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Canard Trajectories in 3D piecewise linear systems

Rafel Prohens and Antonio E. Teruel

Abstract. We present some results on singularly perturbed piecewise linear systems, similar to those obtained by the Geometric Singular Perturbation Theory. Unlike the differentiable case, in the piecewise linear case we obtain the global expression of the slow manifold $\mathcal{S}_\varepsilon$. As a result, we characterize the existence of canard orbits in such systems. Finally, we apply the above theory to a specific case where we show numerical evidences of the existence of a canard cycle.

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1. Introduction and main results

Singularly perturbed systems of ordinary differential equations in standard form write as

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{d\tau} = g(\mathbf{u}, \mathbf{v}, \varepsilon), \quad \varepsilon\dot{\mathbf{v}} = \varepsilon\frac{d\mathbf{v}}{d\tau} = f(\mathbf{u}, \mathbf{v}, \varepsilon), \tag{1.1}$$

where $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^s \times \mathbb{R}^q$ are the state variables, f and g are sufficiently smooth functions and $0 < \varepsilon \ll 1$ is a small parameter. From the above expression, the coordinates of $\mathbf{u}$ are called slow variables, while the coordinates of $\mathbf{v}$ are called fast variables. The variable τ is referred to as the slow time scale. Changing the time τ to the fast time scale $t = \tau/\varepsilon$, system (1.1) writes as

$$\mathbf{u}' = \frac{d\mathbf{u}}{dt} = \varepsilon g(\mathbf{u}, \mathbf{v}, \varepsilon), \quad \mathbf{v}' = \frac{d\mathbf{v}}{dt} = f(\mathbf{u}, \mathbf{v}, \varepsilon). \tag{1.2}$$

Systems (1.1) and (1.2) are differentiable equivalent and their phase portraits are the same. It can be understood that the dynamics of both systems exhibit an slow-fast explicit splitting. In this setting, system (1.1) and (1.2) are called a slow-fast

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systems. Usually, system (1.1) is referred as the slow system while system (1.2) is called the fast system.

Fenichel's geometric theory [11] allows to analyse the dynamics of the perturbed system (1.1) by combining the behaviour of the singular orbits, corresponding to the limiting cases, given by $\varepsilon = 0$. In particular, setting $\varepsilon = 0$ in (1.1), we get the reduced problem

$$\dot{\mathbf{u}} = g(\mathbf{u}, \mathbf{v}, 0), \quad \mathbf{0} = f(\mathbf{u}, \mathbf{v}, 0), \tag{1.3}$$

and, analogously in (1.2), the layer problem

$$\mathbf{u}' = \mathbf{0}, \quad \mathbf{v}' = f(\mathbf{u}, \mathbf{v}, 0). \tag{1.4}$$

The reduced problem is a s-dimensional vector field defined on the set $\mathcal{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{s+q} \mid f(\mathbf{u}, \mathbf{v}, 0) = \mathbf{0}\}$, which is assumed to be an s-dimensional manifold. In what respects to the layer problem, the manifold $\mathcal{S}$ is fulfilled by singular points. We call normally hyperbolic to the singular points $(\mathbf{u}_0, \mathbf{v}_0) \in \mathcal{S}$ for which the eigenvalues of the Jacobian matrix $D_{\mathbf{v}} f(\mathbf{u}_0, \mathbf{v}_0, 0)$ have nonzero real part.

Consider $\mathcal{S}_0 \subset \mathcal{S}$ a compact set such that every point in $\mathcal{S}_0$ is a normally hyperbolic singular point. Such an invariant manifold of equilibria of the layer problem (1.4) is called the critical manifold. From Fenichel's Theorems [11], the critical manifold persists as a locally invariant slow manifold, $\mathcal{S}_\varepsilon$, of the perturbed system (1.1) for ε small enough. Moreover, the restriction of the flow of the perturbed system (1.1) to the slow manifold $\mathcal{S}_\varepsilon$ is a small smooth perturbation of the flow of the reduced problem (1.3). Also it is proved that there exists a stable and an unstable invariant foliation with base $\mathcal{S}_\varepsilon$ with the dynamics along each foliation been a small smooth perturbation of the flow of the layer problem. See also the survey of Jones, [16], for an exposition of a geometric approach to singular perturbation theory.

Roughly speaking, orbits of the perturbed system (1.1) are piecewise composed. Some of these pieces are close to the flow of the reduced problem, while the rest are close to the flow of the layer problem.

A general question is, what remains of this dynamical behaviour when normal hyperbolicity is lost? v.g., how the flow behave in a neighbourhood of a point $(\mathbf{u}_0, \mathbf{v}_0) \in \mathcal{S}$ such that the determinant of the Jacobian matrix $D_{\mathbf{v}} f(\mathbf{u}_0, \mathbf{v}_0, 0)$ is equal zero? Several works are devoted to this subject and different tools and approaches are used. For instance, we refer the reader to the works of Benoît, [2], Dumortier and Roussarie, [9], and Kupra and Szmolyan, [17], [18].

Related to the lost of normal hyperbolicity is the appearance of relaxation oscillation periodic orbits and canard orbits. A canard orbit is a solution of the singularly perturbed system following an attracting branch of the slow manifold, $\mathcal{S}_\varepsilon$, passing close to a non-normally hyperbolic point of the manifold $\mathcal{S}$ and then following a repelling branch of the slow manifold. The analysis of canard orbits can be performed from the study of the linearized system in a neighbourhood of the so called folded singular points [8, 24, 25, 26]. Since systems that we deal with are piecewise linear, we would emphasize that the presence of these points will be not required in the present work. All the needed information for such analysis is obtained from the eigenvalues of the involved matrices.

Then, some of the solutions of the slow-fast systems consist of a mixture of long periods of small changes interspersed by short periods of sudden changes. This mixed dynamical behaviour appears quite naturally in many applications. We refer the introduction of the works [6, 8] for some additional references. In particular in neuroscience this phenomenon can be found related with some models of neurons activity, see [10] for instance. One of these models approaches the bursting activity of spiking neurons. For a detailed exposition on this subject see [14, §9] and the references therein.

Singularly perturbed three dimensional differential systems (1.1) where the slow and fast dynamics have dimension $s = 1$ and $q = 2$, respectively, usually appear in applications. See, for instance, the Hindmarsh-Rose model of bursting neurons [13], the three dimensional Volterra-Gause model of predator-prey model type [12], and [19] for a physical model. See also the works [22, 23].

The model of the self-coupled FitzHugh-Nagumo system, [7], the 3D Hodgk-inHuxley model, [21, 25] and the stellate cell model [26], for instance, are applications of singularly perturbed three dimensional systems where the slow dynamics is two dimensional while the fast dynamic is one dimensional, i.e. $s = 2$ and $q = 1$.

It is worthwhile to observe that most of the works assume smoothness on the manifold $\mathcal{S}$. A question that arise in this setting is, what remains of previous dynamical behaviour when smoothness is no longer present? Under suitable assumptions, in [20] the authors prove the existence of canard cycles in singularly perturbed piecewise differential systems with $s = 2$ and $q = 1$. This fact suggests that canards are not exclusively a differential phenomenon, but rather a geometric one.

In this paper we consider singularly perturbed 3–dimensional piecewise linear differential systems. We use this approach because, there are many works in which versions of piecewise linear differential systems are able to reproduce the dynamical behaviour exhibited by general nonlinear systems. For example, a piecewise linear version of the Michelson system reproduces global dynamic behaviours, [3, 4], as well as bifurcations, [5], that are characteristic of the Michelson system. In particular, we deal with the singularly perturbed piecewise linear differential system

$$\left\{\begin{array}{l} u_1' = \varepsilon(a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1), \\ u_2' = \varepsilon(a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2), \\ v' = u_1 + |v| \end{array}\right. \tag{1.5}$$

where $0 < \varepsilon \ll 1$ and $a_{12} \neq 0$.

The flow of the system (1.5) is formed by the composition of two linear flows, each defined into a half–space, $\{v \geq 0\}$ or $\{v \leq 0\}$. In spite of the fact that the vector field is not differentiable, the flow defined by (1.5) is smooth, even when the orbits cross the common boundary $\{v = 0\}$.

The manifold $\mathcal{S} = \{(u_1, u_2, v) : u_1 + |v| = 0\}$ is made from the union of the two half–planes

$$\begin{aligned} \mathcal{S}^+ &= \{(u_1, u_2, v) : v \geq 0,\, u_1 + v = 0\}, \\ \mathcal{S}^- &= \{(u_1, u_2, v) : v \leq 0,\, u_1 - v = 0\}, \end{aligned} \tag{1.6}$$

which intersect along the folded line $\mathcal{F} = \{(0, u_2, 0) : u_2 \in \mathbb{R}\}$, see Figure 1(a).

As we will see, points in the manifold $\mathcal{S}$, except those contained in the folded line $\mathcal{F}$, are normally hyperbolic singular points of the layer problem. Thus, Fenichel's theory locally applies to this system. Therefore, under the flow of system (1.5), open half–planes $\mathcal{S}^+ \cap \{v > 0\}$ and $\mathcal{S}^- \cap \{v < 0\}$ persist as locally invariant manifolds.

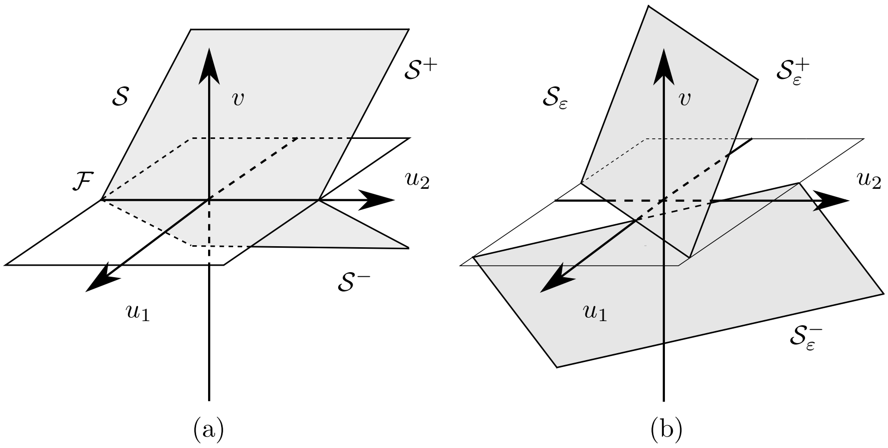

FIGURE 1. (a) Manifold $\mathcal{S}$ made from the union of the half–planes $\mathcal{S}^+$ and $\mathcal{S}^-$, which intersect along the folded line $\mathcal{F}$. (b) The slow manifold $\mathcal{S}_\varepsilon$ for $\varepsilon > 0$ which is the union of the half–planes $\mathcal{S}_\varepsilon^+$ and $\mathcal{S}_\varepsilon^-$.

In next result we claim that these perturbed manifolds are in fact half–planes, denoted by $\mathcal{S}_\varepsilon^+$ and $\mathcal{S}_\varepsilon^-$, also defined on $\{v = 0\}$. Therefore, the slow manifold $\mathcal{S}_\varepsilon$ is the union of these two half–planes, see Figure 1(b). In this result we also give a description of both, the flow defined over $\mathcal{S}_\varepsilon$ and the flow surrounding it. Before present it we introduce some preliminary notation. Set

$$\begin{aligned} d_1 &= a_{11}a_{22} - a_{12}a_{21}, & t_1 &= a_{11} + a_{22}, \\ d_2 &= a_{12}a_{23} - a_{13}a_{21}, & \Delta_1 &= (t_1 - a_{13})^2 - 4(d_1 + d_2), \\ d_3 &= b_1 a_{22} - b_2 a_{12}, & \Delta_2 &= (t_1 + a_{13})^2 - 4(d_1 - d_2). \end{aligned} \tag{1.7}$$

As usual, $d(A, B)$ denotes the distance between two sets A and B in $\mathbb{R}^n$, and $\varphi(t; \mathbf{p})$ denotes the solution of the initial value problem given by the differential system (1.5) and the initial condition $\mathbf{p} \in \mathbb{R}^3$.

Theorem 1.1. *Suppose that* $\Delta_1 \neq 0$ *and* $\Delta_2 \neq 0$. *For* $\varepsilon > 0$ *there exist two real values* $\lambda_1^+ = 1 + a_{13}\varepsilon + O(\varepsilon^2)$ *and* $\lambda_1^- = -1 + a_{13}\varepsilon + O(\varepsilon^2)$, *and two half–planes*

$$\mathcal{S}_\varepsilon^+ = \Big\{ (u_1, u_2, v) \in \mathbb{R}^3 \ : v \geq 0, \left(\lambda_1^+ - \varepsilon a_{22}\right) u_1 + \varepsilon a_{12} u_2 + \left((\lambda_1^+)^2 - \varepsilon t_1 \lambda_1^+ + \varepsilon^2 d_1\right) v = -b_1\varepsilon + \frac{d_3}{\lambda_1^+}\varepsilon^2 \Big\},$$

$$\mathcal{S}_\varepsilon^- = \Big\{ (u_1, u_2, v) \in \mathbb{R}^3 \ : v \leq 0, \left(\lambda_1^- - \varepsilon a_{22}\right) u_1 + \varepsilon a_{12} u_2 + \left((\lambda_1^-)^2 - \varepsilon t_1 \lambda_1^- + \varepsilon^2 d_1\right) v = -b_1\varepsilon + \frac{d_3}{\lambda_1^-}\varepsilon^2 \Big\}$$

such that the manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$ *satisfies next properties.*

a) *The manifold* $\mathcal{S}_\varepsilon$ *is locally invariant by the flow of* (1.5).
b) *If* $\mathcal{S}_0$ *is a compact subset of* $\mathcal{S}$, *then* $d(\mathcal{S}_0, \mathcal{S}_\varepsilon) = O(\varepsilon)$.
c) *The flow over* $\mathcal{S}_\varepsilon$ *defined by* (1.5) *is a regular perturbation of the reduced flow over* $\mathcal{S}$.
d) *If* $\mathbf{p} \in \{v > 0\}$, *then there exists* $t_0 > 0$ *such that for* $t \in (-t_0, t_0)$

$$d(\varphi(t; \mathbf{p}), \mathcal{S}_\varepsilon) = d(\mathbf{p}, \mathcal{S}_\varepsilon) e^{\lambda_1^+ t}.$$

e) *If* $\mathbf{p} \in \{v < 0\}$, *then there exists* $t_0 > 0$ *such that for* $t \in (-t_0, t_0)$

$$d(\varphi(t; \mathbf{p}), \mathcal{S}_\varepsilon) = d(\mathbf{p}, \mathcal{S}_\varepsilon) e^{\lambda_1^- t}.$$

We printout that the manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$ defined in Theorem 1.1 satisfies the same properties to those of Fenichel's theory for smooth vector fields [11, 16]. Moreover, from Theorem 1.1(d) the locally invariant half–plane $\mathcal{S}_\varepsilon^+ \cap \{v > 0\}$ is asymptotically unstable. In fact, while the orbit through a point $\mathbf{p} \in \{v > 0\}$ remains in the positive half-space, it moves away from $\mathcal{S}_\varepsilon^+$ with an exponential rate. Similarly, from Theorem 1.1(e) the locally invariant half–plane $\mathcal{S}_\varepsilon^- \cap \{v < 0\}$ is exponentially stable.

Thus points contained in the intersection $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^- \cap \{v = 0\}$ correspond to orbits passing from the stable branch to the unstable branch of the slow manifold $\mathcal{S}_\varepsilon$, or vice versa. In the first case the orbit is called a primary canard, in the second case it is called a faux–canard.

Next theorem establishes necessary and sufficient conditions on singularly perturbed piecewise linear systems (1.5), for the existence of primary canards.

Theorem 1.2. *Let* $\epsilon > 0$ *and suppose that* $\Delta_1 \neq 0$ *and* $\Delta_2 \neq 0$.

a) *The set* $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^- \cap \{v = 0\}$ *contains a unique point*

$$\mathbf{p}_c = \left(-\frac{d_3}{\lambda_1^+ \lambda_1^-}\varepsilon^2, -\frac{b_1}{a_{12}} + \frac{d_3}{\lambda_1^+ \lambda_1^- a_{12}} (\lambda_1^+ + \lambda_1^- - \varepsilon a_{22})\varepsilon, 0 \right).$$

b) *If* $d_3 > 0$, *then the orbit through* $\mathbf{p}_c$ *is a primary canard.*
c) *If* $d_3 < 0$, *then the orbit through* $\mathbf{p}_c$ *is a faux–canard.*

The rest of the paper is organized in three sections. In Section 2 we deal the dynamic behaviour of the unperturbed systems associated to (1.5), that is, we describe both the layer system and the reduced one. In Section 3 we analyse the singularly perturbed systems (1.5) for $\varepsilon > 0$ and prove Theorem 1.1 and Theorem 1.2. In Section 4 we show an example of a canard orbit in a piecewise linear differential system. Furthermore, by adding suitable new linear regions, we get numerical evidences that allow to conclude that the canard orbit closes to form a periodic orbit, which is called canard cycle.

2. Unperturbed systems

The section is organized in two parts. In the first one we analyse the flow of the layer problem associated to the singularly perturbed system (1.5). In the second part we treat the reduced problem.

By setting $\varepsilon = 0$ in system (1.5), we get the layer system

$$\left\{\begin{array}{l} u_1' = 0, \\ u_2' = 0, \\ v' = u_1 + |v|. \end{array}\right. \tag{2.1}$$

The flow defined by this system is very simple. In fact, orbits are contained in vertical lines and singular points completely fill the piecewise linear manifold $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$, defined in (1.6), see Figure 1(a).

Since layer vector field is locally linear, the spectrum of the Jacobian matrix at any singular point $\mathbf{p} \in \mathcal{S} \cap \{v \neq 0\}$, is $\{0, 0, 1\}$ or $\{0, 0, -1\}$ depending on $\mathbf{p} \in \mathcal{S}^+ \cap \{v > 0\}$ or $\mathbf{p} \in \mathcal{S}^- \cap \{v < 0\}$, respectively. Then $\mathcal{S}^+ \cap \{v > 0\}$ is a repelling normally hyperbolic manifold and $\mathcal{S}^- \cap \{v < 0\}$ is an attracting normally hyperbolic manifold. Singular points on the folded line $\mathcal{F}$ have not defined a Jacobian matrix, so that they are no normally hyperbolic singular points. The local flow surrounding $\mathcal{F}$ follows from (2.1) by noting that $v' > 0$ over the plane $\{u_1 = 0\}$. Then, straight line $\mathcal{F}$ attracts orbits in $\{v < 0\}$ and repels orbits in $\{v > 0\}$, see Figure 2.

Now, we continue by considering the reduced system

$$\left\{\begin{array}{l} \dot{u}_1 = a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1, \\ \dot{u}_2 = a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2, \\ 0 = u_1 + |v|, \end{array}\right. \tag{2.2}$$

which is defined on the manifold $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$. From the last equation in (2.2) and by taking the derivative when $v \neq 0$ we have

$$\dot{v} = -\frac{|v|}{v}\dot{u}_1.$$

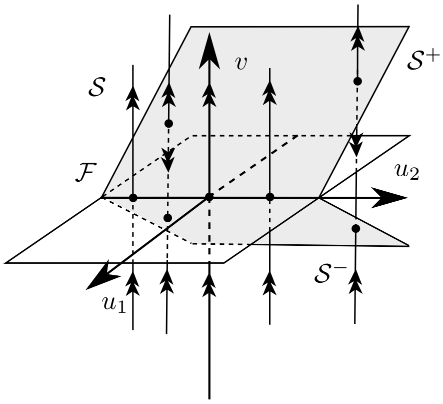

FIGURE 2. Representation of the flow of the layer equation, attracting and repelling normally hyperbolic half–planes $\mathcal{S}^-$ and $\mathcal{S}^+$, and the folded line $\mathcal{F}$.

Thus, the vector field defined by the reduced system on the submanifold $\mathcal{S} \setminus \mathcal{F}$ is given by the piecewise linear function

$$\mathbf{F}(u_1, u_2, v) = \begin{cases} \mathbf{F}^+(u_1, u_2, v) & \text{if } v > 0, \\ \mathbf{F}^-(u_1, u_2, v) & \text{if } v < 0, \end{cases}$$

where

$$\mathbf{F}^+(u_1, u_2, v) = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1 \\ a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2 \\ -a_{11}u_1 - a_{12}u_2 - a_{13}v - b_1 \end{pmatrix}$$

and

$$\mathbf{F}^-(u_1, u_2, v) = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1 \\ a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2 \\ a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1 \end{pmatrix}.$$

The projection map $\pi(u_1, u_2, v) = (u_2, v)$ induces on $\mathbb{R}^2 \setminus \{(u_2, 0) : u_2 \in \mathbb{R}\}$ the discontinuous planar piecewise linear differential system

$$\begin{pmatrix} \dot{u}_2 \\ \dot{v} \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} u_2 \\ v \end{pmatrix} + \mathbf{b}^+ & \text{if } v > 0, \\ \\ A^- \begin{pmatrix} u_2 \\ v \end{pmatrix} + \mathbf{b}^- & \text{if } v < 0, \end{cases} \tag{2.3}$$

where

$$A^+ = \begin{pmatrix} a_{22} & a_{23} - a_{21} \\ -a_{12} & a_{11} - a_{13} \end{pmatrix}, \quad \mathbf{b}^+ = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix},$$

$$A^- = \begin{pmatrix} a_{22} & a_{23} + a_{21} \\ a_{12} & a_{11} + a_{13} \end{pmatrix}, \quad \mathbf{b}^- = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}. \tag{2.4}$$

The flows defined by the systems (2.2) and (2.3) are conjugate in $\mathcal{S} \setminus \mathcal{F}$ and in $\mathbb{R}^2 \setminus \{(u_2, 0) : u_2 \in \mathbb{R}\}$, respectively. On the folded line $\mathcal{F}$, the flow can be obtained by the Filippov extension of the system (2.3) to the boundary $\{v = 0\}$, see [15]; that is $\dot{u}_2 = a_{22}u_2 + b_2, \ \dot{v} = 0$. When $a_{22} \neq 0$ a singular point $\mathbf{e} = (-b_2/a_{22}, 0)$ appears in $\mathcal{F}$. Singular point $\mathbf{e}$ is called a folded singular point.

In differential singularly perturbed systems, folded singular points play an important role in the study of canard trajectories, see [17, 18, 24]. However, for singularly perturbed piecewise linear differential systems its not needed the presence of folded singular points. This fact follows straightforward since the dynamic behaviour of the flow of the system (2.3)

$$\Phi(\tau; \mathbf{p}) = e^{A^{\pm}\tau}\mathbf{p} + \int_0^{\tau} e^{A^{\pm}(\tau - s)}\mathbf{b}^{\pm} ds$$

can be derived from the eigenvalues and the eigenvectors of the matrices A^+ and A^-.

Direct computations show that the eigenvalues β^+ and γ^+ of the matrix A^+, and the eigenvalues β^- and γ^- of the matrix A^- can be written in terms of the values defined in (1.7) as

$$\begin{array}{ll} \beta^+ + \gamma^+ = t_1 - a_{13}, & \beta^+\gamma^+ = d_1 + d_2, \\ \beta^- + \gamma^- = t_1 + a_{13}, & \beta^-\gamma^- = d_1 - d_2. \end{array} \tag{2.5}$$

Therefore if $\Delta_1 \neq 0$, then $\beta^+ \neq \gamma^+$ and the associated eigenvectors are

$$\left(\frac{a_{11} - a_{13} - \beta^+}{a_{12}}, 1 \right), \quad \left(\frac{a_{11} - a_{13} - \gamma^+}{a_{12}}, 1 \right), \tag{2.6}$$

respectively. Similarly if $\Delta_2 \neq 0$, then $\beta^- \neq \gamma^-$ and the corresponding eigenvectors are

$$\left(\frac{-a_{11} - a_{13} + \beta^-}{a_{12}}, 1 \right), \quad \left(\frac{-a_{11} - a_{13} + \gamma^-}{a_{12}}, 1 \right). \tag{2.7}$$

3. Perturbed system

In this section we gives some lemmas to analyse the flow of the perturbed system (1.5) with $\varepsilon > 0$. Moreover, we relate this flow with the flow of the layer problem (2.1) and with the flow of the reduced problem (2.2). At the end of the section we will use this relationship to prove Theorems 1.1 and 1.2.

The flow of (1.5) is defined by the composition of the two linear flows which are associated to the differential system

$$\mathbf{x}' = \begin{cases} A_\varepsilon^+\mathbf{x} + \mathbf{b}_\varepsilon & \text{if } v \geq 0, \\ \\ A_\varepsilon^-\mathbf{x} + \mathbf{b}_\varepsilon & \text{if } v \leq 0, \end{cases} \tag{3.1}$$

where $\mathbf{x} = (u_1, u_2, v)^T$ and

$$A_\varepsilon^\pm = \begin{pmatrix} \varepsilon a_{11} & \varepsilon a_{12} & \varepsilon a_{13} \\ \varepsilon a_{21} & \varepsilon a_{22} & \varepsilon a_{23} \\ 1 & 0 & \pm 1 \end{pmatrix}, \quad \mathbf{b}_\varepsilon = \begin{pmatrix} \varepsilon b_1 \\ \varepsilon b_2 \\ 0 \end{pmatrix}.$$

Hence, locally the flow of (3.1) can be derived from the analysis of the eigenvalues and the eigenvectors of both linear systems. Let us start with the linear system $\mathbf{x}' = A_\varepsilon^+\mathbf{x} + \mathbf{b}_\varepsilon$ defined in the half–space $\{v \geq 0\}$.

Lemma 3.1. *For $\varepsilon > 0$ the eigenvalues of the matrix A_ε^+ expand in power series in ε as*

$$\begin{aligned} \lambda_1^+ &= 1 + a_{13}\varepsilon + O(\varepsilon^2), \\ \lambda_2^+ &= \beta^+\varepsilon + O(\varepsilon^2), \\ \lambda_3^+ &= \gamma^+\varepsilon + O(\varepsilon^2), \end{aligned}$$

where β^+ and γ^+ are the eigenvalues of the matrix A^+ in (2.4)*. Moreover, assuming that $\Delta_1 \neq 0$, the eigenvalues are different for ε small enough and the associated eigenvectors satisfy that*

$$\mathbf{v}_1^+ = \begin{pmatrix} 0 + O(\varepsilon) \\ 0 + O(\varepsilon) \\ 1 \end{pmatrix},$$

$$\mathbf{v}_2^+ = \begin{pmatrix} -1 + O(\varepsilon) \\ \dfrac{a_{11} - a_{13} - \beta^+}{a_{12}} + O(\varepsilon) \\ 1 \end{pmatrix}, \quad \mathbf{v}_3^+ = \begin{pmatrix} -1 + O(\varepsilon) \\ \dfrac{a_{11} - a_{13} - \gamma^+}{a_{12}} + O(\varepsilon) \\ 1 \end{pmatrix},$$

respectively.

Proof. Since the characteristic polynomial of the matrix A_ε^+

$$\lambda^3 - (1 + \varepsilon t_1)\lambda^2 + \varepsilon(t_1 - a_{13} + \varepsilon d_1)\lambda - \varepsilon^2(d_1 + d_2) = 0, \tag{3.2}$$

tends to $\lambda^3 - \lambda^2 = 0$ as ε tends to zero, we conclude that the eigenvalues of the matrix A_ε^+ can be expanded in power series in ε as

$$\begin{aligned} \lambda_1^+ &= 1 + \alpha\varepsilon + O(\varepsilon^2), \\ \lambda_2^+ &= \beta\varepsilon + O(\varepsilon^2), \\ \lambda_3^+ &= \gamma\varepsilon + O(\varepsilon^2), \end{aligned} \tag{3.3}$$

where α is a real number and β, γ are real or complex.

From (3.2) we obtain that

$$\begin{aligned} \lambda_1^+ + \lambda_2^+ + \lambda_3^+ &= 1 + \varepsilon t_1, \\ \lambda_1^+\lambda_2^+ + \lambda_1^+\lambda_3^+ + \lambda_2^+\lambda_3^+ &= \varepsilon(t_1 - a_{13} + \varepsilon d_1), \\ \lambda_1^+\lambda_2^+\lambda_3^+ &= \varepsilon^2(d_1 + d_2). \end{aligned} \tag{3.4}$$

Then, from (3.3) and (3.4) its follows that

$$\begin{aligned} \alpha &= a_{13}, \\ \beta + \gamma &= t_1 - a_{13}, \\ \beta\gamma &= d_1 + d_2. \end{aligned} \tag{3.5}$$

Note that β and γ satisfy the same equations that β^+ and γ^+ in (2.5). Then $\beta = \beta^+$ and $\gamma = \gamma^+$. Moreover, since we assume that $\Delta_1 \neq 0$ it follows that $\beta^+ \neq \gamma^+$.

On the other hand, straightforward computations show that the eigenvector associated to any eigenvalue λ is

$$\begin{pmatrix} \lambda - 1 \\ \dfrac{(\varepsilon a_{11} - \lambda)(1-\lambda) - \varepsilon a_{13}}{\varepsilon a_{12}} \\ 1 \end{pmatrix}.$$

The lemma follows from this and from (3.3), by taking into account that $\alpha = a_{13}, \beta = \beta^+$ and $\gamma = \gamma^+$. □

From Lemma 3.1, we emphasize that the spectrum of the matrix A_ε^+ decomposes into two parts. One consisting on the eigenvalue λ_1^+, which is responsible for the fast dynamic in $\{v \geq 0\}$ when ε tends to zero. The other, is formed by the eigenvalues λ_2^+ and λ_3^+, which tend to zero as ε tends to zero. Now we will see that these eigenvalues are responsible for the slow dynamic in $\{v \geq 0\}$.

Let $\mathbf{w}$ be the eigenvector associated to the eigenvalue λ_1^+ of the matrix $(A_\varepsilon^+)^T$, where superscript T stands for the transpose. Then it follows that

$$\mathbf{w}^T A_\varepsilon^+ = \lambda_1^+ \mathbf{w}^T. \tag{3.6}$$

Since $\lambda_1^+ \neq \lambda_2^+$ and $\lambda_1^+ \neq \lambda_3^+$, from (3.6) it is easy to check that $\mathbf{w}$ is orthogonal to the eigenvectors $\mathbf{v}_2^+$ and $\mathbf{v}_3^+$.

Let Π^+ be the linear span of the eigenvectors $\mathbf{v}_2^+$ and $\mathbf{v}_3^+$, and consider the sets $\mathcal{P}_\varepsilon^+ = \{\mathbf{p} \in \mathbb{R}^3 : A_\varepsilon^+\mathbf{p} + \mathbf{b}_\varepsilon \in \Pi^+\}$ and $\mathcal{S}_\varepsilon^+ = \mathcal{P}_\varepsilon^+ \cap \{v \geq 0\}$.

Lemma 3.2. a) *The set $\mathcal{P}_\varepsilon^+$ is a plane, invariant by the flow of the linear system $\mathbf{x}' = A_\varepsilon^+\mathbf{x} + \mathbf{b}_\varepsilon$.*

b) *The set $\mathcal{S}_\varepsilon^+$ is a half–plane, locally invariant by the flow of* (3.1). *Moreover, points $(u_1, u_2, v) \in \mathcal{S}_\varepsilon^+$ satisfy that $v \geq 0$ and*

$$(\lambda_1^+ - \varepsilon a_{22})u_1 + \varepsilon a_{12}u_2 + \left((\lambda_1^+)^2 - \varepsilon t_1\lambda_1^+ + \varepsilon^2 d_1\right)v = -\varepsilon b_1 + \varepsilon^2\frac{d_3}{\lambda_1^+}.$$

c) *Let* $\mathcal{S}_0^+$ *be a compact subset of* $\mathcal{S}^+$, *then* $d(S_\varepsilon^+, S_0^+) = O(\varepsilon)$.

Proof. Since $\mathbf{w}$ is orthogonal to Π^+, we can write

$$\mathcal{P}_\varepsilon^+ = \left\{\mathbf{p} \in \mathbf{R}^3 : \mathbf{w}^T (A_\varepsilon^+ \mathbf{p} + \mathbf{b}_\varepsilon) = 0\right\} = \left\{\mathbf{p} \in \mathbf{R}^3 : \mathbf{w}^T \mathbf{p} = -\frac{\mathbf{w}^T \mathbf{b}_\varepsilon}{\lambda_1^+}\right\}.$$

Hence $\mathcal{P}_\varepsilon^+$ is a plane orthogonal to the vector $\mathbf{w}$. Therefore, $\mathcal{P}_\varepsilon^+$ is parallel to Π^+. We conclude that $\mathcal{P}_\varepsilon^+$ is invariant by the flow of the linear system. This proves the statement (a) of the lemma.

By solving $\mathbf{w}$ from (3.6) we have

$$\mathbf{w} = \left(\lambda_1^+ - \varepsilon a_{22}, \varepsilon a_{12}, (\lambda_1^+)^2 - \varepsilon t_1 \lambda_1^+ + \varepsilon^2 d_1\right)^T,$$

where $a_{12} \neq 0$, then $\mathbf{w} \neq \mathbf{0}$. Statement (b) follows since

$$\mathbf{w}^T \mathbf{b}_\varepsilon = \varepsilon b_1 \lambda_1^+ - \varepsilon^2 d_3.$$

From statement (b) and by considering that $\lambda_1^+ = 1 + O(\varepsilon)$, if $|u_1|, |u_2|$ and $|v|$ are bounded, then points in the half–plane $\mathcal{S}_\varepsilon^+$ satisfy that $u_1 + v = O(\varepsilon)$. On the other hand, points on $\mathcal{S}^+$ satisfy that $u_1 + v = 0$, which ends the prove of the lemma. □

Hence, the half–plane $\mathcal{S}_\varepsilon^+$ is locally invariant by the flow of the perturbed system (3.1). In particular, orbits in $\mathcal{S}_\varepsilon^+$ remain in $\mathcal{S}_\varepsilon^+$ until they reach the boundary of the half–plane at $\{v = 0\}$. Next, we discuss the behaviour of the flow

$$\varphi(t; \mathbf{p}) = e^{A_\varepsilon^+ t} \mathbf{p} + \int_0^t e^{A_\varepsilon^+ (t-s)} \mathbf{b}_\varepsilon \, ds$$

defined over the locally invariant half–plane $\mathcal{S}_\varepsilon^+$ and surrounding it.

Lemma 3.3. *Let* $\varphi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ *be the flow defined by the system* (3.1).

a) *If* $\mathbf{q} \in \mathcal{S}_\varepsilon^+$, *then there exist* $q_2, q_3 \in \mathbb{R}$ *such that*

$$\varphi(t; \mathbf{q}) = \mathbf{q} + \frac{q_2}{\lambda_2^+}\left(e^{\lambda_2^+ t} - 1\right)\mathbf{v}_2^+ + \frac{q_3}{\lambda_3^+}\left(e^{\lambda_3^+ t} - 1\right)\mathbf{v}_3^+.$$

b) *If* $\mathbf{p} \in \mathbb{R}^3 \cap \{v > 0\}$, *then* $\mathbf{p} = \mathbf{q} + \mathbf{q}_1$ *with* $\mathbf{q} \in \mathcal{S}_\varepsilon^+$ *and* $\mathbf{q}_1 = r\mathbf{v}_1^+$, *and*

$$\varphi(t; \mathbf{p}) = \varphi(t; \mathbf{q}) + r e^{\lambda_1^+ t} \mathbf{v}_1^+.$$

c) *If* $\mathbf{p} \in \mathbb{R}^3 \cap \{v > 0\}$, *then exists* $t_0 > 0$ *such that*

$$d(\varphi(t; \mathbf{p}), \mathcal{S}_\varepsilon^+) = d(\mathbf{p}, \mathcal{S}_\varepsilon^+) e^{\lambda_1^+ t},$$

for $t \in (-t_0, t_0)$.

Proof. Suppose that $\mathbf{q} \in \mathcal{S}_\varepsilon^+$. Since $\mathcal{S}_\varepsilon^+$ is locally invariant, the orbit through $\mathbf{q}$ can be expressed as

$$\varphi(t;\mathbf{q}) = \mathbf{q} + r_2(t)\mathbf{v}_2^+ + r_3(t)\mathbf{v}_3^+,$$

with $r_2(0) = r_3(0) = 0$.

By forcing the function $\varphi(t;\mathbf{q})$ to satisfy the differential equation (3.1) we obtain

$$(\dot r_2(t) - \lambda_2^+ r_2(t) - q_2)\mathbf{v}_2^+ + (\dot r_3(t) - \lambda_3^+ r_3(t) - q_3)\mathbf{v}_3^+ = \mathbf{0},$$

providing that $A_\varepsilon^+\mathbf{q} + \mathbf{b}_\varepsilon = q_2\mathbf{v}_2^+ + q_3\mathbf{v}_3^+$. The functions $r_2(t)$ and $r_3(t)$ can be obtained by integrating the differential equations $\dot r_2(t) - \lambda_2^+ r_2(t) - q_2 = 0$ and $\dot r_3(t) - \lambda_3^+ r_3(t) - q_3 = 0$, which proves the statement (a).

Now, we analyse the behaviour of the flow surrounding the locally invariant half–plane $\mathcal{S}_\varepsilon^+$. Let $\mathbf{p}$ be a point in $\mathbb{R}^3 \cap \{v > 0\}$. Since $\mathbf{v}_1^+$ is not parallel to $\mathcal{S}_\varepsilon^+$, the point $\mathbf{p}$ can be expressed as sum of a point $\mathbf{q}$ in $\mathcal{S}_\varepsilon^+$ and a point $\mathbf{q}_1$ in the straight line $\{r\mathbf{v}_1^+ : r \in \mathbb{R}\}$, i.e. $\mathbf{p} = \mathbf{q} + \mathbf{q}_1$, with $\mathbf{q}_1 = r\mathbf{v}_1^+$. As far as $\varphi(t;\mathbf{p})$ remains in $\{v \geq 0\}$ and $\varphi(t;\mathbf{q})$ remains in $\mathcal{S}_\varepsilon^+$ it can be expressed as

$$\begin{aligned}\varphi(t;\mathbf{p}) &= e^{A_\varepsilon^+ t}\mathbf{q} + \int_0^t e^{A_\varepsilon^+(t-s)}\mathbf{b}_\varepsilon\, ds + re^{A_\varepsilon^+ t}\mathbf{v}_1^+ \\ &= \varphi(t;\mathbf{q}) + re^{\lambda_1^+ t}\mathbf{v}_1^+,\end{aligned}$$

which proves statement (b). From this we also conclude that

$$d(\varphi(t;\mathbf{p}), \mathcal{S}_\varepsilon^+) = d(\mathbf{p}, \mathcal{S}_\varepsilon^+)e^{\lambda_1^+ t},$$

which ends the proof of the lemma. □

Statement (a) in Lemma 3.3 shows that, while orbits remain in $\mathcal{S}_\varepsilon^+$, their behaviours are determined by the eigenvalues λ_2^+ and λ_3^+, and the eigenvectors $\mathbf{v}_2^+$ and $\mathbf{v}_3^+$. Since these eigenvalues tend to zero with ε, the half–plane $\mathcal{S}_\varepsilon^+$ is part of the slow manifold of the perturbed system (3.1). Next lemma relates the flow of the reduced system (2.3) and the flow of the perturbed system on $\mathcal{S}_\varepsilon^+$.

Lemma 3.4. *The flow of the perturbed system* (3.1) *on the locally invariant half–plane $\mathcal{S}_\varepsilon^+$ is a regular perturbation of the flow of the reduced system* (2.2) *on $\mathcal{S}^+$.*

Proof. By rescaling the time variable, i.e. by dividing by ε, the eigenvalues λ_2^+ and λ_3^+ tend to the eigenvalues of the reduced system (2.3) as ε tends to zero, see Lemma 3.1. Moreover, the eigenvectors $\mathbf{v}_2^+$ and $\mathbf{v}_3^+$, after projecting by π, also tend to the eigenvectors of the reduced system as ε tends to zero, see (2.6). This proves the lemma. □

With respect to the dynamic behaviour of the perturbed system (3.1) in the half–space $\{v \leq 0\}$, similar results to those appearing in Lemmas 3.1, 3.2, 3.3 and 3.4 can be established. We omit the proofs of these results because they follow by using similar arguments.

Lemma 3.5. *For $\varepsilon > 0$ the eigenvalues of the matrix A_ε^- expand in power series in ε as*

$$\lambda_1^- = -1 + a_{13}\varepsilon + O(\varepsilon^2),$$
$$\lambda_2^- = \beta^-\varepsilon + O(\varepsilon^2),$$
$$\lambda_3^- = \gamma^-\varepsilon + O(\varepsilon^2),$$

where β^- and γ^- are the eigenvalues of the matrix A^- in (2.4)*. Moreover, assuming that $\Delta_1 \neq 0$, the eigenvalues are different for ε small enough and the associated eigenvectors satisfy that*

$$\mathbf{v}_1^- = \begin{pmatrix} 0 + O(\varepsilon) \\ 0 + O(\varepsilon) \\ 1 \end{pmatrix},$$

$$\mathbf{v}_2^- = \begin{pmatrix} -1 & + O(\varepsilon) \\ \dfrac{a_{11} - a_{13} + \beta^-}{a_{12}} & + O(\varepsilon) \\ 1 & \end{pmatrix}, \quad \mathbf{v}_3^- = \begin{pmatrix} -1 & + O(\varepsilon) \\ \dfrac{a_{11} - a_{13} + \gamma^-}{a_{12}} & + O(\varepsilon) \\ 1 & \end{pmatrix},$$

respectively.

Let Π^- be the linear span of the eigenvectors $\mathbf{v}_2^-$ and $\mathbf{v}_3^-$, and consider the sets $\mathcal{P}_\varepsilon^- = \{\mathbf{p} \in \mathbb{R}^3 : A_\varepsilon^-\mathbf{p} + \mathbf{b}_\varepsilon \in \Pi^-\}$ and $\mathcal{S}_\varepsilon^- = \mathcal{P}_\varepsilon^- \cap \{v \leq 0\}$.

Lemma 3.6. a) *The set $\mathcal{P}_\varepsilon^-$ is a plane, invariant by the flow of the linear system $\mathbf{x}' = A_\varepsilon^-\mathbf{x} + \mathbf{b}_\varepsilon$.*

b) *The set $\mathcal{S}_\varepsilon^-$ is a half–plane, locally invariant by the flow of* (3.1)*. Moreover, points $(u_1, u_2, v) \in \mathcal{S}_\varepsilon^-$ satisfy that $v \geq 0$ and*

$$(\lambda_1^- - \varepsilon a_{22})u_1 + \varepsilon a_{12}u_2 + \left((\lambda_1^-)^2 - \varepsilon t_1\lambda_1^- + \varepsilon^2 d_1\right) v = -\varepsilon b_1 + \varepsilon^2 \frac{d_3}{\lambda_1^-}.$$

c) *Let $\mathcal{S}_0^-$ be a compact subset of $\mathcal{S}^-$, then $d(S_\varepsilon^-, S_0^-) = O(\varepsilon)$.*

Lemma 3.7. *Let $\varphi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the flow defined by the system* (3.1)*.*

a) *If $\mathbf{q} \in \mathcal{S}_\varepsilon^-$, then there exist $q_2, q_3 \in \mathbb{R}$ such that*

$$\varphi(t; \mathbf{q}) = \mathbf{q} + \frac{q_2}{\lambda_2^-}\left(e^{\lambda_2^- t} - 1\right)\mathbf{v}_2^- + \frac{q_3}{\lambda_3^+}\left(e^{\lambda_3^- t} - 1\right)\mathbf{v}_3^-.$$

b) *If $\mathbf{p} \in \mathbb{R}^3 \cap \{v < 0\}$, then $\mathbf{p} = \mathbf{q} + \mathbf{q}_1$ with $\mathbf{q} \in \mathcal{S}_\varepsilon^-$ and $\mathbf{q}_1 = r\mathbf{v}_1^-$, and*

$$\varphi(t; \mathbf{p}) = \varphi(t; \mathbf{q}) + re^{\lambda_1^- t}\mathbf{v}_1^-.$$

c) *If $\mathbf{p} \in \mathbb{R}^3 \cap \{v < 0\}$, then exists $t_0 > 0$ such that*

$$d(\varphi(t; \mathbf{p}), \mathcal{S}_\varepsilon^-) = d(\mathbf{p}, \mathcal{S}_\varepsilon^-)e^{\lambda_1^- t},$$

for $t \in (-t_0, t_0)$.

Lemma 3.8. *The flow of the perturbed system* (3.1) *on the locally invariant half–plane $\mathcal{S}_\varepsilon^-$ is a regular perturbation of the flow of the reduced system* (2.2) *on $\mathcal{S}^-$.*

Proof of Theorem 1.1

Theorem 1.1 is a direct consequence of all the previous lemmas.

Proof of Theorem 1.2

Consider the intersection of the locally invariant half–planes $\mathcal{S}_\varepsilon^+$ and $\mathcal{S}_\varepsilon^-$ and the plane $\{v=0\}$. From Lemmas 3.2(b) and 3.6(b), the intersection points satisfy the following system of linear equations

$$
\begin{aligned}
(\lambda_1^+ - \varepsilon a_{22})u_1 + \varepsilon a_{12}u_2 &= -b_1\varepsilon + \frac{d_3}{\lambda_1^+}\varepsilon^2 \\
(\lambda_1^- - \varepsilon a_{22})u_1 + \varepsilon a_{12}u_2 &= -b_1\varepsilon + \frac{d_3}{\lambda_1^-}\varepsilon^2
\end{aligned}
\tag{3.7}
$$

whose determinant is $\left(\lambda_1^+ - \lambda_1^-\right)\varepsilon a_{12} \neq 0$.

By solving (3.7), we obtain that the flow passes from $\mathcal{S}_\varepsilon^+$ to $\mathcal{S}_\varepsilon^-$, or vice versa, through the point

$$
\mathbf{p}_c = \left(-\frac{d_3}{\lambda_1^+\lambda_1^-}\varepsilon^2, -\frac{b_1}{a_{12}} + \frac{d_3}{\lambda_1^+\lambda_1^- a_{12}}(\lambda_1^+ + \lambda_1^- - \varepsilon a_{22})\varepsilon, 0\right).
$$

The direction of the flow can be obtained from the sign of the third component of the vector field on $\mathbf{p}_c$. Since it depends on the sign of the first component of $\mathbf{p}_c$, see (3.1), we conclude that if $d_3>0$, then the orbit $\gamma_{\mathbf{p}_c}$ through $\mathbf{p}_c$ goes from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^+$. Since $\mathcal{S}_\varepsilon^-$ is the stable branch of the slow manifold $\mathcal{S}_\varepsilon$ and $\mathcal{S}_\varepsilon^+$ is the unstable branch of $\mathcal{S}_\varepsilon$, the orbit $\gamma_{\mathbf{p}_c}$ is a canard. If $d_3<0$, then the orbit goes in the opposite direction, i.e. from the unstable branch $\mathcal{S}_\varepsilon^+$ to the stable branch $\mathcal{S}_\varepsilon^-$. Hence $\gamma_{\mathbf{p}_c}$ is a faux–canard, which proves the theorem.

4. Exemple of canard cycle

In this section we apply Theorem 1.1 and Theorem 1.2 to a particular family of singularly perturbed piecewise linear differential system, (1.5), given by $a_{11} = a_{13} = b_1 = 0$, $a_{12}<0$, $a_{21}=a_{22}=a_{23}=0$ and $b_2=1$, i.e.

$$
\left\{
\begin{array}{l}
u_1' = \varepsilon a_{12}u_2, \\
u_2' = \varepsilon, \\
v' = u_1 + |v|.
\end{array}
\right.
\tag{4.1}
$$

We note that system (4.1) is a piecewise linear version of the differential system $x' = -2\varepsilon\, y,\ y'=\varepsilon,\ z' = x+z^2$, which is considered in [24].

The matrices of the linear problems associated to (4.1) are

$$
A_\varepsilon^+ = \begin{pmatrix} 0 & \varepsilon a_{12} & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_\varepsilon^- = \begin{pmatrix} 0 & \varepsilon a_{12} & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},
$$

and their eigenvalues are $\lambda_1^+ = 1, \lambda_2^+ = \lambda_3^+ = 0$ and $\lambda_1^- = -1, \lambda_2^- = \lambda_3^- = 0$, respectively.

According to Theorem 1.1, the slow manifold $\mathcal{S}_\varepsilon$ of (4.1) is the union of the unstable half–plane

$$\mathcal{S}_\varepsilon^+ = \left\{(u_1,u_2,v)\in\mathbb{R}^3 \ : v\geq 0,\ \ u_1+\varepsilon a_{12}u_2+v=-\varepsilon^2 a_{12}\right\}$$

and the stable half–plane

$$\mathcal{S}_\varepsilon^- = \left\{(u_1,u_2,v)\in\mathbb{R}^3 \ : v\leq 0,\ \ -u_1+\varepsilon a_{12}u_2+v=\varepsilon^2 a_{12}\right\}.$$

These half–planes intersect at a unique point $\mathbf{p}_c = (-\varepsilon^2 a_{12},0,0)$. Since $d_3 = -a_{12} > 0$, from Theorem 1.2, the orbit $\gamma_{\mathbf{p}_c}$ through this point is a primary canard.

Let $\varphi(t;\mathbf{p})$ denote the flow of system (4.1). The expression of $\varphi(t;\mathbf{p})$ can be obtained by integrating the two linear systems associated to (4.1) and by combining the results conveniently. Hence, the first coordinates of $\varphi(t;\mathbf{p})$ can be written as

$$\begin{aligned}\varphi_1(t;\mathbf{p}) &= p_1+\varepsilon t a_{12}p_2+\frac{\varepsilon^2 t^2}{2}a_{12},\\ \varphi_2(t;\mathbf{p}) &= p_2+\varepsilon t,\end{aligned}$$

where $\mathbf{p}=(p_1,p_2,p_3)^T$. Since the plane $\{v=0\}$ separates the two half–spaces where the vector field is linear, it can not be obtained an expression of the third component of flow defined for all time. Next, we present a local expression of $\varphi_3(t;\mathbf{p})$ which is defined in a neighbourhood of the initial time $t=0$. As it is clear, the expression of $\varphi_3(t;\mathbf{p})$ depends on the sign of the third coordinate of the initial condition p_3. Moreover, when $p_3=0$ this expression depends on the direction of the flow at $\mathbf{p}$. This direction points upward when $p_1>0$, and downward when $p_1<0$, see the third equation in (4.1). Therefore, setting $\mathcal{R}_+=\{\mathbf{p}\in\mathbb{R}^3 : p_3 > 0, \text{ or } p_3=0 \text{ and } p_1>0\}$ and $\mathcal{R}_-=\{\mathbf{p}\in\mathbb{R}^3 : p_3<0, \text{ or } p_3=0 \text{ and } p_1<0\}$, it follows that: if $\mathbf{p}\in\mathcal{R}_+$, then

$$\varphi_3(t;\mathbf{p}) = \left(e^t-1\right)p_1+\varepsilon a_{12}\left(e^t-1-t\right)p_2+e^t p_3+\varepsilon^2 a_{12}\left(e^t-1-t-\frac{t^2}{2}\right),$$

for $t\in(t_{-\mathbf{p}},t_{\mathbf{p}})$; and if $\mathbf{p}\in\mathcal{R}_-$, then

$$\varphi_3(t;\mathbf{p}) = \left(1-e^{-t}\right)p_1+\varepsilon a_{12}\left(e^{-t}-1+t\right)p_2+e^{-t}p_3-\varepsilon^2 a_{12}\left(e^{-t}-1+t-\frac{t^2}{2}\right),$$

for $t\in(t_{-\mathbf{p}},t_{\mathbf{p}})$. The endpoints of the intervals of definition $t_{-\mathbf{p}}\leq 0\leq t_{\mathbf{p}}$ correspond to the time in which the solution passes through the separation plane. Assuming that one of these values does not exist, then the corresponding endpoint is infinity.

Since $\mathbf{p}_c\in\mathcal{R}_+$, next proposition is a direct consequence of the expression of the flow shown above.

Proposition 4.1. *The canard orbit* $\gamma_{\mathbf{p}_c}$ *is given by*

$$\varphi(t;\mathbf{p}_c) = \left(-\varepsilon^2 a_{12}\left(1-\frac{t^2}{2}\right),\ \varepsilon t,\ -\varepsilon^2 a_{12}t\left(1+\frac{|t|}{2}\right)\right) \quad \textit{for } t\in\mathbb{R}.$$

From Proposition 4.1 and the expression of the slow manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$, it is easy to check that the canard orbit $\gamma_{\mathbf{p}_c}$ remains in $\mathcal{S}_\varepsilon^+$ for $t > 0$ and in $\mathcal{S}_\varepsilon^-$ for $t < 0$. In Figure 3 we represent the canard orbit $\gamma_{\mathbf{p}_c}$ for the parameters $a_{12} = -1.3$ and $\varepsilon = 1e - 1$.

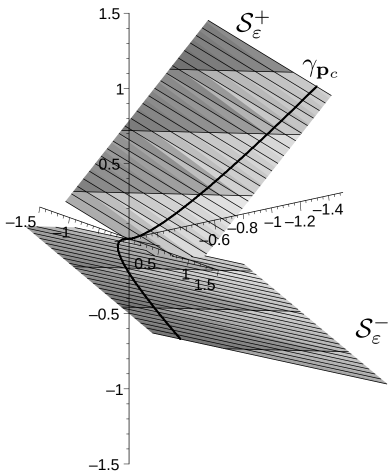

FIGURE 3. Canard orbit $\gamma_{\mathbf{p}_c}$ crossing from the stable half–plane $\mathcal{S}_\varepsilon^-$ through the unstable half–plane $\mathcal{S}_\varepsilon^+$ of the piecewise linear differential system (4.1).

Hence, for $\gamma_{\mathbf{p}_c}$ to be a periodic orbit, it is necessary that $\gamma_{\mathbf{p}_c}$ leaves $\mathcal{S}_\varepsilon^-$ and $\mathcal{S}_\varepsilon^+$ in negative and positive time, respectively. This will be performed by adding to the system (4.1) two new linear pieces. So that, for any arbitrary but fixed positive number η, we consider the four pieces linear differential system

$$\mathbf{x}' = \begin{cases} F^u(\mathbf{x}) & \text{if } v \geq \eta, \\ F^o(\mathbf{x}) & \text{if } |v| \leq \eta, \\ F^l(\mathbf{x}) & \text{if } v \leq -\eta, \end{cases} \tag{4.2}$$

where $\mathbf{x} = (u_1, u_2, v)^T$,

$$F^u(\mathbf{x}) = \begin{pmatrix} \varepsilon a_{12} u_2 + a_1(v - \eta) \\ \varepsilon - a_2^2(v - \eta) \\ u_1 + \eta + a_3^2(v - \eta) \end{pmatrix}, \quad F^l(\mathbf{x}) = \begin{pmatrix} \varepsilon a_{12} u_2 + a_1(v + \eta) \\ \varepsilon + a_2^2(v + \eta) \\ u_1 + \eta - a_3^2(v + \eta) \end{pmatrix},$$

$a_1, a_2, a_3 \in \mathbb{R}$, and $F^o(\mathbf{x})$ is the piecewise linear vector field defined by the differential system (4.1).

Let $\tilde{\varphi}(t; \mathbf{p})$ be the flow defined by the piecewise linear differential system (4.2). It is easy to conclude that $\tilde{\varphi}$ coincides with φ when we restrict it to the central region $\{(u_1, u_2, v) : |v| \leq \eta\}$. Therefore, the slow manifold $\tilde{\mathcal{S}}_\varepsilon$ of the system

(4.2) is the union of the unstable branch

$$\tilde{\mathcal{S}}_\varepsilon^+ = \left\{(u_1, u_2, v) \in \mathbb{R}^3 \ : 0 \le v \le \eta, \ \ u_1 + \varepsilon a_{12} u_2 + v = -\varepsilon^2 a_{12}\right\}$$

and the stable branch

$$\tilde{\mathcal{S}}_\varepsilon^- = \left\{(u_1, u_2, v) \in \mathbb{R}^3 \ : -\eta \le v \le 0, \ \ -u_1 + \varepsilon a_{12} u_2 + v = \varepsilon^2 a_{12}\right\},$$

which are locally invariants by the flow $\tilde{\varphi}$. In fact, the slow manifold has boundaries at $\{v = 0\}$ and at $\{v = \pm\eta\}$ (border planes) through which the flow leaves the manifold, see Figure 4. Moreover, we emphasize that the orbit $\gamma_{\mathbf{p}_c}$ is also a canard in this new setting, and its expression, given in Proposition 4.1, is correct while the orbit remains in the central region $\{|v| \le \eta\}$, that is for

$$|t| \le t^* = \frac{1}{\varepsilon}\sqrt{\varepsilon^2 - \frac{2\eta}{a_{12}}} - 1. \tag{4.3}$$

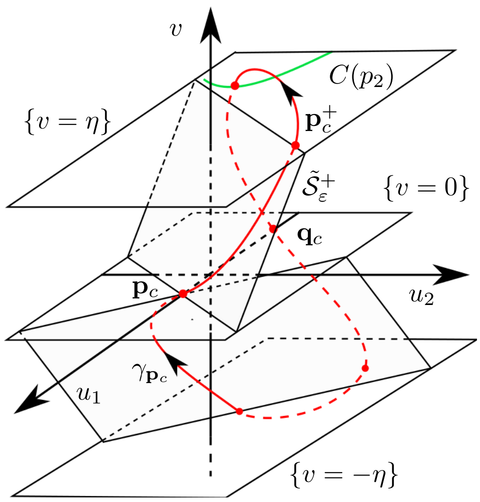

FIGURE 4. Representation of the canard cycle $\gamma_{\mathbf{p}_c}$, slow manifolds $\tilde{\mathcal{S}}_\varepsilon \cup \tilde{\mathcal{S}}_\varepsilon$ and the border planes $\{v = \eta\}, \{v = 0\}$ and $\{v = -\eta\}$, which separate the regions where el system is linear. We highlight the points of intersection of $\gamma_{\mathbf{p}_c}$ with the border planes.

An important thing that allows to get a canard cycle is the fact that system (4.2) is time–reversible with respect to the involution

$$\mathbf{R}(u_1, u_2, v) = (u_1, -u_2, -v).$$

Furthermore, the set of fixed points of involution $\mathbf{R}$ corresponds to the u_1–axis. Since $\mathbf{p}_c$ is on the u_1–axis, a sufficient condition on the orbit $\gamma_{\mathbf{p}_c}$ to be a periodic orbit is that it intersects the u_1–axis at a new point, which we denote by $\mathbf{q}_c$, see Figure 4.

Let us seek now the initial conditions $\mathbf{p} = (p_1, p_2, \eta)$ on the plane $\{v = \eta\}$ such that orbits through them, reach the u_1–axis. That is, $\varphi_2(t; \mathbf{p}) = 0$ and $\varphi_3(t; \mathbf{p}) = 0$, for a convenient time t. Hence, from the expression of the flow φ in the central region $0 \le v \le \eta$ we obtain the system

$$\begin{aligned} &0 = p_2 + \varepsilon t, \\ &0 = (e^t - 1)\, p_1 + \varepsilon a_{12}\, (e^t - 1 - t)\, p_2 + e^t \eta + \varepsilon^2 a_{12} \left(e^t - 1 - t - \tfrac{t^2}{2}\right). \end{aligned}$$

From this, we get the relation $p_1 = C(p_2)$, where

$$C(p_2) = \left(1 - e^{-\frac{p_2}{\varepsilon}}\right)^{-1} \left(e^{-\frac{p_2}{\varepsilon}}\, (\eta + \varepsilon a_{12}(p_2 + \varepsilon)) + \frac{1}{2} a_{12} p_2^2\right). \tag{4.4}$$

Set $\mathbf{p}_c^+ = \tilde{\varphi}(t^*, \mathbf{p}_c)$ where t^* is the value of the time defined in (4.3). Hence, $\mathbf{p}_c^+$ is the point of intersection of the canard orbit $\gamma_{\mathbf{p}_c}$ with the plane $\{v = \eta\}$, see Figure 4. Then

$$\mathbf{p}_c^+ = \left(-\eta - \varepsilon a_{12} \sqrt{\varepsilon^2 - \frac{2\eta}{a_{12}}},\ -\varepsilon + \sqrt{\varepsilon^2 - \frac{2\eta}{a_{12}}},\ \eta\right).$$

Therefore, a sufficient condition on $\gamma_{\mathbf{p}_c}$ to be a periodic orbit is the existence of a value of the time $t_c^+ > 0$ such that the orbit through $\mathbf{p}_c^+$ intersect the plane $\{v = \eta\}$ just at the graph of the function $C(p_2)$, see Figure 4, that is

$$\tilde{\varphi}(t_c^+, \mathbf{p}_c^+) \in \left\{(C(p_2), p_2, \eta) : p_2 \in \mathbb{R}^-\right\}, \tag{4.5}$$

and $\tilde{\varphi}_3(t, \mathbf{p}_c^+) > \eta$ for $t \in (0, t_c^+)$.

Condition (4.5) leads us to a system of two equations

$$\begin{aligned} &\tilde{\varphi}_1(t_c^+; \mathbf{p}_c^+) = C\left(\tilde{\varphi}_2(t_c^+; \mathbf{p}_c^+)\right), \\ &\tilde{\varphi}_3(t_c^+; \mathbf{p}_c^+) = \eta, \end{aligned} \tag{4.6}$$

with seven unknowns $\varepsilon, \eta, a_{12}$, a_1, a_2, a_3 and t_c^+. Now the idea is to fix five of these unknowns and compute the remainder.

We would remark that it is common to think that, systems like this can be solved because the flow is explicitly known in the half–space $\{v > \eta\}$. However, one of the unknowns, t_c^+, appears involved in exponential and trigonometric functions. Thus, their solution often entails an important analytical study. This study is the key point of some references, see for example [3, 4, 5]. Since this analysis goes beyond the objective of this work, we limit ourselves to present a numerical solution of system (4.6).

Setting $\varepsilon = 0.1$, $a_{12} = -1.3$, $\eta = 1$, $a_1 = -1$ and $a_2 = 3.4$, and solving (4.6) for a_3 and t_c^+, we obtain the values $a_3 \approx 0.583695486652$ and $t_c^+ \approx 2.3372454$.

Figure 5 contain different views of the canard cycle $\gamma_{\mathbf{p}_c}$ obtained from the previous computations. The top row represents the projection of $\gamma_{\mathbf{p}_c}$ on the planes (u_2, v), (u_1, v) and (u_1, u_2), respectively. The bottom row represents, first, a three dimensional view of the canard cycle and, second, a graph of the variable v versus the time t.

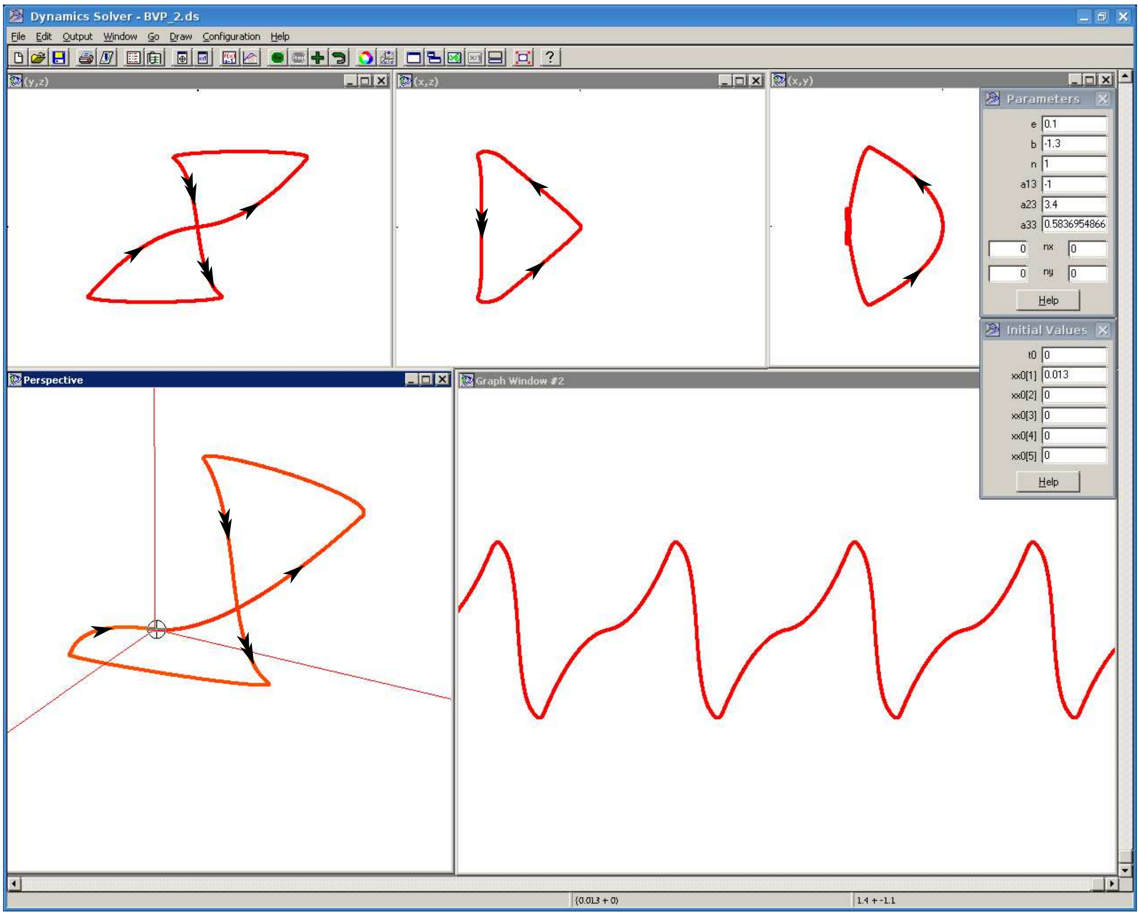

FIGURE 5. Screenshot of the Dynamics Solver [1] environment. In this figure appear different views of the canard cycle $\gamma_{\mathbf{P}_c}$ exhibited by system (4.2) when $\varepsilon = 0.1$, $a_{12} = -1.3$, $\eta = 1$, $a_1 = -1$ and $a_2 = 3.4$.

We recall that fast dynamic takes place on perturbed vertical straight lines, while the slow dynamic follows the slow manifold $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^+ \cup \mathcal{S}_\varepsilon^-$. In the (u_1, v) projection, one can observe the portion of the canard orbit following the fast manifold and the portion which follows the slow manifold. As shown in that drawing, the growth of the variable v during the slow phase is lost during the fast phase. We turn now to the drawing which represents the variable v versus time. As it can be observed, along a period, the time interval in which the variable v increases (the slow phase) is, approximately, one tenth of the time interval in which the variable v decreases (the fast phase). This is because $\varepsilon = 0.1$.

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