This is a preprint of: "On the number of limit cycles for a generalization of Liénard polynomial differential systems", Jaume Llibre, Clàudia Valls, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 23(3), 1350048 (16 pages), 2013.
DOI: [10.1142/S021812741350048X]

ON THE NUMBER OF LIMIT CYCLES FOR A GENERALIZATION OF LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE
Departament de Matemàtiques
Universitat Autònoma de Barcelona, 08193, Bellaterra, Barcelona, Catalonia, Spain, e-mail: jllibre@mat.uab.cat
CLAUDIA VALLS
Departamento de Matemática
Instituto Superior Técnico, 1049–001 Lisboa, Portugal
e-mail: cvalls@math.ist.utl.pt

February 2, 2012

We study the number of limit cycles of the polynomial differential systems of the form

$$\dot{x} = y - g_1(x), \quad \dot{y} = -x - g_2(x) - f(x)y,$$

where $g_1(x) = \varepsilon g_{11}(x)+\varepsilon^2 g_{12}(x)+\varepsilon^3 g_{13}(x)$, $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f(x) = \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x)$ where g_{1i}, g_{2i}, f_{2i} have degree k, m and n respectively for each $i = 1, 2, 3$, and ε is a small parameter. Note that when $g_1(x) = 0$ we obtain the generalized Liénard polynomial differential systems. We provide an upper bound of the maximum number of limit cycles that the previous differential system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order.

1. Introduction and statement of the main results

The second part of the 16th Hilbert's problem wants to find an upper bound for the maximum number of limit cycles that a polynomial vector field of a fixed degree can have. In this paper we will try to give a partial answer to this problem for the class of polynomial differential systems

$$\dot{x} = y - g_1(x), \quad \dot{y} = -x - g_2(x) - f(x)y, \tag{1}$$

where

$$\begin{aligned} g_1(x) &= \varepsilon g_{11}(x) + \varepsilon^2 g_{12}(x) + \varepsilon^3 g_{13}(x), \\ g_2(x) &= \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x), \\ f(x) &= \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x), \end{aligned}$$

where g_{1i}, g_{2i}, f_i have degree k, m and n respectively for each $i = 1, 2, 3$, and ε is a small parameter. When $g_1(x) = 0$ the differential system (1) coincides with the generalized Liénard polynomial differential systems. The classical Liénard polynomial differential systems are

$$\dot{x} = y, \quad \dot{y} = -x - f(x)y, \tag{2}$$

where $f(x)$ is a polynomial in the variable x of degree n. For these systems [Lins et al., 1977] stated the conjecture that if $f(x)$ has degree $n \geq 1$ then

gsd http://www.gsd.uab.cat

system (2) has at most $[n/2]$ limit cycles. Here $[x]$ denotes the integer part function of $x \in \mathbb{R}$. They proved this conjecture for $n = 1, 2$. The conjecture for $n = 3$ has been proved recently by [Li & Llibre, 2012]. For $n \geq 5$ the conjecture is not true, see [De Maesschalck & Dumortier, 2011] and [Dumortier et al., 2007]. So it remains to know if the conjecture is true or not for $n = 4$.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point (i.e., from a Hopf bifurcation), that are called *small amplitude limit cycles*, see for instance [Lloyd, 1988]. There are partial results concerning the maximum number of small amplitude limit cycles for Liénard polynomial differential systems. The number of small amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have.

There are many results concerning the existence of small amplitude limit cycles for the following generalization of the classical Liénard polynomial differential system (2)

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \tag{3}$$

where $g(x)$ and $f(x)$ are polynomials in the variable x of degrees m and n, respectively. We denote by $H(m, n)$ the maximum number of limit cycles that systems (3) can have. This number is usually called the *Hilbert number* for systems (3).

(i) [Liénard, 1928] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (3) has a unique limit cycle.

(ii) [Rychkov, 1975] proved that if $m = 1$ and $F(x)$ is an odd polynomial of degree five, then equation (3) has at most two limit cycles.

(iii) [Lins et al., 1977] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.

(iv) [Coppel, 1998] proved that $H(2, 1) = 1$.

(v) [Dumortier & Rousseau, 1990] and [Dumortier & Li, 1996] shown that $H(3, 1) = 1$.

(vi) [Dumortier & Li, 1997] proved that $H(2, 2) = 1$.

(vii) [Lloyd & Lynch, 1988] proved that $H(1, 3) = 1$.

Up to now and as far as we know only for these four cases ((iii)-(vii)) the Hilbert number for systems (3) has been determined.

The maximum number of small amplitude limit cycles for systems (3) is denoted by $\hat{H}(m, n)$. [Blows & Lloyd, 1984], [Lloyd & Lynch, 1988] and [Lynch, 1995] have used inductive arguments in order to prove the following results.

(I) If g is odd then $\hat{H}(m, n) = [n/2]$.

(II) If f is even then $\hat{H}(m, n) = n$, whatever g is.

(III) If f is odd then $\hat{H}(m, 2n+1) = [(m-2)/2]+n$.

(IV) If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

[Christopher & Lynch, 1999], [Lynch, 1998], [Lynch, 1999], [Lynch & Christopher, 1999] have developed a new algebraic method for determining the Liapunov quantities of systems (3) and proved some other bounds for $\hat{H}(m, n)$ for different m and n.

(V) $\hat{H}(m, 2) = [(2m + 1)/3]$.

(VI) $\hat{H}(2, n) = [(2n + 1)/3]$.

(VII) $\hat{H}(m, 3) = 2[(3m + 2)/8]$ for all $1 < m \leq 50$.

(VIII) $\hat{H}(3, n) = 2[(3n + 2)/8]$ for all $1 < m \leq 50$.

(IX) $\hat{H}(4, k) = \hat{H}(k, 4)$, $k = 6, 7, 8, 9$ and $\hat{H}(5, 6) = \hat{H}(6, 5)$.

[Gasull & Torregrosa, 1998] obtained upper bounds for $\hat{H}(7, 6)$, $\hat{H}(6, 7)$, $\hat{H}(7, 7)$ and $\hat{H}(4, 20)$.

[Yu & Han, 2006] give some accurate values of $\hat{H}(m, n) = \hat{H}(n, m)$, for $n = 4$, $m = 10, 11, 12, 13$; $n = 5$, $m = 6, 7, 8, 9$; $n = 6$, $m = 5, 6$, see also [Llibre et al., 2009] for a table with all the specific values.

[Llibre et al., 2009] compute the maximum number of limit cycles $\tilde{H}_k(m, n)$ of systems (3) which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of order k, for $k = 1, 2, 3$.

In [Llibre & Valls, 2011] the authors studied using the averaging theory of first and second order the more general system

$$\begin{aligned} \dot{x} = &\ y - \varepsilon(g_{11}(x) + f_{11}(x)y) - \\ &\ \varepsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} = &\ -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \\ &\ \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{aligned} \tag{4}$$

where g_{1i}, f_{1i}, g_{2i}, f_{2i} have degree k, l, m and n respectively for each $i = 1, 2$, and ε is a small parameter. Using the averaging method of first and second order they proved the following result.

Theorem 1.1. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (4) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order is:*

$$\begin{aligned} \max \quad & \{\mu + [(m-1)/2],\ \mu + [k/2], \\ & [(n-1)/2] + [m/2], \\ & [l/2] + [m/2] - 1, \\ & [(n-1)/2] + [(k-1)/2] + 1, \\ & [l/2] + [(k-1)/2]\}, \end{aligned} \tag{5}$$

with $\mu = \min\{[n/2], [(l-1)/2]\}$.

[Alavez-Ramirez et al., 2012] studied system (1) with $g_{13}(x) = g_{23}(x) = f_{23}(x) = 0$ and they proved the following result.

Theorem 1.2. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (4) with $f_{11}(x) = f_{12}(x) = 0$ bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order is*

$$\lambda_0 = \left[\frac{1}{2}\left(\max\{O(m+n), E(k+m) - 1\} - 1\right)\right],$$

where $O(l)$ is the largest odd integer $\leq l$, and $E(l)$ is the largest even integer $\leq l$.

In the present paper we study system (1), i.e. we extend the results of [Alavez-Ramirez et al., 2012] because in [Alavez-Ramirez et al., 2012] first $g_{13}(x) = g_{23}(x) = f_3(x) = 0$, and additionally the study of the limit cycles coming from averaging of second and third order is made under some restrictive conditions. Using the averaging method of third order we will show our main result:

Theorem 1.3. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (1) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order is λ_1 equal to*

$$\begin{aligned} \max\{& 2m + 2[k/2] - 2, \\ & m + 2[(k-1)/2], \\ & m + 2[(n-1)/2] + 2[m/2] + 2, \\ & 4[(n-1)/2] + 2[(m-1)/2] + 1\}. \end{aligned} \tag{6}$$

Note that $\lambda_0 < \lambda_1$. The proof of Theorem 1.3 is given in Section 3.

The results that we shall use from the averaging theory of third order for computing limit cycles are presented in Section 2.

2. The averaging theory of third order

The averaging theory for studying specifically limit cycles up to third order in ε was developed in [Buică & J. Llibre, 2004]. It is summarized as follows.

Consider the differential system

$$\begin{aligned} \dot{x} = &\ \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \\ &\ \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \end{aligned} \tag{7}$$

where $F_1, F_2, F_3 \colon \mathbb{R} \times D \to \mathbb{R}$, $R \colon \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following conditions hold.

(i) $F_1(t,\cdot) \subset C^2(D)$, $F_2(t,\cdot) \subset C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} \colon D \to \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(z) = &\ \frac{1}{T}\int_0^T F_1(s,z)\,ds, \\ F_{20}(z) = &\ \frac{1}{T}\int_0^T [D_z F_1(s,z) y_1(s,z) + \\ &\ F_2(s,z)]\,ds, \end{aligned}$$

$$
\begin{aligned}
F_{30}(z) = \; & \frac{1}{T}\int_0^T \Big(\frac{1}{2}\frac{\partial^2 F_1}{\partial z^2}(s,z) y_1(s,z)^2 + \\
& \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z) y_2(s,z) + \\
& \frac{\partial F_2}{\partial z}(s,z) y_1(s,z) + F_3(s,z)\Big)\, ds,
\end{aligned}
$$

where

$$
\begin{aligned}
y_1(s,z) = \; & \int_0^s F_1(t,z)\, dt, \\
y_2(s,z) = \; & 2\int_0^s \left(\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)\, dr \right. \\
& \left. + F_2(t,z)\right) dt.
\end{aligned}
$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\phi(\cdot, \varepsilon)$ of the system such that $\phi(0, a_\varepsilon) \to a_\varepsilon$ when $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30} \colon V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition in order that this inequality is true is that the Jacobian of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} are identically zero and F_{30} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

3. Proof of Theorem 1.3

We shall need the third order averaging theory to prove Theorem 1.3. We write system (1) in polar coordinates (r, θ) where

$$
x = r\cos\theta, \quad y = r\sin\theta, \quad r > 0.
$$

If we write

$$
\begin{aligned}
f_1(x) = \; & \sum_{i=0}^n a_i x^i, \\
f_2(x) = \; & \sum_{i=0}^n c_i x^i,, \\
f_3(x) = \; & \sum_{i=0}^n p_i x^i,
\end{aligned}
$$

$$
\begin{aligned}
g_{11}(x) = \; & \sum_{i=0}^k b_{i,1} x^i, \\
g_{12}(x) = \; & \sum_{i=0}^k d_{i,1} x^i, \\
g_{13}(x) = \; & \sum_{i=0}^k q_{i,1} x^i,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
g_{21}(x) = & \sum_{i=0}^m b_{i,2} x^i, \\
g_{22}(x) = & \sum_{i=0}^m d_{i,2} x^i, \\
g_{23}(x) = & \textstyle\sum_{i=0}^m q_{i,2} x^i,
\end{aligned}
$$

then system (1) becomes

$$
\begin{aligned}
\dot{r} = \; & -\varepsilon(A + \varepsilon B + \varepsilon^2 C), \\
\dot{\theta} = \; & -1 - \frac{\varepsilon}{r}(A_1 + \varepsilon B_1 + \varepsilon^2 C_1),
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
A = \; & \sum_{i=0}^n a_i r^{i+1} \cos^i\theta \sin^2\theta + \\
& \sum_{i=0}^m b_{i,2} r^i \cos^i\theta \sin\theta + \\
& \sum_{i=0}^k b_{i,1} r^i \cos^{i+1}\theta, \\
B = \; & \sum_{i=0}^n c_i r^{i+1} \cos^i\theta \sin^2\theta + \\
& \sum_{i=0}^m d_{i,2} r^i \cos^i\theta \sin\theta + \\
& \sum_{i=0}^k d_{i,1} r^i \cos^{i+1}\theta,
\end{aligned}
$$

$$
\begin{aligned}
C = \ & \sum_{i=0}^{n} p_i r^{i+1}\cos^i\theta\sin^2\theta + \\
& \sum_{i=0}^{m} q_{i,2} r^{i}\cos^i\theta\sin\theta + \\
& \sum_{i=0}^{k} q_{i,1} r^{i}\cos^{i+1}\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 = \ & \sum_{i=0}^{n} a_{i,} r^{i+1}\cos^{i+1}\theta\sin\theta + \\
& \sum_{i=0}^{m} b_{i,2} r^{i}\cos^{i+1}\theta - \\
& \sum_{i=0}^{k} b_{i,1} r^{i}\cos^{i}\theta\sin\theta, \\
B_1 = \ & \sum_{i=0}^{n} c_{i} r^{i+1}\cos^{i+1}\theta\sin\theta + \\
& \sum_{i=0}^{m} d_{i,2} r^{i}\cos^{i+1}\theta - \\
& \sum_{i=0}^{k} d_{i,1} r^{i}\cos^{i}\theta\sin\theta, \\
C_1 = \ & \sum_{i=0}^{n} p_{i} r^{i+1}\cos^{i+1}\theta\sin\theta + \\
& \sum_{i=0}^{m} q_{i,2} r^{i}\cos^{i+1}\theta - \\
& \sum_{i=0}^{k} q_{i,1} r^{i}\cos^{i}\theta\sin\theta,
\end{aligned}
$$

Now taking θ as the new independent variable, system (9) becomes

$$
\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + \varepsilon^3 F_3(r,\theta) + O(\varepsilon^4),
$$

where

$$
\begin{aligned}
F_1(r,\theta) = \ & A, \quad F_2(r,\theta) = B - \frac{1}{r}AA_1, \\
F_3(r,\theta) = \ & C - \tfrac{1}{r}(BA_1 + AB_1) + \frac{1}{r^2}A_1^2 A.
\end{aligned} \tag{10}
$$

In [Alavez-Ramirez et al., 2012] the authors took the sufficient conditions

$$
b_{2i+1,1} = 0 \quad \text{and} \quad a_{2j,2} = 0, \tag{11}
$$

for $i = 0, \ldots, [(k-1)/2]$ and $j = 0, \ldots, [n/2]$ to have $F_{10} = 0$. Furthermore, using this condition they computed F_{20} and they took

$$
\begin{aligned}
& d_{2i+1,1} = c_{2i} = 0 \quad \text{for } i = 0, \ldots, [(k-1)/2] \\
& \qquad\qquad \text{and } j = 0, \ldots, [n/2], \text{ and} \\
& \text{either } b_{i,1} = a_j = 0 \text{ for } i = 0, \ldots, k \text{ and} \\
& \qquad\qquad j = 0, \ldots, n, \\
& \text{or } b_{i,2} = 0 \text{ for } i = 0, \ldots, m,
\end{aligned}
$$

thus obtaining an upper bound for the number of limit cycles. However a close look to $F_{20}(r)$ obtained in [Alavez-Ramirez et al., 2012] imply that it is only necessary to take

$$
\begin{aligned}
& \sum_{i=0}^{[(k-1)/2]} d_{2i+1,1} + \sum_{i=0}^{[n/2]} \frac{c_{2i}}{2i+1} + \\
& \sum_{i=0}^{[k/2]} \sum_{j=0}^{[m/2]} C_{ij} b_{2i,1} b_{2j,2} + \\
& \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} D_{ij} a_{2i+1} b_{2j,2} = 0,
\end{aligned} \tag{12}
$$

for some constants C_{ij} and D_{ij}.

We shall work only with the necessary conditions (11) and (12), and consequently we improve the results of [Alavez-Ramirez et al., 2012] in two ways. First because we work with this necessary conditions and second because we consider in the differential system (1) terms up to order ε^3 while in [Alavez-Ramirez et al., 2012] they consider only up to order ε^2.

In order to apply the third order averaging method we need to compute the corresponding function $F_{30}(r)$ that we rewrite as

$$
F_{30}(r) = F_{30}^1(r) + F_{30}^2(r) + F_{30}^3(r) + F_{30}^4(r)
$$

with

$$
\begin{aligned}
F_{30}^1(r) = \ & \frac{1}{4\pi}\int_0^{2\pi} \frac{\partial^2 F_1}{\partial r^2}(r,\theta) y_1(r,\theta)^2 \, d\theta, \\
F_{30}^2(r) = \ & \frac{1}{4\pi}\int_0^{2\pi} \frac{\partial F_1}{\partial r}(r,\theta) y_2(r,\theta) \, d\theta, \\
F_{30}^3(r) = \ & \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial F_2}{\partial r}(r,\theta) y_1(r,\theta) \, d\theta, \\
F_{30}^4(r) = \ & \frac{1}{2\pi}\int_0^{2\pi} F_3(r,\theta) \, d\theta.
\end{aligned} \tag{13}
$$

It was proved in [Alavez-Ramirez et al., 2012] that using the integrals of the Appendix, or in

[Llibre & Valls, 2011] that $y_1 = y_1(\theta, r)$ is equal to

$$\sum_{i=0}^{[(n-1)/2]} a_{2i+1} r^{2i+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) + \sum_{i=0}^{m} \frac{b_{i,2}}{i+1} r^i (1-\cos^{i+1}\theta) + \sum_{i=0}^{[k/2]} b_{2i,1} r^{2i} \sum_{s=0}^{i} \gamma_{i,s} \sin((2s+1)\theta), \tag{14}$$

where

$$\tilde{\gamma}_{i,l} = \begin{cases} \gamma_{i,l} - \gamma_{i+1,l}, & 0 \le l \le i, \\ -\gamma_{i+1,i+1}, & l = i+1, \end{cases}$$

and $F_1(\theta, r)$ is

$$\sum_{i=0}^{[(n-1)/2]} a_{2i+1} r^{2i+2} \cos^{2i+1}\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta + \sum_{i=0}^{[k/2]} b_{2i,1} r^{2i} \cos^{2i+1}\theta,$$

finally $\dfrac{\partial}{\partial r} F_1(\theta, r)$ is

$$\sum_{i=0}^{[(n-1)/2]} (2i+2) a_{2i+1} r^{2i+1} \cos^{2i+1}\theta \sin^2\theta + \sum_{i=0}^{m} i b_{i,2} r^{i-1} \cos^i\theta \sin\theta + \sum_{i=0}^{[k/2]} 2i b_{2i,1} r^{2i-1} \cos^{2i+1}\theta.$$

We also note that $F_2(r, \theta)$ is equal to

$$\begin{aligned} &\sum_{i=0}^{n} c_i r^{i+1} \cos^i\theta (1-\cos^2\theta) + \sum_{i=0}^{m} d_{i,2} r^i \cos^i\theta \sin\theta + \sum_{i=0}^{k} d_{i,1} r^i \cos^{i+1}\theta - \\ &\sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(n-1)/2]} a_{2i+1} a_{2j+1} r^{2i+2j+3} \cos^{2i+2j+3} \cdot \sin\theta (1-\cos^2\theta) - \\ &2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{m} a_{2i+1} b_{j,2} r^{2i+j+1} \cos^{2i+j+2}\theta \cdot (1-\cos^2\theta) + \\ &\sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[k/2]} a_{2i+1} b_{2j,1} r^{2i+2j+1} \cos^{2i+2j+1}\theta \sin\theta (1-2\cos^2\theta) - \\ &\sum_{i=0}^{m} \sum_{j=0}^{m} b_{i,2} b_{j,2} r^{i+j-1} \cos^{i+j+1}\theta \sin\theta + \\ &\sum_{i=0}^{m} \sum_{j=0}^{[k/2]} b_{i,2} b_{2j,1} r^{i+2j-1} \cos^{i+2j}\theta (1-2\cos^2\theta) + \\ &\sum_{i=0}^{[k/2]} \sum_{j=0}^{[k/2]} b_{2i,1} b_{2j,1} r^{2i+2j-1} \cos^{2i+2j+1}\theta \sin\theta. \end{aligned} \tag{15, 16}$$

The proof of Theorem 1.3 will be a direct consequence of the next auxiliary lemmas.

Lemma 3.1. *The integral $F_{30}^1(r)$ is a polynomial in the variable r of degree*

$$\lambda_2 = 2(m + \max\{[(n-1)/2], [k/2]-1\}).$$

(For an explicit expression of the polynomial $F_{30}^1(r)$ we refer the reader to the proof of Lemma 3.1).

Proof. We first note that $\dfrac{\partial^2}{\partial r^2} F_1(\theta, r)$ is equal to

$$\sum_{i=0}^{[(n-1)/2]} (2i+2)(2i+1) a_{2i+1} r^{2i} \cos^{2i+1}\theta (1-\cos^2\theta) + \sum_{i=0}^{m} i(i-1) b_{i,2} r^{i-2} \cos^i\theta \sin\theta + \sum_{i=0}^{[k/2]} 2i(2i-1) b_{2i,1} r^{2i-2} \cos^{2i+1}\theta,$$

and $y_1(r, \theta)^2$ is equal to

$$\begin{aligned} &\sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(n-1)/2]} a_{2i+1} a_{2j+1} r^{2i+2j+4} \cdot \sum_{s=0}^{i+1} \sum_{r=0}^{j+1} \tilde{\gamma}_{i,s} \tilde{\gamma}_{j,r} \sin((2s+1)\theta) \sin((2r+1)\theta) + \\ &2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{m} a_{2i+1} \frac{b_{j,2}}{j+1} r^{2i+j+2} \cdot \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} (1-\cos^{j+1}\theta) \sin((2s+1)\theta) + \end{aligned}$$

$$2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]} a_{2i+1}b_{2j,1}r^{2i+2j+2}.$$
$$\sum_{s=0}^{i+1}\sum_{r=0}^{j}\tilde{\gamma}_{i,s}\gamma_{j,r}\sin((2s+1)\theta)\sin((2r+1)\theta)+$$
$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{b_{i,2}b_{j,2}r^{i+j}}{(i+1)(j+1)}(1-\cos^{i+1}\theta)(1-\cos^{j+1}\theta)+$$
$$2\sum_{i=0}^{m}\sum_{j=0}^{[k/2]}\frac{b_{i,2}b_{2j,1}}{(i+1)}r^{i+2j}\sum_{r=0}^{j}\gamma_{j,r}(1-\cos^{i+1}\theta)\cdot$$
$$\sin((2r+1)\theta)+\sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]} b_{2i,1}b_{2j,1}r^{2i+2j}.$$
$$\sum_{s=0}^{i}\sum_{r=0}^{j}\gamma_{i,s}\gamma_{j,r}\sin((2s+1)\theta)\sin((2r+1)\theta).$$

Hence, using the formulae in the appendix and the explicit formula of $F_{30}^1(r)$ given in (13) we obtain that $F_{30}^1(r)$ is equal to

$$\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{m}\sum_{j=0}^{m}\delta_1 a_{2t+1}b_{i,2}b_{j,2}r^{2t+i+j}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{m}\delta_2 b_{2t,2}a_{2i+1}b_{j,2}r^{2t+2i+j}+$$
$$\sum_{t=0}^{[(m-1)/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]}\delta_3 b_{2t+1,2}a_{2i+1}b_{2j,2}r^{2t+2i+2j+1}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{m}\sum_{j=0}^{[k/2]}\delta_4 b_{2t,2}b_{i,2}b_{2j,1}r^{2t+i+2j-2}+$$
$$\sum_{t=0}^{[(m-1)/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]}\delta_5 b_{2t+1,2}b_{2i,2}b_{2j,1}r^{2t+2i+2j-1}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{m}\sum_{j=0}^{m}\delta_6 b_{2t,1}b_{i,2}b_{j,2}r^{2t+i+j-2},$$

where δ_i for $i = 1,\dots,6$ are constants depending on t, i and j. ■

Lemma 3.2. *The integral $F_{30}^3(r)$ is a polynomial in the variable r of degree λ_3 equal to*

$$\max\{m+2[(k-1)/2],\; m+2[(n-1)/2]+2[m/2]+2,\; m+2[m/2]+2[k/2]-2\}.$$

(For an explicit expression of the polynomial $F_{30}^3(r)$ we refer the reader to the proof of Lemma 3.2).

Proof. We first note that $\partial F_2(r,\theta)/\partial r$ is equal to

$$\sum_{i=0}^{n}(i+1)c_i r^i\cos^i\theta(1-\cos^2\theta)+$$
$$\sum_{i=0}^{m} id_{i,2}r^{i-1}\cos^i\theta\sin\theta+\sum_{i=0}^{k} id_{i,1}r^{i-1}\cos^{i+1}\theta-$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(n-1)/2]}(2i+2j+3)a_{2i+1}a_{2j+1}r^{2i+2j+2}.$$
$$\cos^{2i+2j+3}\sin^3\theta-2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{m}(2i+j+1)a_{2i+1}\cdot$$
$$b_{j,2}r^{2i+j}\cos^{2i+j+2}\theta\sin^2\theta+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]}(2i+2j+1)a_{2i+1}b_{2j,1}r^{2i+2j}.$$
$$\cos^{2i+2j+1}\theta\sin\theta(1-2\cos^2\theta)-$$
$$\sum_{i=0}^{m}\sum_{j=0}^{m}(i+j-1)b_{i,2}b_{j,2}r^{i+j-2}\cos^{i+j+1}\theta\sin\theta+$$
$$\sum_{i=0}^{m}\sum_{j=0}^{[k/2]}(i+2j-1)b_{i,2}b_{2j,1}r^{i+2j-2}\cos^{i+2j}\theta\cdot$$
$$(1-2\cos^2\theta)+\sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]}(2i+2j-1)b_{2i,1}b_{2j,1}\cdot$$
$$r^{2i+2j-2}\cos^{2i+2j+1}\theta\sin\theta.$$

Hence, using the formulae in the appendix and the explicit formula of $F_{30}^3(r)$ given in (13) we obtain that $F_{30}^3(r)$ is equal to

$$\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{[m/2]}\rho_1 a_{2t+1}d_{2i,2}r^{2t+2i+1}+$$
$$\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\rho_2 a_{2t+1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j+1}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[k/2]}\rho_3 b_{2t,2}d_{2i,1}r^{2t+2i-1}+$$
$$\sum_{t=0}^{m}\sum_{i=0}^{[(k-1)/2]}\rho_4 b_{t,2}d_{2i+1,1}r^{t+2i}+$$
$$\sum_{t=0}^{m}\sum_{i=0}^{[n/2]}\rho_5 b_{t,2}c_{2i}r^{t+2i}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\rho_6 b_{2t,2}c_{2i+1}r^{2t+2i+1}+$$
$$\sum_{t=0}^{m}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]}\rho_7 b_{t,2}a_{2i+1}b_{2j,2}r^{t+2i+2j+2}+$$

$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]}\rho_8 b_{2t,2}a_{2i+1}b_{2j+1,2}r^{2t+2i+2j+3}+$$
$$\sum_{t=0}^{m}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]}\rho_9 b_{t,2}a_{2i}b_{2j+1,2}r^{t+2i+2j-2}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]}\rho_{10} b_{2t,2}b_{2i+1,2}b_{2j,1}r^{2t+2i+2j-1}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\rho_{11} b_{2t,1}d_{2i,2}r^{2t+2i-1}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\rho_{12} b_{2t,1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j-1},$$

where ρ_i for $i = 1, \ldots, 12$ are constants depending on t, i and j. ■

Lemma 3.3. *The integral $F_{30}^4(r)$ is a polynomial in the variable r of degree λ_4 equal to*

$$\begin{aligned}\max\{&2[n/2]+2[(m-1)/2]+1,\\ &2[(m-1)/2]+2[(k-1)/2]+1,\\ &2[(n-1)/2]+2[m/2]+2[(m-1)/2]+1,\\ &2[m/2]+2[(m-1)/2]+2[k/2]-1\}.\end{aligned}$$

(For an explicit expression of the polynomial $F_{30}^4(r)$ we refer the reader to the proof of Lemma 3.3).

Proof. We will first compute in F_3 all the terms that have non-zero integral from 0 to 2π. To do it, we note that in view of (10) F_3 is equal to

$$C - \frac{1}{r}(BA_1 + AB_1) + \frac{1}{r^2}AA_1^2.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} C(\theta, r)\,d\theta = \sum_{i=0}^{[n/2]}\zeta_1 p_{2i}r^{2i+1}+ \sum_{i=0}^{[(k-1)/2]}\zeta_2 q_{2i+1,1}r^{2i+1},$$

where the constants ζ_1, ζ_2 depend on i. Furthermore, using the formulae of the Appendix we conclude that

$$\frac{1}{2\pi r}\int_0^{2\pi}(AB_1 + BA_1)(\theta, r)\,d\theta =$$
$$\sum_{i=0}^{[n/2]}\sum_{j=0}^{[(m-1)/2]}\zeta_3 a_{2i}d_{2j+1,2}r^{2i+2j+1}+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]}\zeta_4 a_{2i+1}d_{2j,2}r^{2i+2j+1}+$$
$$\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(n-1)/2]}\zeta_5 b_{2i,2}c_{2j+1}r^{2i+2j+1}+$$
$$\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[n/2]}\zeta_6 b_{2i+1,2}c_{2j}r^{2i+2j+1}+$$
$$\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]}\zeta_7 b_{2i,2}d_{2j,1}r^{2i+2j-1}+$$
$$\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[(k-1)/2]}\zeta_8 b_{2i+1,2}d_{2j+1,1}r^{2i+2j+1}+$$
$$\sum_{i=0}^{[k/2]}\sum_{j=0}^{[m/2]}\zeta_9 b_{2i,1}d_{2j,2}r^{2i+2j-1},$$

where the constants ζ_l for $l = 3, \ldots, 9$ depend on i, j. Finally, since A_1^2/r^2 is equal to

$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(n-1)/2]} a_{2i+1}a_{2j+1}r^{2i+2j+2}\cos^{2i+2j+4}\theta\cdot$$
$$(1-\cos^2\theta) + 2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{m} a_{2i+1}b_{j,2}r^{2i+j}\cdot$$
$$\cos^{2i+j+3}\theta\sin\theta - 2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]} a_{2i+1}b_{2j,1}r^{2i+2j}\cdot$$
$$\cos^{2i+2j+2}\theta(1-\cos^2\theta) + \sum_{i=0}^{m}\sum_{j=0}^{m} b_{i,2}b_{j,2}r^{i+j-2}\cdot$$
$$\cos^{i+j+2}\theta - 2\sum_{i=0}^{m}\sum_{j=0}^{[k/2]} b_{i,2}b_{2j,1}r^{i+2j-2}\cos^{i+2j+1}\theta\cdot$$
$$\sin\theta + \sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]} b_{2i,1}b_{2j,1}r^{2i+2j-2}\cos^{2i+2j}\theta\cdot$$
$$(1-\cos^2\theta),$$

using the formulae of the Appendix we conclude that

$$\frac{1}{2\pi r^2}\int_0^{2\pi}(AA_1^2)(\theta, r)\,d\theta =$$
$$\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\zeta_{10} a_{2t+1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j+1}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]}\zeta_{11} b_{2t,2}a_{2i+1}b_{2j+1,2}r^{2t+2i+2j+1}+$$
$$\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]}\zeta_{12} b_{2t,2}b_{2i+1,2}b_{2j,1}r^{2t+2i+2j-1}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\zeta_{13} b_{2t,1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j-1},$$

where the constants ζ_l for $l = 10, \ldots, 13$ depend on t, i, j. ■

Lemma 3.4. *The integral $F_{30}^2(r)$ is a polynomial in the variable r of degree λ_5 equal to*

$$\begin{aligned}\max\{&m + 2[(k-1)/2],\\ &m + 2[(n-1)/2] + 2[m/2],\\ &m + 2[m/2] + 2[k/2] - 2,\\ &4[(n-1)/2] + 2[(m-1)/2] + 1\}.\end{aligned}$$

(For an explicit expression of the polynomial $F_{30}^2(r)$ we refer the reader to the proof of Lemma 3.4).

Proof. We note that $F_{30}^2(r)$ is equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial F_1}{\partial r}(\theta,r)\int_0^{\theta}F_2(\psi,r)\,d\psi\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial F_1}{\partial r}(\theta,r)\int_0^{\theta}\frac{\partial F_1}{\partial r}(\psi,r)y_1(\psi,r)\,d\psi\,d\theta.$$

We will first compute the terms in

$$\hat{F}_{30}^2(r) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial F_1}{\partial r}(\theta,r)\int_0^{\theta}F_2(\psi,r)\,d\psi\,d\theta.$$

For this we note that $\int_0^{\theta}F_2(\psi,r)\,d\psi$ is equal to

$$\begin{aligned}&\sum_{i=0}^{[n/2]} c_{2i}r^{2i+1}\Big(\beta_{i,0}\theta + \sum_{s=1}^{i}\beta_{i,s}\sin(2s\theta)\Big) + \\ &\sum_{i=0}^{[(n-1)/2]} c_{2i+1}r^{2i+2}\sum_{s=0}^{i}\gamma_{i,s}\sin((2s+1)\theta) + \\ &\sum_{i=0}^{m}\frac{d_{i,2}}{i+1}r^{i}(1-\cos^{i+1}\theta) + \\ &\sum_{i=0}^{[k/2]} d_{2i,1}r^{2i}\sum_{s=0}^{i}\gamma_{i,s}\sin((2s+1)\theta) + \\ &\sum_{i=0}^{[(k-1)/2]} d_{2i+1,1}r^{2i+1}\Big(\beta_{i,0}\theta + \sum_{s=1}^{i+1}\beta_{i,s}\sin(2s\theta)\Big) - \\ &\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(n-1)/2]} a_{2i+1}a_{2j+1}r^{2i+2j+3}\cdot \\ &\Big(\frac{1-\cos^{2i+2j+4}\theta}{2i+2j+4} - \frac{1-\cos^{2i+2j+6}\theta}{2i+2j+6}\Big) - \\ &2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]} a_{2i+1}b_{2j+1,2}r^{2i+2j+2}\cdot \\ &\sum_{s=0}^{i+j+2}\gamma^*_{i+j+2,s}\sin((2s+1)\theta) - \\ &2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} a_{2i+1}b_{2j,2}r^{2i+2j+1}\Big(\beta^*_{i+j+2,0}\theta + \\ &\sum_{s=1}^{i+j+2}\beta^*_{i+j+2,s}\sin(2s\theta)\Big) + \\ &\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]} a_{2i+1}b_{2j,1}r^{2i+2j+1}\Big(\frac{1-\cos^{2i+2j+2}\theta}{2i+2j+2} - \\ &\frac{1-\cos^{2i+2j+4}\theta}{2i+2j+4}\Big) - \\ &\sum_{i=0}^{m}\sum_{j=0}^{m} b_{i,2}b_{j,2}r^{i+j-1}\frac{1-\cos^{i+j+2}\theta}{i+j+2} + \\ &\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]} b_{2i+1,2}b_{2j,1}r^{2i+2j}\sum_{s=0}^{i+j+1}\bar{\gamma}_{i+j+1,s}\cdot \\ &\sin((2s+1)\theta) + \sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} b_{2i,2}b_{2j,1}r^{2i+2j-1}\cdot \\ &\Big(\bar{\beta}_{i+j+1,0}\theta + \sum_{s=1}^{i+j+1}\bar{\beta}_{i+j+1,s}\sin(2s\theta)\Big) + \\ &\sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]} b_{2i,1}b_{2j,1}r^{2i+2j-1}\frac{1-\cos^{2i+2j+2}\theta}{2i+2j+2}.\end{aligned}$$

Now using the non-zero formulae in the appendix we conclude that $\hat{F}_{30}^2(r)$ is equal to

$$\begin{aligned}&\sum_{t=0}^{[(m-1)/2]}\sum_{i=0}^{[m/2]}\mu_1 a_{2t}d_{2i,2}r^{2t+2i+1} + \\ &\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\mu_2 a_{2t+1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j+1} + \\ &\sum_{t=0}^{m}\sum_{i=0}^{[n/2]}\mu_3 b_{t,2}c_{2i}r^{t+2i} + \sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\mu_4 b_{2t,2}c_{2i+1}\cdot \\ &r^{2t+2i+1} + \sum_{t=0}^{[m/2]}\sum_{i=0}^{[k/2]}\mu_5 b_{2t,2}d_{2i,1}r^{2t+2i-1} + \\ &\sum_{t=0}^{m}\sum_{i=0}^{[(k-1)/2]}\mu_6 b_{t,2}d_{2i+1,1}r^{t+2i} + \\ &\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]}\mu_7 b_{2t,2}a_{2i+1}b_{2j+1,2}r^{2t+2i+2j+1} + \\ &\sum_{t=0}^{m}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]}\mu_8 b_{t,2}a_{2i+1}b_{2j,2}r^{t+2i+2j} + \\ &\sum_{t=0}^{[m/2]}\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]}\mu_9 b_{2t,2}b_{2i+1,2}b_{2j,1}r^{2t+2i+2j-1} +\end{aligned}$$

$$\sum_{t=0}^{m}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]}\mu_{10}b_{t,2}b_{2i,2}b_{2j,1}r^{t+2i+2j-2}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\mu_{11}b_{2t,1}d_{2i,2}r^{2t+2i-1}+$$
$$\sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]}\mu_{12}b_{2t,1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j-1},$$

where the constants μ_i for $i=1,\dots,12$ depend on t,i,j.

Now we compute $\int_0^\theta y_1(r,\psi)\frac{\partial F_1}{\partial r}(r,\psi)\,d\psi$. We first note that $y_1(r,\psi)\frac{\partial F_1}{\partial r}(r,\psi)$ is equal to

$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(n-1)/2]}a_{2i+1}(2j+2)a_{2j+1}r^{2i+2j+3}\cdot$$
$$\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\cos^{2j+1}\psi(1-\cos^2\psi)\sin((2s+1)\psi)+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{m}a_{2i+1}jb_{j,2}r^{2i+j+1}\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\cos^j\psi\cdot$$
$$\sin\psi\sin((2s+1)\psi)+\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]}a_{2i+1}2jb_{2j,1}\cdot$$
$$r^{2i+2j+1}\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\cos^{2j+1}\psi\sin((2s+1)\psi)+$$
$$\sum_{i=0}^{m}\sum_{j=0}^{[(n-1)/2]}\frac{b_{i,2}}{i+1}(2j+2)a_{2j+1}r^{i+2j+1}(1-\cos^{i+1}\psi)\cdot$$
$$\cos^{2j+1}\psi(1-\cos^2\psi)+\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{b_{i,2}}{i+1}jb_{j,2}r^{i+j-1}\cdot$$
$$(1-\cos^{i+1}\psi)\cos^j\psi\sin\psi+\sum_{i=0}^{m}\sum_{j=0}^{[k/2]}\frac{b_{i,2}}{i+1}2jb_{2j,1}\cdot$$
$$r^{i+2j-1}(1-\cos^{i+1}\psi)\cos^{2j+1}\psi+\sum_{i=0}^{[k/2]}\sum_{j=0}^{[(n-1)/2]}b_{2i,1}\cdot$$
$$(2j+2)a_{2j+1}r^{2i+2j+1}\sum_{s=0}^{i}\gamma_{i,s}\cos^{2j+1}\psi(1-\cos^2\psi)\cdot$$
$$\sin((2s+1)\psi)+\sum_{i=0}^{[k/2]}\sum_{j=0}^{m}b_{2i,1}jb_{j,2}r^{2i+j-1}\cdot$$
$$\sum_{s=0}^{i}\gamma_{i,s}\cos^j\psi\sin\psi\sin((2s+1)\psi)+$$
$$\sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]}b_{2i,1}2jb_{2j,1}r^{2i+2j-1}\sum_{s=0}^{i}\gamma_{i,s}\cos^{2j+1}\psi\cdot$$
$$\sin((2s+1)\psi).$$

Hence using the formulae in the appendix we deduce that $\int_0^\theta y_1(r,\psi)\frac{\partial F_1}{\partial r}(r,\psi)\,d\psi$ is equal to

$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(n-1)/2]}a_{2i+1,2}(2j+2)a_{2j+1}r^{2i+2j+3}\cdot$$
$$\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\sum_{r=1}^{j+s+2}\hat{R}_{j+s+2,r}\cos(2r\theta)+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]}a_{2i+1}(2j+1)b_{2j+1,2}r^{2i+2j+2}\cdot$$
$$\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\sum_{r=1}^{j+s+1}U_{j+s+1,r}\sin((2r+1)\theta)+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]}a_{2i+1}2jb_{2j,2}r^{2i+2j+1}\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\cdot$$
$$\Big(W_{j+s+1}\theta+\sum_{r=1}^{j+s+1}W_{j+s+1,r}\sin(2r\theta)\Big)+$$
$$\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[k/2]}a_{2i+1}2jb_{2j,1}r^{2i+2j+1}\cdot$$
$$\textstyle\sum_{s=0}^{i+1}\tilde{\gamma}_{i,s}\sum_{r=1}^{j+s+1}R_{j+s,r}\cos(2r\theta)+$$
$$\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(n-1)/2]}\frac{b_{2i,2}}{2i+1}(2j+2)a_{2j+1}r^{2i+2j+1}\cdot$$
$$\Big(\beta^*_{i+j+2}\theta+\sum_{r=1}^{i+j+2}\beta^*_{i+j+2,r}\sin(2r\theta)\Big)+$$
$$\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[(n-1)/2]}\frac{b_{2i+1,2}}{2i+2}(2j+2)a_{2j+1}r^{2i+2j+2}\cdot$$
$$\textstyle\sum_{r=1}^{i+j+2}\gamma^*_{i+j+2,r}\sin((2r+1)\theta)+$$
$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{b_{i,2}}{i+1}jb_{j,2}r^{i+j-1}\Big(\frac{1-\cos^{j+1}\theta}{j+1}-$$
$$\frac{1-\cos^{i+j+2}\theta}{i+j+2}\Big)+\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]}\frac{b_{2i,2}}{2i+1}2jb_{2j,1}r^{2i+2j-1}\cdot$$
$$\Big(\bar{\beta}_{i+j+1}\theta+\sum_{r=1}^{i+j+1}\bar{\beta}_{i+j+1,r}\sin(2r\theta)\Big)+$$
$$\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]}\frac{b_{2i+1,2}}{2i+2}2jb_{2j,1}r^{2i+2j}\sum_{r=1}^{i+j+1}\bar{\gamma}_{i+j+1,r}\cdot$$
$$\sin((2r+1)\theta)+\sum_{i=0}^{[k/2]}\sum_{j=0}^{[(n-1)/2]}b_{2i,1}(2j+2)a_{2j+1}\cdot$$
$$r^{2i+2j+1}\sum_{s=0}^{i+1}\gamma_{i,s}\sum_{r=1}^{j+s+2}\hat{R}_{j+s+2,r}\cos(2r\theta)+$$

$$\sum_{i=0}^{[k/2]}\sum_{j=0}^{[m/2]} b_{2i,1}2jb_{2j,2}r^{2i+2j-1}\sum_{s=0}^{i+1}\gamma_{i,s}\Big(W_{j+s+1}\theta+ \sum_{r=1}^{j+s+1} W_{j+s+1,r}\sin(2r\theta)\Big) + \sum_{i=0}^{[k/2]}\sum_{j=0}^{[(m-1)/2]} b_{2i,1}\cdot (2j+1)b_{2j+1,2}r^{2i+2j}\sum_{s=0}^{i+1}\gamma_{i,s}\sum_{r=1}^{j+s+1} U_{j+s+1,r}\cdot \sin((2r+1)\theta) + \sum_{i=0}^{[k/2]}\sum_{j=0}^{[k/2]} b_{2i,1}2jb_{2j,1}r^{2i+2j-1}\cdot \sum_{s=0}^{i+1}\gamma_{i,s}\sum_{r=1}^{j+s+1} R_{j+s,r}\cos(2r\theta).$$

Hence using the zero formulae in the appendix we compute $\bar{F}^2_{30}(r)$ and we obtain that it is equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial F_1}{\partial r}(\theta,r)\int_0^{\theta}\frac{\partial F_1}{\partial r}(\psi,r)y_1(\psi,r)\,d\psi\,d\theta,$$

and this integral is

$$\sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{m}\sum_{j=0}^{[m/2]} \nu_1 a_{2t+1}b_{i,2}b_{2j,2}r^{2t+i+2j}+ \sum_{t=0}^{[(n-1)/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]} \nu_2 a_{2t+1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j+1}+ \sum_{t=0}^{[m/2]}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(m-1)/2]} \nu_3 b_{2t,2}a_{2i+1}b_{2j+1,2}r^{2t+2i+2j+1}+ \sum_{t=0}^{m}\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} \nu_4 b_{t,2}a_{2i+1}b_{2j,2}r^{t+2i+2j}+ \sum_{t=0}^{m}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(n-1)/2]} \nu_5 b_{t,2}b_{2i,2}a_{2j+1}r^{t+2i+2j}+ \sum_{t=0}^{[m/2]}\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[(n-1)/2]} \nu_6 b_{2t,2}b_{2i+1,2}a_{2j+1}r^{2t+2i+2j+1}+ \sum_{t=0}^{m}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} \nu_7 b_{t,2}b_{2i,2}b_{2j,1}r^{t+2i+2j-2}+ \sum_{t=0}^{[m/2]}\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[k/2]} \nu_8 b_{2t,2}b_{2i+1,2}b_{2j,1}r^{2t+2i+2j-1}+ \sum_{t=0}^{m}\sum_{i=0}^{[k/2]}\sum_{j=0}^{[m/2]} \nu_9 b_{t,2}b_{2i,1}b_{2j,2}r^{t+2i+2j-2}+ \sum_{t=0}^{[m/2]}\sum_{i=0}^{[k/2]}\sum_{j=0}^{[(m-1)/2]} \nu_{10} b_{2t,2}b_{2i,1}b_{2j+1,2}r^{2t+2i+2j-1}+ \sum_{t=0}^{[k/2]}\sum_{i=0}^{m}\sum_{j=0}^{[m/2]} \nu_{11} b_{2t,1}b_{i,2}b_{2j,2}r^{2t+i+2j-2}+ \sum_{t=0}^{[k/2]}\sum_{i=0}^{[m/2]}\sum_{j=0}^{[(m-1)/2]} \nu_{12} b_{2t,1}b_{2i,2}b_{2j+1,2}r^{2t+2i+2j-1},$$

where the constants ν_i for $i = 1, \ldots, 12$ depend on t, i, j. ■

Appendix: Formulae

In this appendix we recall some formulae that will be used during the paper, see for more details [Abramowitz & Stegun, 1964]. For $i \geq 0$ we have

$$\int_0^{2\pi}\cos^{2i+1}\theta\sin^2\theta\,d\theta = \int_0^{2\pi}\cos^{i}\theta\sin\theta\,d\theta = \int_0^{2\pi}\cos^{2i+1}\theta\,d\theta = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2i}\theta\,d\theta = \alpha_i$$

where α_i is a non-zero constant,

$$\int_0^{\theta}\cos^{2i}\phi\,d\phi = \frac{1}{2^{2i}}\begin{pmatrix} 2i \\ i \end{pmatrix}\theta + \frac{1}{2^{2i}}\sum_{l=1}^{i}\begin{pmatrix} 2i \\ i+1 \end{pmatrix}\cdot \frac{1}{l}\sin(2l\theta) = \beta_{i,0}\theta + \sum_{l=1}^{i}\beta_{i,l}\sin(2l\theta),$$

$$\int_0^{\theta}\cos^{2i+1}\phi\,d\phi = \frac{1}{2^{2i}}\sum_{l=0}^{i}\begin{pmatrix} 2i+1 \\ i-1 \end{pmatrix}\frac{1}{2l+1}\cdot \sin((2l+1)\theta) = \sum_{l=0}^{i}\gamma_{i,l}\sin((2l+1)\theta),$$

$$\int_0^{\theta}\cos^{i}\phi\sin\phi\,d\phi = \frac{1}{i+1}(1-\cos^{i+1}\theta),$$

$$\int_0^{\theta}\cos^{2i}\psi\sin((2s+1)\psi)\,d\psi = \sum_{r=1}^{i+s} P_{i+s,r}\cos((2r+1)\theta),$$

$$\int_0^{\theta}\cos^{2i}\psi\sin(2s\psi)\,d\psi = \sum_{r=1}^{i+s} Q_{i+s,r}\cos(2r\theta),$$

$$\int_0^{\theta}\cos^{2i+1}\psi\sin((2s+1)\psi)\,d\psi = \sum_{r=1}^{i+s+1} R_{i+s,r}\cos(2r\theta),$$

$$\int_0^{\theta}\cos^{2i+1}\psi\sin(2s\psi)\,d\psi = \sum_{r=1}^{i+s} T_{i+s,r}\cos((2r+1)\theta),$$

$$\int_0^\theta \cos^{2i+1}\psi \sin\psi \sin((2s+1)\psi)\,d\psi = \sum_{r=1}^{i+s+1} U_{i+s+1,r}\sin((2r+1)\theta),$$

$$\int_0^\theta \cos^{2i+1}\psi \sin\psi \sin(2s\psi)\,d\psi = V_{i+s+1}\theta + \sum_{r=1}^{i+s+1} V_{i+s+1,r}\sin(2r\theta),$$

$$\int_0^\theta \cos^{2i}\psi \sin\psi \sin((2s+1)\psi)\,d\psi = W_{i+s+1}\theta + \sum_{r=1}^{i+s+1} W_{i+s+1,r}\sin(2r\theta),$$

$$\int_0^\theta \cos^{2i}\psi \sin\psi \sin(2s\psi)\,d\psi = \sum_{r=1}^{i+s} Z_{i+s,r}\sin((2r+1)\theta),$$

$$\int_0^{2\pi} \cos^{i}\theta \sin\theta \cos(j\theta)\,d\theta = \int_0^{2\pi} \cos^{2i}\theta \cos((2j+1)\theta)\,d\theta = \int_0^{2\pi} \cos^{2i+1}\theta \cos(2j\theta)\,d\theta = 0,\ j\in\mathbb{Z}$$

$$\int_0^{2\pi} \cos^{2i}\theta \cos(2j\theta)\,d\theta = D_{i,j},$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \cos((2j+1)\theta)\,d\theta = E_{i,j},$$

where $D_{i,j}$, $E_{i,j}$ are nonzero constants,

$$\int_0^{2\pi} \theta\cos^{2i}\theta\,d\theta = F_i, \quad \int_0^{2\pi} \theta\cos^{i}\theta\sin\theta\,d\theta = G_i,$$

where F_i, G_i are nonzero constants,

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin((2l+1)\theta)\,d\theta = \int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin(2l\theta)\,d\theta = 0, \quad l\geq 0,$$

$$\int_0^{2\pi} \cos^{i}\theta \sin((2l+1)\theta)\,d\theta = \int_0^{2\pi} \cos^{i}\theta \sin(2l\theta)\,d\theta = 0, \quad l\geq 0,$$

$$\int_0^{2\pi} \theta\cos^{2i+1}\theta\,d\theta = \int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin((2l+1)\theta)\,d\theta = \int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin(2l\theta)\,d\theta = 0, \quad l\geq 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin((2l+1)\theta)\,d\theta = C_{i,l}, \quad l\geq 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin(2l\theta)\,d\theta = K_{i,l}, \quad l\geq 1,$$

where $C_{i,j}$ and $K_{i,l}$ are nonzero constants,

$$\int_0^{2\pi} \cos^{i}\theta \sin\theta \sin(r\theta)\sin(s\theta)\,d\theta = 0, \quad r,s\in\mathbb{N}.$$

In the remainder formulae $r,s\in\mathbb{Z}$:

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin((2r+1)\theta)\sin((2s+1)\theta)\,d\theta = 0,$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin(2r\theta)\sin(2s\theta)\,d\theta = 0,$$

$$\int_0^{2\pi} \cos^{2i}\theta \sin((2r+1)\theta)\sin(2s\theta)\,d\theta = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2i}\theta \sin((2r+1)\theta)\sin((2s+1)\theta)\,d\theta = \Delta_{i,r,s},$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2i}\theta \sin(2r\theta)\sin(2s\theta)\,d\theta = \Gamma_{i,r,s},$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin((2r+1)\theta)\sin(2s\theta)\,d\theta = \Upsilon_{i,r,s},$$

where $\Delta_{i,r,s}$, $\Gamma_{i,r,s}$ and $\Upsilon_{i,r,s}$ are non-zero constants.

Acknowledgments

The first author has been supported by the grants MICINN/FEDER MTM 2009-03437, AGAUR 2009SGR 410 and ICREA Academia. The second author has been supported by the grant AGAUR PIV-DGR-2010 and by FCT through CAMGSD.

References

Abramowitz, M., & Stegun, I., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series 55, Washington.

Alavez-Ramirez, J., Blé, G., Llibre, J. & and Lopez-Lopez, J., [2012] "On the maximum number of limit cycles of a class of generalized Liénard differential systems," *to appear in Int. J. Bifurcation and Chaos* , .

Blows, T.R., & Lloyd, N.G., [1984] "The number of small-amplitude limit cycles of Liénard equations," *Math. Proc. Camb. Phil. Soc.* **95**, 359–366.

Buică, A., & Llibre, J., [2004] "Averaging methods for finding periodic orbits via Brouwer degree," *Bull. Sci. Math.* **128**, 7–22.

Christopher, C.J., & Lynch, S., [1999] "Small-amplitude limit cycle bifurcations for Liénard systems with quadratic or cubic dapimg or restoring forces," *Nonlinearity* **12**, 1099–1112.

Coppel, W.A., [1998] "Some quadratic systems with at most one limit cycles," *Dynamics Reported* **2**, 61–68.

De Maesschalck, P., & Dumortier, F., [2011] "Classical Liénard equation of degree $n \geq 6$ can have $\left[\frac{n-1}{2}\right]+2$ limit cycles," *J. Differential Equations* **250**, 2162–2176.

Dumortier, F., Panazzolo, D., & Roussarie, R., [2007] "More limit cycles than expected in Liénard systems," *Proc. Amer. Math. Soc.* **135**, 1895–1904.

Dumortier,F., & Li, C., [1996] "On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations," *Nonlinearity* **9**, 1489–1500.

Dumortier,F., & Li, C., [1997] "Quadratic Liénard equations with quadratic damping," *J. Differential Equations* **139**, 41–59.

Dumortier, F., & Rousseau, C., [1990] "Cubic Liénard equations with linear dapimg," *Nonlinearity* **3**, 1015–1039.

Gasull, A., & Torregrosa,J., [1998] "Small–Amplitude limit cycles in Liénard systems via multiplicity," *J. Differential Equations* **159**, 1015–1039.

Guckenheimer, J. & Holmes, P., [1990] *Nonlinear Osciallations, Dynamical Systems, and Bifurcations of Vector Fields*, Revised and corrected reprint of the 1983 original, Applied Mathematical Sciences, Vol. 42, Springer-Verlag, New York.

Li, C., & Llibre, J., [2012] "Uniqueness of limit cycles for Liénard differential equations of degree four," *J. Differential Equations* **252**, 3142–3162.

Liénard, A., [1928] "Étude des oscillations entrenues," *Revue Génerale de l' Électricité* **23**, 946–954.

Lins, A., de Melo, W., & Pugh, C.C., [1977] "On Liénard's Equation," *Lecture Notes in Math.* **597**, 335–357.

Llibre, J., [2002] "Averaging theory and limit cycles for quadratic systems," *Radovi Matematicki* **11**, 215–228.

Llibre, J., Mereu, A.C.,& Teixeira, M.A. [2009] "Limit cycles of the generalized polynomial Liénard differential equations," *Math. Proceed. Camb. Phyl. Soc.* **148**, 363–383.

Llibre, J., & Valls, C., [2011] "On the number of limit cycles of a class of polynomial differential systems," *preprint* , .

Lloyd, N.G., [1988] "Limit cycles of polynomial systems – some recent developments," *London Math. Soc. Lecture Note Ser.* **127**, 192–234.

Lloyd, N.G., & Lynch, S., [1988] "Small–amplitude limit cycles of certain Liénard systems," *Proc. Royal Soc. London Ser. A* **418**, 199–208.

Lynch, S., [1995] "Limit cycles if generalized Liénard equations," *Applied Math. Letters* **8**, 15–17.

Lynch, S., [1998] "Generalized quadratic Liénard equations," *Applied Math. Letters* **11**, 7–10.

Lynch, S., [1999] "Generalized cubic Liénard equations," *Applied Math. Letters* **12**, 1–6.

Lynch, S., & Christopher, C.J., [1999] "Limit cycles in highly non-linear differential equations," *J. Sound Vib.* **224**, 505–517.

Rychkov, G.S., [1975] "The maximum number of limit cycle of the system $\dot{x} = y - a_1x^3 - a_2x^5, \dot{y} = -x$ is two," *Differential'nye Uravneniya* **11**, 380–391.

Verhulst, F., [1991] *Nonlinear Differential Equations and dynamical Systems* Universitext, Springer–Verlag, New York.

Yu, P., & Han, M., [2006] "Limit cycles in generalized Liénard systems," *Chaos, Solitons and Fractals* **30**, 1048–1068.