y^2 - z^2$ has finitely many invariant non degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant non degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} m_2 + 5m_3 & \text{if } m_2 \le m_3, \\ 5m_2 + m_3 & \text{if } m_2 \ge m_3. \end{array}$$

Proof. Let $W = \langle 1, x, y, z\rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by equation (4). The degrees of the monomials of the polynomial $\varepsilon_W(\mathcal{X})$ are $3m_2+3m_3-2$, m_2+5m_3-2, $2m_2+4m_3-2$, $4m_2+2m_3-2$ $5m_2+m_3-2$, $5m_2+m_3-1$, $3m_2+3m_3-1$, m_2+5m_3-1, $4m_2+2m_3-1$, $2m_2+4m_3-1$, $5m_2+m_3$, $4m_2+2m_3$, $3m_2+3m_3$, $2m_2+4m_3$ and m_2+5m_3. The degree of $\varepsilon_W(\mathcal{X})$ is the maximum of these values. Observe that if $m_2 \le m_3$, $5m_2+m_3 \le 4m_2+2m_3 \le 3m_2+3m_3 \le 4m_3+2m_2 \le m_2+5m_3$. So the degree of $\varepsilon_W(\mathcal{X})$ is m_2+5m_3 if $m_2 \le m_3$, or $5m_2+m_3$ if $m_2 \ge m_3$. These values provide the bound for the maximum number of invariant planes (counting their multiplicities) that intersect the hyperbolic paraboloid $x = y^2 - z^2$ stated in the theorem. □

Proposition 54. *The bound provided by Theorem 53 decreased by four can be reached.*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on the hyperbolic paraboloid $x = y^2 - z^2$ whose associated differential system is $\dot{x} = -4y^3 z^{m-2}$, $\dot{y} = -y^2\, z^{m-2}$, $\dot{z} = y^3\, z^{m-3}$, with $m \in \mathbb{N}$, $m \geq 3$. According with Theorem 53 the maximum number of invariant non degenerate conics of $\mathcal{X}$ is at most $6m$. The extactic polynomial (4) is $\varepsilon_W(\mathcal{X}) = 12y^{13}z^{6m-17}(y^2+z^2)$. The plane $y = 0$ is invariant with cofactor $k = -yz^{m-2}$, also $z = 0$ is an invariant plane with $k = y^3 z^{m-4}$. Therefore, we obtain $6m - 4$ invariant real planes, counting their multiplicities. Since these planes have not the form of the ones which appear in (10) or (11), they intersect the paraboloid in non degenerate conics, so $\mathcal{X}$ reaches the bound of Theorem 53 decreased by four. □

Since we work with real polynomial vector fields and their invariant real curves, in the proof of Proposition 54 does not consider the two complex planes obtained from the factor $y^2 + z^2$ in the extactic polynomial. However, the planes $y \pm iz = 0$ are invariant, so if we think in the vector field as a complex one, and we consider the two invariant complex conics obtained from the intersection of these planes and the paraboloid we get the bound $6m-2$ for the number of invariant non degenerate conics, which is nearer to the bound $6m$ provided by Theorem 53. In fact $y^2+z^2 = 0$ is an invariant degenerate conic.

10. Results on the elliptic paraboloid

We shall work with the following canonical form $x = y^2 + z^2$ of a elliptic paraboloid $\mathcal{Q}$.

Proposition 55. *The intersection of the elliptic paraboloid $\mathcal{Q}$ with a plane is a single point, a parabola, or an ellipse.*

Proof. Consider a plane $ax + by + cz + d = 0$ with $c \neq 0$, that intersects the elliptic paraboloid $x = y^2 + z^2$, then $z = -(ax + by + d)/c$. Replacing it in $x = y^2 + z^2$ we obtain the equation of the intersection conic $a^2x^2+2abxy+(b^2+c^2)y^2+(2ad-c^2)x+2bdy+d^2 = 0$. Applying Proposition 2 to this conic we get $d_1 = -c^4(b^2+c^2-4ad)/4$ and $d_2 = a^2c^2$. As $c \neq 0$, $d_1 = 0$ only if $4ad = b^2 + c^2$. In this case also $a \neq 0$, otherwise $b = c = 0$ which is a contradiction. Thus we obtain $d_2 > 0$ and so the intersection conic is a single point. If $d_1 \neq 0$, the conic obtained is a parabola if $a = 0$, and it is an ellipse if $a \neq 0$.

Now consider $c = 0$. The equation of the plane is $ax + by + d = 0$. Assuming $b \neq 0$, we solve y in the above equation and replacing it at $x = y^2 + z^2$ we obtain the equation of the intersection conic $a^2x^2 + b^2z^2 + (2ad - b^2)x + d^2 = 0$ and the determinants $d_1 = -b^4(b^2-4ad)/4$ and $d_2 = a^2b^2$. As $b \neq 0$, the conic is degenerate ($d_1 = 0$) if $b^2 = 4ad$, and consequently a and d are not zero; so $d_2 > 0$ and therefore the intersection conic is a single point. If $d_1 \neq 0$, the intersection conic is a parabola if $a = 0$, and it is an ellipse if $a \neq 0$.

If $c = b = 0$, we obtain the plane $ax + d = 0$. Let $a \neq 0$, then $x = -d/a$ and replacing it in $x = y^2 + z^2$, we have the intersection conic $ay^2 + az^2 + d = 0$ and the determinants $d_1 = a^2d$ and $d_2 = a^2 > 0$. As $a \neq 0$, $d_1 = 0$ whenever $d = 0$, and in this case the intersection conic is a single point, so the plane $x = 0$ intersects $\mathcal{Q}$ only at a point. If $d \neq 0$, then $d_1 \neq 0$ and therefore the intersection conic is an ellipse. □

10.1. **Invariant non degenerate conics.** Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ defined on an elliptic paraboloid $\mathcal{Q}$. In this section we find an upper bound for the maximum number of invariant parabolas and ellipses of $\mathcal{X}$ obtained by intersecting an invariant plane of $\mathcal{X}$ with $\mathcal{Q}$.

If $\mathcal{X} = (P, Q, R)$ is the polynomial vector field defined on the elliptic paraboloid $y^2 + z^2 - x = 0$, then $P = 2(yQ + zR)$ and $\mathcal{X} = (2(yQ + zR), Q, R)$.

Proposition 56. *There are polynomial vector fields $\mathcal{X}$ defined on the elliptic paraboloid $\mathcal{Q}$ that have infinitely many invariant non degenerate conics, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be the polynomial vector field on $\mathcal{Q}$ such that its associated differential system is $\dot{x} = 2yQ$, $\dot{y} = Q$, $\dot{z} = 0$. Since $\dot{z} = 0$, the planes $f = z - a = 0$, with $a \in \mathbb{R}$, are invariant by $\mathcal{X}$. All the planes $f = 0$ intersect $\mathcal{Q}$ in a parabola. So, the vector field $\mathcal{X}$ has infinitely many invariant non degenerate conics. $\square$

Theorem 57. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the elliptic paraboloid $x = y^2 + z^2$ has finitely many invariant non degenerate conics, taking into account their multiplicities. Then,*

(a) the maximum number of parabolas contained in invariant planes of $\mathcal{X}$, taking into account their multiplicities, is at most

$$\begin{array}{ll} 2m_2 + m_3 & \textit{if } m_2 \geq m_3, \\ m_2 + 2m_3 & \textit{if } m_2 \leq m_3, \end{array}$$

(b) and the maximum number of parabolas and ellipses contained in invariant planes of $\mathcal{X}$, taking into account their multiplicities, is at most

$$\begin{array}{ll} 5m_2 + m_3 & \textit{if } m_2 \geq m_3, \\ m_2 + 5m_3 & \textit{if } m_2 \leq m_3. \end{array}$$

Proof. From the proof of Proposition 55 it follows that the intersection of the elliptic paraboloid $x = y^2 + z^2$ with a plane can be a parabola when the plane is parallel the x axis. So, in order to study the invariant parabolas we only need to consider these planes. Let $W = \langle 1, y, z \rangle$, then

$$\varepsilon_W(\mathcal{X}) = -R^2 Q_z + QRR_z - QRQ_y + Q^2 R_y - 2yQRQ_x - 2zR^2 Q_x + 2yQ^2 R_x + 2zQRR_x.$$

The degree of the monomials of $\varepsilon_W(\mathcal{X})$ are $m_2 + 2m_3 - 1$, $2m_2 + m_3 - 1$, $2m_2 + m_3$ and $m_2 + 2m_3$. So, the degree of $\varepsilon_W(\mathcal{X})$ is $m_2 + 2m_3$ if $m_2 \leq m_3$, or $2m_2 + m_3$ if $m_2 \geq m_3$. Therefore, the maximum number of real invariant planes that intersect $\mathcal{Q}$ is $m_2 + 2m_3$ if $m_2 \leq m_3$, or $2m_2 + m_3$ if $m_2 \geq m_3$. Hence, we obtain the bound provided in statement (a) of the theorem.

Now we find a bound for the maximum number of invariant parabolas and ellipses obtained by intersecting $\mathcal{Q}$ with the plane $ax + by + cz + d = 0$. Let $W = \langle 1, x, y, z \rangle$. The degrees of the monomials of the polynomial $\varepsilon_W(\mathcal{X})$ are $3m_2 + 3m_3 - 2$, $m_2 + 5m_3 - 2$, $2m_2 + 4m_3 - 2$, $4m_2 + 2m_3 - 2$ $5m_2 + m_3 - 2$, $5m_2 + m_3 - 1$, $3m_2 + 3m_3 - 1$, $m_2 + 5m_3 - 1$, $4m_2 + 2m_3 - 1$, $2m_2 + 4m_3 - 1$, $5m_2 + m_3$, $4m_2 + 2m_3$, $3m_2 + 3m_3$, $2m_2 + 4m_3$ and $m_2 + 5m_3$. Note that if $m_2 \leq m_3$, we have the following inequalities $5m_2 + m_3 \leq 4m_2 + 2m_3 \leq 3m_2 + 3m_3 \leq 4m_3 + 2m_2 \leq m_2 + 5m_3$. So, the degree of $\varepsilon_W(\mathcal{X})$ is $m_2 + 5m_3$ if $m_2 \leq m_3$, and in a similar way we have that the degree of $\varepsilon_W(\mathcal{X})$ is $5m_2 + m_3$ if $m_2 \geq m_3$. Since the degree of $\varepsilon_W(\mathcal{X})$ provides a bound for the maximum number of invariant planes (counting their multiplicities), statement (b) of the theorem follows. $\square$

Proposition 58. *The bound of statement* (a) *of Theorem 57 decreased by one and the bound of statement* (b) *provided by Theorem 57 decreased by two, can be reached.*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on the elliptic paraboloid $x = y^2 + z^2$ whose associated differential system is $\dot{x} = 2(y + z^{m+1})$, $\dot{y} = 1$, $\dot{z} = z^m$. According with Theorem 57 the maximum number of invariant parabolas that $\mathcal{X}$ can have is at most $2m$. The extactic polynomial given by (4) is $\varepsilon_W(\mathcal{X}) = mz^{2m-1}$. The plane $z = 0$ is invariant by $\mathcal{X}$ with cofactor $k = z^{m-1}$ and its multiplicity is $2m - 1$, i.e we obtain the bound minus one provided by Theorem 57(a).

Let $\mathcal{X}$ be the polynomial vector field defined on the elliptic paraboloid with associated differential system $\dot{x} = 4(x-1)^{m-2}y^2z$, $\dot{y} = (x-1)^{m-2}yz$, $\dot{z} = (x-1)^{m-2}y^2$. The maximum number of invariant non degenerate conics of $\mathcal{X}$ given by Theorem 57(b) is $6m$. We verify that $\mathcal{X}$ reaches this bound minus two. The extactic polynomial (4) is $\varepsilon_W(\mathcal{X}) = -12(x-1)^{6m-12}y^7(y-z)z(y+z)$. The plane $x = 1$ is invariant with cofactor $k = 4(x-1)^{m-3}y^2z$. Also $y = 0$ is an invariant plane with cofactor $k = (x-1)^{m-2}z$. The planes $y \pm z = 0$ are invariant with $k = \pm(x-1)^{m-2}y$ respectively; finally $z = 0$ is an invariant plane with cofactor $k = (x-1)^{m-1}y^2$. Therefore, counting their multiplicities, we obtain $6m - 2$ non degenerate conics contained in these invariant planes. □

10.2. **Invariant degenerate conics.** From Proposition 55, the degenerate conics are obtained when invariant planes are tangent to the elliptic paraboloid $\mathcal{Q}$, they are single points.

Proposition 59. *There are polynomial vector fields* $\mathcal{X}$ *defined on the elliptic paraboloid* $\mathcal{Q}$ *that have infinitely many invariant degenerate conics taking into account their multiplicities.*

Proof. The polynomial differential system $\dot{x} = 2xy$, $\dot{y} = x$, $\dot{z} = 0$ leaves invariant the elliptic paraboloid $x = y^2 + z^2$. From the proof of Proposition 55 we see that the tangent planes to the elliptic paraboloid can be of the form $ax + by + cz + d = 0$, and hence to obtain invariant degenerate conics we have to consider $W = \langle 1, x, y, z \rangle$. The extactic polynomial (4) is zero. So, the multiplicity of the invariant plane $x = 0$ is infinite. Moreover, this invariant plane is tangent at the origin to the elliptic paraboloid. So the proposition is proved. □

Theorem 60. *Assume that a polynomial vector field* $\mathcal{X}$ *in* $\mathbb{R}^3$ *of degree* $\boldsymbol{m} = (m_1, m_2, m_3)$ *defined on the elliptic paraboloid* $x = y^2 + z^2$ *has finitely many degenerate conics taking into account their multiplicities. The maximum number of degenerate conics contained in invariant planes of* $\mathcal{X}$*, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_2 + m_3 & \text{if } m_2 \geq m_3, \\ m_2 + 5m_3 & \text{if } m_2 \leq m_3. \end{array}$$

Moreover, this bound decreased by five, can be reached.

Proof. To find the bound of the maximum number of invariant degenerate conics of $\mathcal{X}$ we consider $W = \langle 1, x, y, z \rangle$, which is the same space used in the proof of Theorem 57(b) to obtain the bound of the invariant non degenerate conics. So, the bound obtained in both cases is the same.

Let $\mathcal{X}$ be the polynomial vector field defined on the elliptic paraboloid with associated differential system $\dot{x} = 2x^{m-1}(xy+z)$, $\dot{y} = x^m$, $\dot{z} = x^{m-1}$. The maximum number of invariant degenerate conics of $\mathcal{X}$ is $6m-1$. We verify that $\mathcal{X}$ reaches this bound minus five. The extactic polynomial given by (4) is $\varepsilon_W(\mathcal{X}) = 4x^{6m-6}(1+2x^2+x^4+2xy^2-4x^3y^2+2yz-10x^2yz-6xz^2)$. The plane $x=0$ is invariant by $\mathcal{X}$ with cofactor $k = 2x^{m-2}(xy+z)$ and its multiplicity is $6m-6$. Therefore, we obtain the bound minus five for the maximum number of invariant degenerate conics of $\mathcal{X}$. □

11. Results on the ellipsoid

We will use the following canonical form $x^2+y^2+z^2-1=0$ of an ellipsoid $\mathcal{Q}$, i.e. the sphere.

Proposition 61. *The intersection of the sphere $\mathcal{Q}$ with a plane is a single point, or a circle.*

Proof. It follows using the same arguments of the proof of Proposition 55. □

11.1. Invariant circles.

Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ defined on $\mathcal{Q}$. In this section we find an upper bound for the maximum number of these invariant circles of $\mathcal{X}$. If $\mathcal{X} = (P,Q,R)$, we have $zR = -(xP+yQ)$, i.e. $\mathcal{X} = (P,Q,-(xP+yQ)/z)$.

Proposition 62. *There are polynomial vector fields defined on the sphere $\mathcal{Q}$ that have infinitely many invariant circles, taking into account their multiplicities.*

Proof. Let $\mathcal{X}$ be the polynomial vector field defined on the sphere $\mathcal{Q}$ whose associated differential system is $\dot{x} = z$, $\dot{y} = 0$, $\dot{z} = -x$. Since $\dot{y} = 0$ the planes $f = y + a$, with $a \in \mathbb{R}$ are invariant by $\mathcal{X}$. These planes intersect $\mathcal{Q}$ in circles, so we obtain infinitely many invariant circles for the vector field $\mathcal{X}$. □

Theorem 63. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the sphere $x^2+y^2+z^2 = 1$ has finitely many invariant circles, taking into account their multiplicities. Then, the maximum number of invariant circles of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1+m_2-3 & \text{if } m_1 \geq m_2, \\ m_1+5m_2-3 & \text{if } m_1 \leq m_2. \end{array}$$

Proof. Let $W = \langle 1, x, y, z\rangle$. Then, the extactic polynomial of $\mathcal{X}$ associated to W is given by the equation (4). The degree of the monomials of $\varepsilon_W(\mathcal{X})$ are $5m_1+m_2-3$, $4m_1+2m_2-3$, $3m_1+3m_2-3$, $2m_1+4m_2-3$ and m_1+5m_2-3. We can see that if $m_1 \geq m_2$, the following inequalities are satisfied $5m_1+m_2-3 \geq 4m_1+2m_2-3 \geq 3m_1+3m_2-3 \geq 2m_1+4m_2-3 \geq m_1+5m_2-3$. So, the degree of $\varepsilon_W(\mathcal{X})$ is $5m_1+m_2-3$ if $m_1 \geq m_2$, and in a similar way we obtain that the degree of $\varepsilon_W(\mathcal{X})$ is m_1+5m_2-3 if $m_1 \leq m_2$. Therefore, the maximum number of invariant planes (counting their multiplicities) that intersect $\mathcal{Q}$ is $5m_1+m_2-3$ if $m_1 \geq m_2$, or m_1+5m_2-3 if $m_1 \leq m_2$. Hence the theorem follows. □

Proposition 64. *The bound provided by Theorem 63 decreased by four is reached.*

Proof. Let $\mathcal{X} = (P, Q, R)$ the polynomial vector field defined on the sphere $\mathcal{Q}$ with $P = z^{m-3}y^2x$, $Q = z^{m-3}x^2y$ and $R = -2x^2y^2z^{m-4}$, $m \in \mathbb{N}$ and $m \geq 5$. According with Theorem 63 the maximum number of invariant degenerate conics of $\mathcal{X}$ is at most $6m - 3$, taking into account their multiplicities. Now we verify that $\mathcal{X}$ reaches this bound decreased in four. The extactic polynomial (4) is $\varepsilon_W(\mathcal{X}) = 6x^7(x-y)\,y^7(x+y)\,z^{6m-23}(2x^2+z^2)(2y^2+z^2)$. The plane $x = 0$ is invariant by $\mathcal{X}$ with cofactor $k = y^2z^{m-3}$, $y = 0$ also is invariant with cofactor $k = x^2z^{m-3}$ and $z = 0$ is invariant with cofactor $k = -2x^2y^2z^{m-5}$. The planes $x \pm y = 0$ are invariant with cofactor $k = \pm xyz^{m-3}$ respectively. Counting all these planes with their multiplicities we obtain the bound $6m - 7$ for the maximum number of invariant circles of $\mathcal{X}$ that corresponds to the bound of Theorem 63 minus four. □

In the proof of Proposition 64 we do not consider the four complex planes that are obtained of the factors $2x^2 + z^2$ and $2y^2 + z^2$ in the extactic polynomial. However, the planes $\sqrt{2}x \pm iz = 0$ and $\sqrt{2}y \pm iz = 0$ are invariant, so taking into account the complex conics resulting from its intersection with the sphere we would get the bound $6m - 3$ for the maximum number of invariant circles of $\mathcal{X}$ which coincides with the bound provided by Theorem 63. In fact $2x^2 + z^2 = 0$ and $2y^2 + z^2 = 0$ are two invariant degenerate conics.

11.2. **Invariant degenerate conics.** According with Proposition 61, the degenerate conics appear when we have invariant planes tangent to the sphere $\mathcal{Q}$, they are single points.

Proposition 65. *There are polynomial vector fields $\mathcal{X}$ defined on the sphere $\mathcal{Q}$ that have infinitely many invariant degenerate conics taking into account their multiplicities.*

Proof. The polynomial differential system $\dot{x} = z(x-1)$, $\dot{y} = 0$, $\dot{z} = -x(x-1)$ leaves invariant the sphere $x^2 + y^2 + z^2 - 1 = 0$. Then, according with the proof of Proposition 61, to get planes tangents to the sphere, we consider $W = \langle 1, x, y, z\rangle$. The associated extactic polynomial (4) is zero. So, the multiplicity of the invariant plane $x = 1$ is infinite. Moreover, this invariant plane is tangent at $(1, 0, 0)$ to $\mathcal{Q}$. So the proposition is proved. □

Theorem 66. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the sphere $x^2 + y^2 + z^2 = 1$ has finitely many invariant degenerate conics taking into account their multiplicities. Then, the maximum number of invariant degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1 + m_2 - 3 & \text{if } m_1 \geq m_2, \\ m_1 + 5m_2 - 3 & \text{if } m_1 \leq m_2. \end{array}$$

Moreover, this bound minus seven can be reached.

Proof. We observe that the result of this theorem coincides with the bound of Theorem 63, this happens because in both cases $W = \langle 1, x, y, z\rangle$ and so we obtain the same extactic polynomials whose degree produces the bound required. So, the proof follows the same steps than the proof of Theorem 63.

Let $\mathcal{X}$ be the polynomial vector field defined on the sphere with associated differential system $\dot{x} = z(x+1)^{m-1}$, $\dot{y} = -z(x+1)^{m-2}$, $\dot{z} = -(x+1)^{m-2}(x^2+x-y)$. The maximum number of invariant degenerate conics of $\mathcal{X}$ is $6m - 4$. We verify that $\mathcal{X}$

reaches this bound minus seven. The extactic polynomial given by (4) is $\varepsilon_W(\mathcal{X}) = -(x+1)^{6m-11}z(3x^3+9x^4+9x^5+3x^6-9x^2y-18x^3y-9x^4y+9xy^2+9x^2y^2-3y^3+6xz^2+14x^2z^2+13x^3z^2+5x^4z^2-6yz^2-7xyz^2-4x^2yz^2-y^2z^2+z^4+4xz^4+2x^2z^4)$. The plane $x+1=0$ is invariant by $\mathcal{X}$ with cofactor $k = z(x+1)^{m-2}$ and its multiplicity is $6m-11$. Therefore, we obtain the bound minus seven for the maximum number of invariant degenerate conics of $\mathcal{X}$. □

12. Results on the hyperboloid of two sheets

We will use the following canonical form $x^2+y^2-z^2+1=0$ of a hyperboloid of two sheets $\mathcal{Q}$.

Proposition 67. *The intersection of a hyperboloid of two sheets $\mathcal{Q}$ with a plane is a single point, a parabola, a ellipse or a hyperbola.*

Proof. Using the same arguments of the proof of Proposition 55 this proposition can be proved. □

12.1. **Invariant non degenerate conics.** Define $\mathcal{X}$ as a polynomial vector field in $\mathbb{R}^3$ on the hyperboloid of two sheets $x^2+y^2-z^2+1=0$. We find an upper bound for the maximum number of invariant non degenerate conics of $\mathcal{X}$ living in invariant planes. If $\mathcal{X} = (P,Q,R)$, then $zR = xP+yQ$, i.e. $\mathcal{X} = (P,Q,(xP+yQ)/z)$.

Proposition 68. *There are polynomial vector fields defined on the hyperboloid of two sheets that have infinitely many invariant non degenerate conics, taking into account their multiplicities.*

Proof. Consider the differential system $\dot{x} = z$, $\dot{y} = 0$, $\dot{z} = x$. As $\dot{y} = 0$, $f = y+a$, with $a \in \mathbb{R}$ and $a \neq 0$ are invariant planes of the polynomial vector field $\mathcal{X}$ associated to system. The planes $f = 0$ intersect $\mathcal{Q}$ in hyperbolas. So, $\mathcal{X}$ has infinitely many invariant non degenerate conics. □

The following theorem provides a bound for the maximum number of invariant non degenerate conics of vector fields defined on the hyperboloid of two sheets. Given the similarity between the equation of the hyperboloid of one sheet and the hyperboloid of two sheets, we obtain the same result for the case of vector fields defined on the hyperboloid of one sheet (Theorem 43) and the proof is essentially the same, so we do not write it.

Theorem 69. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on a hyperboloid of two sheets $x^2+y^2-z^2+1=0$ has finitely many invariant non degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant non degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1+m_2-3 & \textit{if } m_1 \geq m_2, \\ m_1+5m_2-3 & \textit{if } m_1 \leq m_2. \end{array}$$

Proposition 70. *The bound provided by Theorem 69 decreased in one can be reached.*

Proof. It is similar to the proof of Proposition 44. □

12.2. **Invariant degenerate conics.** From Proposition 67, the degenerate conics appear when invariant planes are tangent to the hyperboloid of two sheets $\mathcal{Q}$, and they are single points.

Proposition 71. *There are polynomial vector fields $\mathcal{X}$ defined on the hyperboloid of two sheets $\mathcal{Q}$ that have infinitely many invariant degenerate conics taking into account their multiplicities.*

Proof. Consider $\dot{x} = 0$, $\dot{y} = z(z-1)$, $\dot{z} = y(z-1)$ the polynomial differential system associated to a vector field $\mathcal{X}$ that leaves invariant the hyperboloid of two sheets $x^2+y^2-z^2+1=0$. Then, the associated extactic polynomial $\varepsilon_W(\mathcal{X})$ with $W = \langle 1, x, y, z\rangle$ is zero. So, the multiplicity of the invariant plane $z=1$ is infinite. Moreover, this invariant plane is tangent to $\mathcal{Q}$ at the point $(0,0,1)$. So the proposition is proved. □

Theorem 72. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^3$ of degree $\mathbf{m} = (m_1, m_2, m_3)$ defined on the hyperboloid of two sheets $x^2+y^2-z^2+1=0$ has finitely many invariant degenerate conics, taking into account their multiplicities. Then, the maximum number of invariant degenerate conics of $\mathcal{X}$ contained in invariant planes, taking into account their multiplicities, is at most*

$$\begin{array}{ll} 5m_1+m_2-3 & \text{if } m_1 \geq m_2, \\ m_1+5m_2-3 & \text{if } m_1 \leq m_2. \end{array}$$

Moreover, this bound minus eight can be reached.

Proof. Note that the bound of Theorems 69 and 72 for the case of non degenerate conics and degenerate conics respectively is the same. This happens because in both cases $W = \langle 1, x, y, z\rangle$, and so we obtain the same extactic polynomials whose degree produces the bound required.

Let $\mathcal{X}$ be the polynomial vector field defined on the sphere with associated differential system $\dot{x} = z(z+1)^{m-2}$, $\dot{y} = z(z+1)^{m-1}$, $\dot{z} = (z+1)^{m-2}(x+y+yz)$. The maximum number of invariant degenerate conics of $\mathcal{X}$ is $6m-4$. We verify that $\mathcal{X}$ reaches this bound minus eight. The extactic polynomial given by (4) is $\varepsilon_W(\mathcal{X}) = (z+1)^{6m-12}(2x^4+8x^3y+12x^2y^2+8xy^3+2y^4+6x^3yz+18x^2y^2z+18xy^3z+6y^4z-2x^2z^2-4xyz^2-2y^2z^2+6x^2y^2z^2+12xy^3z^2+6y^4z^2+x^2z^3-4xyz^3-5y^2z^3+2xy^3z^3+2y^4z^3-4z^4+2x^2z^4+4xyz^4-2y^2z^4-8z^5+3xyz^5+2y^2z^5-8z^6+y^2z^6-4z^7-z^8)$. The plane $z+1=0$ is invariant by $\mathcal{X}$ with cofactor $k=(z+1)^{m-3}(x+y+yz)$ and its multiplicity is $6m-12$. Therefore, we obtain the bound minus eight for the maximum number of invariant degenerate conics of $\mathcal{X}$. □

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