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Computer-assisted techniques for the verification of the Chebyshev property of Abelian integrals

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Abstract. We develop techniques for the verification of the Chebyshev property of Abelian integrals. These techniques are a combination of theoretical results, analysis of asymptotic behavior of Wronskians, and rigorous computations based on interval arithmetic. We apply this approach to tackle a conjecture formulated by Dumortier and Roussarie in [*Birth of canard cycles,* Discrete Contin. Dyn. Syst. **2** (2009), 723–781], which we are able to prove for $q \leqslant 2$.

1 Introduction and setting of the problem

The present paper addresses the problem of verifying when a collection of Abelian integrals form an *extended complete Chebyshev system* (ECT-system for short). This problem arises in the context of the second part of *Hilbert's 16th problem* [16], that asks about the maximum number and location of limit cycles of a planar polynomial vector field of degree d. Solving this problem even for the case $d = 2$ seems to be out of reach at the present state of knowledge (see [19, 23] for a survey of the recent results on the subject). Arnold [2] proposed a weaker version of this problem, the so-called *infinitesimal Hilbert's 16th problem.* Let ω be a real 1-form with polynomial coefficients of degree at most d. Consider a real polynomial H of degree $d+1$ in the plane. A closed connected component of a level curve $H = h$ is called an oval of H and denoted by γ_h. These ovals form continuous families, and the infinitesimal Hilbert's problem is to find an upper bound $V(d)$ of the number of real zeros of the *Abelian integral*

$$I(h) = \int_{\gamma_h} \omega.$$

The bound must depend on the degree d only, i.e., it should be uniform with respect to the choice of the polynomial H, the family of ovals $\{\gamma_h\}$ and the form ω. Zeros of Abelian integrals are related to limit cycles in the following way. Consider a small deformation of a Hamiltonian vector field $X_\varepsilon = X_H + \varepsilon Y$, where

$$X_H = -H_y\partial_x + H_x\partial_y \text{ and } Y = P\partial_x + Q\partial_y.$$

The first approximation in ε of the displacement function of the Poincaré map of X_ε is given by $I(h) = \int_{\gamma_h} \omega$ with $\omega = Pdy - Qdx$. Thus the number of zeros of $I(h)$, counted with multiplicity, provides an upper bound for the number of ovals of H that generate limit cycles of X_ε for $\varepsilon \approx 0$. The function $I(h)$ can be decomposed into a linear combination

$$\alpha_0 I_0(h) + \alpha_1 I_1(h) + \ldots + \alpha_{n-1} I_{n-1}(h)$$

where each α_k depends on the coefficients of P and Q, which are considered as parameters, and each I_k is an Abelian integral with $\omega = x^i y^j dx$ or $\omega = x^i y^j dy$. Therefore the problem is equivalent to finding an upper bound for the number of isolated zeros of any function in the linear span of I_0, $I_1 \dots, I_{n-1}$. This is where the notion of an ECT-system (see Definition 3.1) comes into play.

In the literature there is an abundance of papers dealing with zeros of Abelian integrals (see for instance [6, 15, 8, 12, 30, 43] and references therein). In many cases, it is essential to show that a collection of Abelian integrals has some kind of Chebyshev property. The techniques and arguments to tackle these problems are usually very long and highly non-trivial. For instance, in some papers (e.g. [5, 17, 31]) the authors study the geometrical properties of the so-called *centroid curve* using that it satisfies a Riccati equation (which is itself deduced from a Picard-Fuchs system). In other papers (e.g. [9, 10, 11]), the authors use complex analysis and algebraic topology (analytic continuation, argument principle, monodromy, Picard-Lefschetz formula, ...). The present paper addresses the applicability of a criterion appearing in [13], which greatly simplifies the verification of the Chebyshev property. Let us briefly explain this criterion.

Suppose that $H(x,y) = A(x) + B(x)y^{2m}$ is an analytic function in some open subset of the plane that has a local minimum at the origin. Then there exists a punctured neighbourhood $\mathscr{P}$ of the origin foliated by ovals $\gamma_h \subset \{H(x,y) = h\}$. We set $H(0,0) = 0$; then the set of ovals γ_h inside $\mathscr{P}$ is parameterized by the energy levels $h \in (0, h_0)$ for some positive h_0. The projection of $\mathscr{P}$ on the x-axis is an interval (x_ℓ, x_r) with $x_\ell < 0 < x_r$. Under these assumptions A has a zero of even multiplicity at $x = 0$, and it is easy to verify that there exist an analytic involution σ such that

$$A(x) = A\big(\sigma(x)\big) \text{ for all } x \in (x_\ell, x_r). \tag{1}$$

Recall that a map σ is an *involution* if $\sigma^2 = Id$ and $\sigma \neq Id$. Given a function κ defined on $(x_\ell, x_r) \setminus \{0\}$, we define its σ-*balance* as

$$\mathscr{B}_\sigma\big(\kappa\big)(x) = \kappa(x) - \kappa\big(\sigma(x)\big).$$

Following this notation we can state the result in [13, Theorem B] as follows. In the statement CT-system stands for *complete Chebyshev system*, see Definition 3.1.

Theorem 1.1. *Let $f_0, f_1, \dots, f_{n-1}$ be analytic functions on (x_ℓ, x_r), and consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \ i = 0, 1, \dots, n-1.$$

Let σ be the involution associated to A and define $\ell_i := \mathscr{B}_\sigma\Big(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\Big)$. If $\big(\ell_0, \ell_1, \dots, \ell_{n-1}\big)$ is a CT-system on $(0, x_r)$ and $s > m(n-2)$, then $(I_0, I_1, \dots, I_{n-1})$ is an ECT-system on $(0, h_0)$.

Let us stress, see Definition 3.1, that the difference between ECT-system and CT-system is that the first one takes the multiplicity of the zeros into account, while the second does not. Note in particular that an ECT-system is a CT-system. A collection of functions is an ECT-system if and only if none of the leading principal minors of its Wronskian vanishes (see Lemma 3.3). Since a CT-system is not so easy to characterize, when applying Theorem 1.1 we usually check if $\big(\ell_0, \ell_1, \dots, \ell_{n-1}\big)$ is an ECT-system. The applicability of the criterion comes from the fact that, instead of computing a Wronskian of Abelian integrals, we compute a Wronskian of *almost* explicit functions. We say "almost" because σ is defined only implicitly, by means of the relation in (1). In the polynomial setting, i.e., in case when A, B and f_i are polynomials, we can circumvent this problem by computing resultants. Indeed, in this case, by taking the derivative in (1), we can express $\sigma^{(i)}(x)$ in terms of a rational function depending on x and $y = \sigma(x)$. This in turn enables us to write the Wronskian of $\big(\ell_0, \ell_1, \dots, \ell_k\big)$, up to a nonvanishing factor, as $R_k\big(x, \sigma(x)\big)$ where $R_k(x,y)$ is a polynomial. If the Wronskian vanishes at $x_0 \in (0, x_r)$, then the polynomials $R_k(x,y)$ and $A(x) - A(y)$ have a common zero at (x_0, y_0) with $y_0 = \sigma(x_0)$. Accordingly, to prove that the Wronskian does not vanish, it suffices to show that the resultant of both polynomials, for instance with respect to y, has no zeros on $(0, x_r)$. This can be easily verified by applying Sturm's Theorem. This is the approach used in the examples of application of Theorem 1.1 that appear in [13, Section 4]. All the subsequent papers applying the criterion deal with the polynomial setting, and follow the same approach (see [3, 4, 14, 21, 24, 36, 37, 38, 40, 41]).

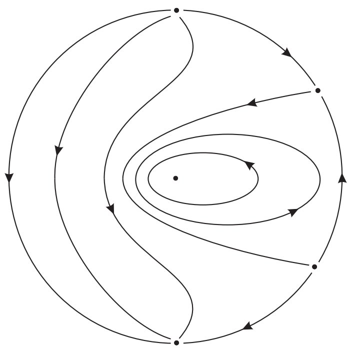

Figure 1: Phase portrait of the differential system (2) in the Poincaré disc.

The first aim of this paper is to show the applicability of Theorem 1.1 in the non-polynomial setting, and to this end we will rely upon computer-assisted methods based on interval arithmetic, see [35]. In fact we shall study a problem in which the involved functions are non-algebraic. More precisely, we tackle the conjecture posed by Dumortier and Roussarie in [7] where the authors consider singular perturbation problems occurring in planar slow-fast systems depending on parameters. They investigate the number of limit cycles that appear near a slow-fast Hopf point, i.e., its cyclicity. Their main results (Theorem 1.2, 1.5, and 1.8 in [7]) show that under very general conditions this cyclicity is finite and, modulo the following conjecture, provide its precise upper bound.

Conjecture. *For each integer* $i \geqslant 0$, *let us define* $\bar{J}_i(h) = \int_{\gamma_h} y^{2i-1}dx$, *where* $\gamma_h \subset \{A(x) + B(x)y^2 = h\}$ *with* $A(x) = \frac{1}{2} - e^{-2x}(x + \frac{1}{2})$ *and* $B(x) = e^{-2x}$. *Then* $\big(\bar{J}_0, \bar{J}_1, \dots, \bar{J}_n\big)$ *is an ECT-system on* $\big[0, \frac{1}{2}\big)$ *for* $n \geqslant 0$.

The second aim of this paper is to show the validity of the conjecture for $n = 0, 1$ and 2, i.e., we prove:

Theorem A. $\big(\bar{J}_0, \bar{J}_1, \bar{J}_2\big)$ *is an ECT-system on* $\big[0, \frac{1}{2}\big)$.

It is important to point out that, although Theorem A does not prove the conjecture for all n, it has noteworthy implications. Indeed, thanks to Theorem A, by applying [7, Theorem 1.5] one can explicitly bound the cyclicity of a *smooth* system with a slow-fast Hopf point of codimension 1 or 2. Similarly, thanks to Theorem A as well, [7, Theorem 1.8] gives the bound for the cyclicity of an *analytic* system with a slow-fast Hopf point of order 1 or 2. One can easily verify that $H(x, y) = A(x) + B(x)y^2$ is a first integral of the planar polynomial differential system

$$\text{(2)} \qquad \begin{cases} \dot{x} = -y, \\ \dot{y} = x + y^2. \end{cases}$$

Figure 1 displays its phase portrait in the Poincaré disc.

Let us outline the proof of Theorem A. First we show, see Corollary 3.5, that $\big(\bar{J}_0, \bar{J}_1, \bar{J}_2\big)$ is an ECT-system on $[0, \varepsilon]$ for some $\varepsilon > 0$. This follows by using analytic arguments that are rather standard. Having established this result, we construct a computer-assisted proof of that $\big(\bar{J}_0, \bar{J}_1, \bar{J}_2\big)$ is an ECT-system on $\big(0, \frac{1}{2}\big)$. Together, these two results prove Theorem A. Let us briefly explain the approach we follow for the second result. Since Theorem 1.1 can not be applied directly to the $\big(\bar{J}_0, \bar{J}_1, \bar{J}_2\big)$ because the associated 1-forms do not fulfill the assumptions, we show that it is satisfied for the functions

$$h^2 \bar{J}_i(h) = J_i(h) := \int_{\gamma_h} f_i(x) y^3 dx,$$

where f_0, f_1 and f_2 are the analytic functions given in Lemma 4.2. Then, since $m = 1$ and $s = 2$, we have that $\ell_i = \mathscr{B}_\sigma\Big(\frac{f_i}{A' B^{3/2}}\Big)$ for $i = 0, 1, 2$. The projection on the x-axis of the family of ovals associated to $H(x, y) = A(x) + B(x)y^2$ is the interval (x_ℓ, x_r) with $x_\ell = -\frac{1}{2}$ and $x_r = +\infty$, see Figure 1. Thus, according to Theorem 1.1, it suffices to show that (ℓ_0, ℓ_1, ℓ_2) is an ECT-system on $(0, x_r)$. As a matter of fact, since σ is a diffeomorphism that maps $(0, x_r)$ to $(x_\ell, 0)$, these σ-balanced functions form an ECT-system on $(0, x_r)$ if, and only if, they form an ECT-system on $(x_\ell, 0)$, see Remark 4.3. For convenience we will show the latter. To this end, see Lemma 3.3, we must compute the Wronskian of $\big(\ell_0, \ell_1, \ell_2\big)$ and check that none of its leading principal minors vanish on $(x_\ell, 0)$. The first order principal minor is ℓ_0 and the proof for this case is completely analytical, see Proposition 4.4. The approach for the two other is analogous, so we will focus on the third order one for the sake of brevity. It turns out that the functions ℓ_i are analytic on $(x_\ell, 0)$, meromorphic at $x = 0$ and not even defined at $x = x_\ell$, so the strategy is to split the interval into three subintervals, namely $I_1 = (x_\ell, x_\ell + \varepsilon_1)$, $I_2 = [x_\ell + \varepsilon_1, -\varepsilon_2]$ and $I_3 = (-\varepsilon_2, 0)$, where ε_1 and ε_2 are positive numbers to be specified. We first quantify them by analyzing the behaviour of the Wronskian near the endpoints $x = -\frac{1}{2}$ and $x = 0$. This gives us only qualitative results, but combining them with computer-assisted methods we can specify neighbourhoods of the endpoints, i.e., quantify ε_i, so that the Wronskian does not vanish on $I_1 = (x_\ell, x_\ell + \varepsilon_1)$ and $I_3 = (-\varepsilon_2, 0)$. This is carried out in Lemma 4.9. Next, once these ε_i are fixed, we show in Lemma 4.10 that the Wronskian is nonvanishing on I_2. Since the interval is compact, and the Wronskian is analytic, no analytic estimates are required here; standard computer-assisted techniques can handle the problem. In practice the interval I_2 is subdivided into many small subintervals. On each such subinterval we enclose the range of the Wronskian, and verify that the range enclosure does not contain zero.

Due to the fact that the proof we present here is a combination of theoretical and computer-assisted arguments, the proofs stated in this paper are presented using two formats: this written paper and some computer codes. This written paper contains the theoretical results and the explanations of the theory behind the computer-assisted proofs. The computer-assisted proofs are presented in separated files. These are available from **www2.math.uu.se/∼figueras/preprints/Chebyshev.tar.gz**, where there is a compressed file containing all `C` codes and a `README.txt` file with the specifications for the compilation and execution of the codes.

Some words must be said about the implementation of the codes. The interval library we have used is `MPFI`, see [33]. This library let us perform rigorous computation with arbitrary precision intervals. This is important for the validations we present because the functions that we need to evaluate have big oscillations, and normal interval libraries with only `double` or `long double` precision do not suffice.

The paper is organized as follows. In Section 2, for reader's convenience, we make a short description of the fundamentals of interval arithmetic. In Section 3 we give the definitions that will be used henceforth and among other preliminary results we prove that $\big(\bar{J}_0, \bar{J}_1, \ldots, \bar{J}_n\big)$ is an ECT-system on $[0, \varepsilon]$ for some $\varepsilon > 0$. Taking $n = 2$ shows Theorem A on $[0, \varepsilon]$. Finally in Section 4 we give the proof of Theorem A on $(0, \frac{1}{2})$.

2 Computer-assisted proofs and interval analysis

The ability to enclose the range of a given function is a basic feature of set-valued computations. This can be performed on a computer, and will produce a mathematically guaranteed result, see [35] and references therein. In what follows, we will briefly describe the fundamentals of interval arithmetic. For a concise reference on this topic, see e.g. [1, 22, 27, 28]. For early papers on the topic, see [34, 39, 42].

Let $\mathbb{IR}$ denote the set of closed intervals of the real line. For any element $\boldsymbol{a} \in \mathbb{IR}$, we adapt the notation $\boldsymbol{a} = [\underline{a}, \bar{a}]$. If $\star$ is one of the operators $+, -, \times, \div$, we define arithmetic operations on elements of $\mathbb{IR}$ by

$$\boldsymbol{a} \star \boldsymbol{b} = \{a \star b \colon a \in \boldsymbol{a}, b \in \boldsymbol{b}\},$$

except that $\boldsymbol{a} \div \boldsymbol{b}$ is undefined when $0 \in \boldsymbol{b}$. Working exclusively with closed intervals, we can describe the resulting interval in terms of the endpoints of the operands:

$$\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \\
\boldsymbol{a} - \boldsymbol{b} &= [\underline{a} - \bar{b}, \bar{a} - \underline{b}] \\
\boldsymbol{a} \times \boldsymbol{b} &= [\min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}), \max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})] \\
\boldsymbol{a} \div \boldsymbol{b} &= \boldsymbol{a} \times [1/\bar{b}, 1/\underline{b}], \quad \text{if } 0 \notin \boldsymbol{b}.
\end{aligned}$$

When computing with finite precision (which we do in computer-assisted proofs), directed rounding must also be taken into account, see e.g. [22, 28]. It follows immediately from the definitions that interval addition and multiplication are both associative and commutative. The distributive law, however, does *not* always hold. As an example, we have

$$[-1, 1]([-1, 0] + [3, 4]) = [-1, 1][2, 4] = [-4, 4]$$

whereas

$$[-1, 1][-1, 0] + [-1, 1][3, 4] = [-1, 1] + [-4, 4] = [-5, 5].$$

This unusual property is important to keep in mind when representing functions as part of a computer program. Interval arithmetic satisfies a weaker rule than the distributive law, which we refer to as *sub-distributivity*:

$$\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) \subseteq \boldsymbol{a}\boldsymbol{b} + \boldsymbol{a}\boldsymbol{c}.$$

Another key feature of interval arithmetic is that it is *inclusion monotonic*, i.e., if $\boldsymbol{a} \subseteq \boldsymbol{a}'$, and $\boldsymbol{b} \subseteq \boldsymbol{b}'$, then

$$\boldsymbol{a} \star \boldsymbol{b} \subseteq \boldsymbol{a}' \star \boldsymbol{b}'.$$

It is not hard to extend simple functions to the interval realm. As an example, we may define the extension of the exponential function as

$$\exp(\mathbf{x}) = \big[\exp(\underline{x}), \exp(\bar{x})\big].$$

In general, when we extend a real-valued function f to an interval-valued one F, we demand that it satisfies the *inclusion principle*

$$\text{Range}(f; \mathbf{x}) = \{f(x) \colon x \in \mathbf{x}\} \subseteq F(\mathbf{x}).$$

This is all we need in order to prove that a function (e.g. the Wronskian mentioned above) is nonvanishing on a closed interval. Indeed, if we have $0 \notin F(\mathbf{x})$ then we know that f is non-zero on the domain $\mathbf{x}$.

In order to avoid overestimation of the range, we can use the inclusion monotonicity, and split the interval into smaller pieces: $\mathbf{x} = \cup_{i=1}^N \mathbf{x}_i$. We then have

$$\text{Range}(f; \mathbf{x}) \subseteq \cup_{i=1}^N F(\mathbf{x}_i) \subseteq F\big(\cup_{i=1}^N \mathbf{x}_i\big) = F(\mathbf{x}).$$

For sufficiently smooth functions, we can make the overestimation of the range arbitrarily small by taking a sufficiently fine subdivision of the domain $\mathbf{x}$.

For a complicated function f, we might prefer working with a polynomial approximation p. It is then important that we can bound the error $f - p$ over the domain of interest. For Taylor expansions, we have the following enclosure property: If $x, \check{x} \in \mathbf{x}$ then

$$f(x) \in f(\check{x}) + f'(\check{x})(x - \check{x}) + \cdots + \frac{f^{(n-1)}(\check{x})}{(n-1)!}(x - \check{x})^{n-1} + \frac{1}{n!}F^{(n)}(\mathbf{x})(x - \check{x})^n.$$

This means that bounding the range of $f^{(n)}$ over the domain $\mathbf{x}$ (e.g. via its interval extension) will provide a bound on the approximation error. As an example, taking $f(x) = \sin x$ and $\check{x} = 0$, we have for all $x \in \mathbb{R}$:

$$\sin x \in x + [-1, 1]\frac{x^3}{3!},$$

which gives good bounds for small x. We can use this type of information to prove statements of the following kind: "$f\colon [a,b] \longrightarrow \mathbb{R}$ is positive on $(a,b]$, and zero at a". In the case $f(x) = \sin x$ and $[a,b] = [0,1]$, we have $\sin x \in x - [-1,1]x^3/3! = x(1-[-1,1]x^2/6) \in x(1-[-1,1]/6) = x[5/6, 7/6]$. Thus we get a cone based at the origin enclosing $\sin x$ for all $x \in [0,1]$. It follows that $\sin x$ is positive on $(0,1]$, and vanishes at the origin. Naturally, we are interested in more complicated functions f, but the same procedure applies.

3 Definitions and preliminary results

We give now the definitions and theoretical results that will be used throughout the paper. In this section I denotes a real interval with a nonempty interior.

Definition 3.1 Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I.

(a) The ordered set of functions $(f_0, f_1, \ldots, f_{n-1})$ is a *complete Chebyshev system* (in short, CT-system) on I if, for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \ldots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on I.

(b) The ordered set of functions $(f_0, f_1, \ldots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on I if, for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \ldots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on I counted with multiplicity.

(Let us note that in these abbreviations "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev.) □

It is clear that if $(f_0, f_1, \ldots, f_{n-1})$ is an ECT-system on I, then it is a CT-system on I.

Definition 3.2 Let $f_0, f_1, \ldots, f_{k-1}$ be analytic functions on I. The *Wronskian* of $(f_0, f_1, \ldots, f_{k-1})$ at $x \in I$ is

$$W\big[f_0, f_1, \cdots, f_{k-1}\big](x) = \det\Big(f_j^{(i)}(x)\Big)_{0\leqslant i,j\leqslant k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

□

The next result, see [20, 26], provides an easy characterization of the ECT-systems.

Lemma 3.3. *$(f_0, f_1, \ldots, f_{n-1})$ is an ECT-system on I if and only if for each $k = 1, 2, \ldots, n$,*

$$W\big[f_0, f_1, \ldots, f_{k-1}\big](x) \neq 0 \text{ for all } x \in I.$$

Note that, although the Abelian integrals $\bar{J}_i(h)$ are only defined for $h \in \big(0, \frac{1}{2}\big)$, the conjecture of Dumortier and Roussarie refers to the interval $\big[0, \frac{1}{2}\big)$. The level curves $h = 0$ and $h = \frac{1}{2}$ are, respectively, a non-degenerated center and an unbounded polycycle, see Figure 1. We begin by showing that these Abelian integrals can be analytically extended to $h = 0$.

Lemma 3.4. *For each $i \geqslant 0$, $\bar{J}_i(h) = h^i\big(\Delta_i + \ell_i(h)\big)$ where $\Delta_i \neq 0$ and ℓ_i is an analytic function on $h=0$ with $\ell_i(0)=0$.*

Proof. The oval γ_h intersects with the x-axis at two points, $x_-(h) < 0 < x_+(h)$, which are roots of $A(x)=h$. Since $A(x) = \frac{1}{2} - e^{-2x}(x+\frac{1}{2}) = \frac{1}{2}x^2 + \mathrm{o}(x^2)$, the function $g(x) := \mathrm{sgn}(x)\sqrt{A(x)}$ is an analytic diffeomorphism on (x_ℓ, x_r). Moreover $x_\pm(h) = g^{-1}(\pm\sqrt{h})$. Therefore it follows that

$$
\begin{aligned}
\frac{\bar{J}_i(h)}{h^i} &= \frac{1}{h^i}\int_{\gamma_h} y^{2i-1}dx = \frac{2}{h^i}\int_{x_-(h)}^{x_+(h)} \left(\frac{h-A(x)}{B(x)}\right)^{i-\frac{1}{2}} dx \\
&= 2\int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})} \left(\frac{h-A(x)}{hB(x)}\right)^{i-\frac{1}{2}} \frac{dx}{\sqrt{h}} = 2\int_{-1}^{1} \frac{\big(g^{-1}\big)'(\sqrt{h}u)}{\big(B^{i-\frac{1}{2}}\big)\big(g^{-1}(\sqrt{h}u)\big)} \frac{du}{(1-u^2)^{\frac{1}{2}-i}},
\end{aligned}
$$

where in the last equality we perform the coordinate transformation given by $g(x) = \sqrt{h}u$ and we take $A(x) = g(x)^2$ into account. Note at this point that the function

$$
F(z) := \frac{\big(g^{-1}\big)'(z)}{\big(B^{i-\frac{1}{2}}\big)\big(g^{-1}(z)\big)}
$$

is analytic at $z=0$. Let $F(z) = \sum_{n\geqslant 0}\alpha_n z^n$ be its Taylor series at $z=0$. Then

$$
\frac{\bar{J}_i(h)}{h^i} = 2\int_{-1}^{1}\sum_{n=0}^{\infty}\alpha_n h^{n/2}\frac{u^n du}{(1-u^2)^{\frac{1}{2}-i}} = 2\sum_{n=0}^{\infty}\alpha_{2n}h^n\int_{-1}^{1}\frac{u^{2n}du}{(1-u^2)^{\frac{1}{2}-i}},
$$

and this proves the analyticity of $\bar{J}_i(h)$ at $h=0$. In addition, since $g'(0) = \frac{1}{\sqrt{2}}$ and $B(0)=1$, we get that

$$
\lim_{h\longrightarrow 0}\frac{\bar{J}_i(h)}{h^i} = 2\sqrt{2}\int_{-1}^{1}\frac{du}{(1-u^2)^{\frac{1}{2}-i}} = 2\sqrt{2\pi}\,\frac{\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} =: \Delta_i,
$$

where Γ is the Gamma function. This completes the proof of the result. ∎

Corollary 3.5. *For each $n \geqslant 0$ there exists $\varepsilon > 0$ such that $\big(\bar{J}_0, \bar{J}_1, \ldots, \bar{J}_n\big)$ is an ECT-system on $[0,\varepsilon]$.*

Proof. For each integer $i \geqslant 0$, we will say that a function f belongs to the set $\mathscr{R}_i$ if $f(x) = x^i s(x)$ with s being any analytic function at $x=0$ with $s(0)\neq 0$. Note in particular that if $f\in\mathscr{R}_0$, then $f(0)\neq 0$. In addition one can easily verify that if $g\in\mathscr{R}_i$ and $h\in\mathscr{R}_j$ with $i>j$, then $\left(\frac{g}{h}\right)'\in\mathscr{R}_{i-j-1}$.

According to this notation Lemma 3.4 shows that $\bar{J}_i\in\mathscr{R}_i$ for all $i\geqslant 0$. Fix k such that $0\leqslant k\leqslant n$ and consider any function ℓ in the linear span of $\bar{J}_0, \bar{J}_1, \ldots, \bar{J}_k$, i.e.,

$$
\ell(h) = \alpha_0\bar{J}_0(h) + \alpha_1\bar{J}_1(h) + \ldots + \alpha_k\bar{J}_k(h),
$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_k\in\mathbb{R}$. For the sake of convenience, let us rewrite the above equality as

$$
\ell^0 = \alpha_0 J_0^0 + \alpha_1 J_1^0 + \ldots + \alpha_k J_k^0.
$$

Since $J_0^0\in\mathscr{R}_0$, there exists $\varepsilon_0>0$ such that $J_0^0(h)\neq 0$ for all $h\in[-\varepsilon_0,\varepsilon_0]$. This enables us to perform the first step of the division-derivation algorithm,

$$
\ell^1 := \left(\frac{\ell^0}{J_0^0}\right)' = \alpha_1\left(\frac{J_1^0}{J_0^0}\right)' + \alpha_2\left(\frac{J_2^0}{J_0^0}\right)' + \ldots + \alpha_k\left(\frac{J_k^0}{J_0^0}\right)'.
$$

Given any analytic function f, let us denote the number of zeros of f on the interval $[-\delta,\delta]$ counted with multiplicity by $\mathcal{Z}(f,\delta)$. Thus by applying Rolle's Theorem we can assert that $\mathcal{Z}(\ell^0,\varepsilon_0)\leqslant\mathcal{Z}(\ell^1,\varepsilon_0)+1$. Note moreover that $J_i^1:=\left(\frac{J_i^0}{J_0^0}\right)'\in\mathscr{R}_{i-1}$. Therefore, since $J_1^1(h)\neq 0$ for all $h\in[-\varepsilon_1,\varepsilon_1]$ with $0<\varepsilon_1<\varepsilon_0$, we can perform the second step of the division-derivation algorithm to obtain

$$\ell^2:=\left(\frac{\ell^1}{J_1^1}\right)'=\alpha_2\left(\frac{J_2^1}{J_1^1}\right)'+\alpha_3\left(\frac{J_3^1}{J_1^1}\right)'+\ldots+\alpha_k\left(\frac{J_k^1}{J_1^1}\right)',$$

so that $\mathcal{Z}(\ell^0,\varepsilon_1)\leqslant\mathcal{Z}(\ell^2,\varepsilon_1)+2$. As before, $J_i^2:=\left(\frac{J_i^1}{J_1^1}\right)'\in\mathscr{R}_{i-2}$ and this allows us to perform the next step of the division-derivation algorithm. Following the obvious notation, in the kth step of the division-derivation algorithm we get

$$\ell^k:=\left(\frac{\ell^{k-1}}{J_{k-1}^{k-1}}\right)'=\alpha_k\left(\frac{J_k^{k-1}}{J_{k-1}^{k-1}}\right)',$$

with $J_k^k:=\left(\frac{J_k^{k-1}}{J_{k-1}^{k-1}}\right)'\in\mathscr{R}_0$. Thus ℓ^k does not vanish on $[-\varepsilon_k,\varepsilon_k]$ with $0<\varepsilon_k<\varepsilon_{k-1}$ and, on the other hand, $\mathcal{Z}(\ell^0,\varepsilon_{k-1})\leqslant\mathcal{Z}(\ell^k,\varepsilon_{k-1})+k$. Note that $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_k$ only depend on $\bar{J}_0,\bar{J}_1,\ldots,\bar{J}_k$. In particular they do not depend on ℓ and so, due to $\mathcal{Z}(\ell^k,\varepsilon_k)=0$, this shows that any function in the linear span of $\bar{J}_0,\bar{J}_1,\ldots,\bar{J}_k$ has at most k zeros on $[0,\varepsilon_k]$ counted with multiplicity. Accordingly the result follows by taking $\varepsilon=\varepsilon_n$. ■

The proof of the following result can be found in [18, 32].

Lemma 3.6. *Let $f_0,f_1,\ldots,f_n$ be analytic functions on I such that $W[f_0,\ldots,f_{n-2},f_{n-1}]$ is nonvanishing. Then it holds that*

$$\left(\frac{W[f_0,\ldots,f_{n-2},f_n]}{W[f_0,\ldots,f_{n-2},f_{n-1}]}\right)'=\frac{W[f_0,\ldots,f_n]\,W[f_0,\ldots,f_{n-2}]}{(W[f_0,\ldots,f_{n-2},f_{n-1}])^2}.$$

The authors in [7] state their conjecture in terms of a definition of ECT-system (cf. [7, Definition 7.4]) which is different from Definition 3.1. For the sake of completeness we conclude the present section by showing the equivalence between both definitions.

Lemma 3.7. *$(f_0,f_1,\ldots,f_{n-1})$ is an ECT-system on I if and only if one can construct inductively the sequence $\mathcal{F}^1,\mathcal{F}^2,\ldots,\mathcal{F}^{n-1}$ with $\mathcal{F}^k=\{f_k^k,f_{k+1}^k,\ldots,f_{n-1}^k\}$ such that:*

(a) Setting $f_i^1:=\frac{f_i}{f_0}$ for $i=1,2,\ldots,n-1$, the functions

$$f_i^{k+1}:=\frac{\left(f_i^k\right)'}{\left(f_k^k\right)'}\ \text{ for } k=1,2,\ldots,n-1 \text{ and } i=k+1,\ldots,n-1,$$

are analytic on I, and

(b) f_0 and $\left(f_k^k\right)'$, for $k=1,2,\ldots,n-1$, do not vanish on I.

Proof. By applying Lemma 3.6, on account of $(f_i^1)'=\left(\frac{f_i}{f_0}\right)'=\frac{W[f_0,f_i]}{W[f_0]^2}$, it turns out that

$$(f_i^2)'=\left(\frac{\left(f_i^1\right)'}{\left(f_1^1\right)'}\right)'=\left(\frac{W[f_0,f_i]}{W[f_0,f_1]}\right)'=\frac{W[f_0,f_1,f_i]\,W[f_0]}{W[f_0,f_1]^2}.$$

Consequently, by Lemma 3.6 again,

$$(f_i^3)'=\left(\frac{\left(f_i^2\right)'}{\left(f_2^2\right)'}\right)'=\left(\frac{W[f_0,f_1,f_i]}{W[f_0,f_1,f_2]}\right)'=\frac{W[f_0,f_1,f_2,f_i]\,W[f_0,f_1]}{W[f_0,f_1,f_2]^2}.$$

Thus, by applying Lemma 3.6 as before, for an arbitrary k we obtain

$$(f_i^k)' = \left(\frac{(f_i^{k-1})'}{(f_{k-1}^{k-1})'}\right)' = \left(\frac{W[f_0, f_1, \ldots, f_{k-2}, f_i]}{W[f_0, f_1, \ldots, f_{k-1}]}\right)' = \frac{W[f_0, f_1, \ldots, f_{k-1}, f_i]\, W[f_0, f_1, \ldots, f_{k-2}]}{W[f_0, f_1, \ldots, f_{k-1}]^2}.$$

Now, taking these equalities with $i = k$, the result follows from Lemma 3.3. ■

4 Proof of the Theorem A on $\left(0, \frac{1}{2}\right)$

The Abelian integrals $\bar{J}_0$, $\bar{J}_1$ and $\bar{J}_2$ defined in Section 1 do not fulfill the hypothesis in Theorem 1.1. We shall use the next result, which follows from [13, Lemma 4.1], to obtain another set of Abelian integrals with the desired structure.

Lemma 4.1. *Let γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$.*

(a) If F/A' is analytic on (x_ℓ, x_r), then

$$\int_{\gamma_h} F(x)y^{k-2}dx = \int_{\gamma_h} \mathscr{U}_F(x)y^k dx \quad \textit{where } \mathscr{U}_F = \tfrac{2}{k}B^{k/2}\left(\tfrac{FB^{1-k/2}}{A'}\right)'.$$

(b) If G is analytic on (x_ℓ, x_r), then

$$\int_{\gamma_h} G(x)y^k dx = \int_{\gamma_h} \mathscr{D}_G(x)y^{k-2}dx \quad \textit{where } \mathscr{D}_G = \tfrac{k}{2}A'B^{\frac{k}{2}-1}\textstyle\int GB^{\frac{-k}{2}}dx.$$

By applying the previous result we can now prove:

Lemma 4.2. *For $i = 0, 1, 2$, $\bar{J}_i(h) = \dfrac{1}{h^2}\displaystyle\int_{\gamma_h} f_i(x)y^3 dx$, where*

$$f_0(x) = \frac{1}{12x^4}\Big((32x^4 + 12x^3 + 15x^2 + 7x + 3)e^{-4x} + (8x^3 - 6x^2 - 2x - 6)e^{-2x} + 3x^2 - 5x + 3\Big),$$
$$f_1(x) = \frac{1}{12x^2}\Big((32x^3 - 12x^2 - x - 1)e^{-4x} + (8x^2 - 2x + 2)e^{-2x} + 3x - 1\Big),$$
$$f_2(x) = \frac{1}{12}\Big((32x^2 - 68x + 3)e^{-4x} + (8x - 6)e^{-2x} + 3\Big).$$

Proof. For convenience, let us define $I(h) = a\bar{J}_0(h) + b\bar{J}_1(h) + c\bar{J}_2(h)$. Then

$$h^2 I(h) = \int_{\gamma_h} \big(A(x) + B(x)y^2\big)^2 \left(\frac{a}{y} + by + cy^3\right) dx = \int_{\gamma_h} \left(\sum_{i=0}^{4} g_i(x)y^{2i-1}\right) dx,$$

where $g_0 := aA^2$, $g_1 := A(2aB + bA)$, $g_2 := aB^2 + 2bAB + cA^2$, $g_3 := B(bB + 2cA)$ and $g_4 := cB^2$. Now, by applying Lemma 4.1 with $\{F = g_0, k = 1\}$ and $\{G = g_4, k = 7\}$ we get

$$h^2 I(h) = \int_{\gamma_h} \big(\hat{g}_1 y + g_2 y^3 + \hat{g}_3 y^5\big) dx,$$

where $\hat{g}_1 := g_1 + \mathscr{U}_{g_0}$ and $\hat{g}_3 := g_3 + \mathscr{D}_{g_4}$. Next, by Lemma 4.1 again, in this case taking $\{F = \hat{g}_1, k = 3\}$ and $\{G = \hat{g}_3, k = 5\}$, we obtain

$$h^2 I(h) = \int_{\gamma_h} \hat{g}_2 y^3 dx, \text{ where } \hat{g}_2 := g_2 + \mathscr{U}_{\hat{g}_1} + \mathscr{D}_{\hat{g}_3}.$$

(Let us point out that g_0/A' and $\hat{g}_1/A'$ are analytic because one can verify that g_0 and $\hat{g}_1$ vanish at $x=0$, so that we can apply (a) in Lemma 4.1). Finally some computations show that $\hat{g}_2 = af_0 + bf_1 + cf_2$ with the functions given in the statement. This proves the result. ■

Remark 4.3 Consider the functions f_0, f_1 and f_2 in the statement of Lemma 4.2. Then $h^2\bar{J}_i(h) = J_i(h)$, where

$$J_i(h) := \int_{\gamma_h} f_i(x)y^3dx \text{ for } i=0,1,2.$$

Hence $(\bar{J}_0, \bar{J}_1, \bar{J}_2)$ is an ECT-system on $(0,h_0)$ if and only if (J_0,J_1,J_2) is an ECT-system on $(0,h_0)$. Setting $g_i := \frac{f_i}{A'B^{3/2}}$ for $i=0,1,2$, the latter will follow by Theorem 1.1 once we show that $\big(\mathscr{B}_\sigma(g_0), \mathscr{B}_\sigma(g_1), \mathscr{B}_\sigma(g_2)\big)$ is an ECT-system on $(0,x_r)$.

Note on the other hand that if $\varphi\colon I \longrightarrow J$ is a diffeomorphism, then $\big(f_0, f_1, \ldots, f_{n-1}\big)$ is an ECT-system on J if and only if $\big(f_0\circ\varphi, f_1\circ\varphi, \ldots, f_{n-1}\circ\varphi\big)$ is an ECT-system on I. Consequently, since σ is maps $(0,x_r)$ to $(x_\ell,0)$ and $\mathscr{B}_\sigma(g_i\circ\sigma) = -\mathscr{B}_\sigma(g_i)$, for convenience we shall verify that $\big(\mathscr{B}_\sigma(g_0), \mathscr{B}_\sigma(g_1), \mathscr{B}_\sigma(g_2)\big)$ is an ECT-system on $(x_\ell,0)$. On account of Lemma 3.3 this is equivalent to showing that all the leading principal minors of its Wronskian are nonvanishing on $(x_\ell,0)$. Setting $\ell_i = \mathscr{B}_\sigma(g_i)$ for the sake of shortness, the rest of the section is devoted to prove this. □

Proposition 4.4. *ℓ_0 is nonvanishing on $(-\frac{1}{2},0)$.*

Proof. We claim that $xg_0(x) > 0$ for all $x\in(-\frac{1}{2},+\infty)$. Recall that $g_0 = \frac{f_0}{A'B^{3/2}}$ where, for the sake of convenience, we rewrite the numerator as $f_0(x) = \frac{n(x)}{12x^4e^{2x}}$ with

$$n(x) = (32x^4+12x^3+15x^2+7x+3)e^{-2x} + (3x^2-5x+3)e^{2x} + 8x^3 - 6x^2 - 2x - 6.$$

On account of $xA'(x)>0$, the claim will follow if we show that f_0 is positive on $(-\frac{1}{2},+\infty)$. To this end let us first note that $n(x) = 32x^4 - \frac{164}{3}x^5 + \mathrm{o}(x^5)$, so that n has a zero of multiplicity four at $x=0$. By applying Sturm's Theorem we can assert that $p(x) = 32x^4+12x^3+15x^2+7x+3$ has no real zeros. Consequently, the function $\kappa(x) = \frac{n(x)}{p(x)e^{-2x}}$ is well defined and has exactly the same roots, counted with multiplicity, as n. One can verify that

$$\kappa'(x) = \frac{4xs(x)}{p(x)^2e^{-2x}},$$

where $s(x) = q_0(x) + q_1(x)e^{2x}$ with $q_0(x) = 128x^6 - 112x^5 + 88x^4 - 29x^3 + 172x^2 + 8x + 12$ and $q_1(x) = 96x^5 - 172x^4 + 192x^3 - 84x^2 + 16x - 12$. By Sturm's Theorem again it follows that q_0 does not vanish, so that $t(x) = \frac{s(x)}{q_0(x)}$ is well defined. Moreover by Rolle's Theorem the function n has at most two more zeros, counted with multiplicity, than t. A computation shows that

$$t'(x) = \frac{4xe^{2x}p(x)r(x)}{q_0(x)^2},$$

where

$$r(x) = 192x^6 - 680x^5 + 1326x^4 - 1665x^3 + 1541x^2 - 735x + 192.$$

Since by Sturm's Theorem once again, r does not vanish, by applying Rolle's Theorem it follows that t has at most two real zeros, counted with multiplicity. Accordingly, n has at most four real zeros counted with multiplicity. Since $n(x) = 32x^4 - \frac{164}{3}x^5 + \mathrm{o}(x^5)$, this implies that $f_0(x)>0$ for all $x\in\mathbb{R}$, so the claim is true. Finally, taking the claim into account, since $\sigma(x)>0$ for all $x\in(-\frac{1}{2},0)$, it turns out that $g_0\big(\sigma(x)\big) > 0 > g_0(x)$ for all $x\in(-\frac{1}{2},0)$. Thus $\mathscr{B}_\sigma\big(g_0\big)(x) = g_0(x) - g_0\big(\sigma(x)\big) < 0$ for all $x\in(-\frac{1}{2},0)$, and this completes the proof. ■

To prove that $W[\ell_0,\ell_1]$ and $W[\ell_0,\ell_1,\ell_2]$ do not vanish on $(x_\ell,0)$ we follow the same strategy. As we explain in Section 1, we divide $(x_\ell,0)$ in three subintervals, namely $I_1=(x_\ell,x_\ell+\varepsilon_1)$, $I_2=[x_\ell+\varepsilon_1,-\varepsilon_2]$ and $I_3=(-\varepsilon_2,0)$. In the proof for I_1 and I_3 we perform a combination of asymptotic analysis with rigorous quantitative computer-assisted proofs, whereas for I_2 we use a rather standard computer-assisted technique. Indeed, since the functions ℓ_i are analytic on $(x_\ell,0)$, we can rigorously enclose the range of the Wronskian on small subintervals, and verify that it is nonvanishing. For the sake of brevity we shall only give in full the proof for $W[\ell_0,\ell_1,\ell_2]$ because the proof for $W[\ell_0,\ell_1]$ follows exactly the same lines. To this end we begin by proving some lemmas that provide upper and lower bounds of the involution σ, which recall that is defined by means of $A(x)=A\big(\sigma(x)\big)$ with $A(x)=\frac{1}{2}-e^{-2x}(x+\frac{1}{2})$.

Lemma 4.5. *Define* $\sup(x)=\sqrt{\frac{2e^{2x}}{2x+1}-2}$ *and* $\inf(x)=x-\frac{1}{2}\log(2x+1)$. *Then* $0<\inf(x)<\sigma(x)<\sup(x)$ *for all* $x\in(-\frac{1}{2},0)$.

Proof. Let us first note that $f(x)<A(x)<g(x)$ for all $x>0$ with $f(x):=\frac{x^2}{2x^2+4}$ and $g(x):=\frac{1}{2}e^{-2x}$. Hence $f\big(\sigma(x)\big)<A\big(\sigma(x)\big)=A(x)<g\big(\sigma(x)\big)$ for all $x\in(-\frac{1}{2},0)$. Thus, since f and g are monotonous increasing on $(0,+\infty)$, this implies $\inf(x):=g^{-1}\big(A(x)\big)<\sigma(x)<f^{-1}\big(A(x)\big)=:\sup(x)$ for all $x\in(-\frac{1}{2},0)$. ■

Lemma 4.6. *Define* $f(z,x)=2(x-z)+\log(2z+1)-\log(2x+1)$ *and fix* $x_0\in(-\frac{1}{2},0)$. *If* $f(z_0,x_0)$ *is positive (respectively, negative) then* $z_0>\sigma(x_0)$ *(respectively,* $\sigma(x_0)>z_0$*).*

Proof. Clearly, $\sigma(x_0)>0$. In addition, since A is monotonous increasing on $(0,+\infty)$, note that $\sigma(x_0)<z_0$ if, and only if, $A\big(\sigma(x_0)\big)<A(z_0)$. By definition this inequality writes as $A(x_0)<A(z_0)$, which taking logarithms is equivalent to $f(z_0,x_0)>0$. The other assertion follows from the same type of arguments. ■

Let us point out that, in the statement of the following result, μ is an interval-valued function.

Lemma 4.7. *Define* $\mu(x)=-x+[1,\frac{5}{3}]x^2$ *and* $I_3=(-0.1,0)$. *Then* $\sigma(I)\subset\mu(I)$ *for every interval* $I\subset I_3$.

Proof. Setting $a_1=1$ and $a_2=\frac{5}{3}$, let us define $\sigma_i(x):=-x+a_ix^2$. Then, by Lemma 4.6 it suffices to prove that $f(\sigma_1(x),x)>0$ and $f(\sigma_2(x),x)<0$ for all $x\in I_3$. Both inequalities can be proved by means of elementary analytic arguments. Let us show the second one for instance. Some computations show that

$$g(x):=f\big(\sigma_2(x),x\big)=4x-\frac{10}{3}x^2+\log\left(\frac{10x^2-6x+3}{6x+3}\right),$$

and we can assert that $g(x)<0$ for all $x\in I_3$ because $g(0)=0$ and its derivative, which is given by

$$g'(x)=\frac{4x^2\left(9+70x-100x^2\right)}{3\,(2x+1)\,(3-6x+10x^2)},$$

is positive for $x\in I_3$. The first inequality follows exactly the same way and holds in fact on $(-\frac{1}{2},0)$. ■

We shall also use the so-called *Fujiwara's bound* (see for instance [25]).

Lemma 4.8 (Fujiwara). *The roots of the polynomial* $p(z)=c_0+c_1z+\ldots+c_nz^n$, *with* $c_n\neq 0$, *have modulus smaller than*

$$2\max\left\{\left|\frac{c_{n-1}}{c_n}\right|,\left|\frac{c_{n-2}}{c_n}\right|^{\frac{1}{2}},\ldots,\left|\frac{c_1}{c_n}\right|^{\frac{1}{n-1}},\left|\frac{c_0}{2c_n}\right|^{\frac{1}{n}}\right\}.$$

Lemma 4.9. $W[\ell_0,\ell_1,\ell_2]$ *is a nonvanishing on* $I_1=(-\frac{1}{2},-\frac{1}{2}+10^{-23})$ *and* $I_3=(-0.1,0)$.

Proof. Some straight-forward computations show that

$$W[\ell_0,\ell_1,\ell_2](x)=\frac{D\big(x,e^x,\sigma(x),e^{\sigma(x)}\big)}{2^9 3^3 x^{12} e^{6x}\sigma(x)^{15}}, \tag{3}$$

with D being a polynomial. We set $y=e^x$, $z=\sigma(x)$ and $w=e^{\sigma(x)}$. Then $A(x)=A\big(\sigma(x)\big)$ writes as

$$\frac{2x+1}{y^2}=\frac{2z+1}{w^2}.$$

Taking this relation into account and introducing $u=\sqrt{z+\frac{1}{2}}$ we obtain

$$D(x,y,z,w)=D\left(x,y,z,\sqrt{\frac{2z+1}{2x+1}}y\right)=D\left(x,y,u^2-\frac{1}{2},\frac{uy}{\sqrt{x+\frac{1}{2}}}\right)=\frac{S(x,y,u)}{(x+\frac{1}{2})^{\frac{21}{2}}}, \tag{4}$$

where $S(x,e^x,u)=\sum_{i=0}^{33}a_i(x)u^i$ with the functions a_i continuous at $x=-\frac{1}{2}$. One can check that $a_i(-\frac{1}{2})=0$ for $i=28,29,\dots,33$. More precisely

$$\lim_{x\longrightarrow-\frac{1}{2}}\frac{a_i(x)}{(x+1/2)^{n_i}}=\Delta_i, \tag{5}$$

where $n_{33}=n_{31}=n_{29}=6$, $n_{32}=n_{30}=n_{28}=\frac{5}{2}$, and

$$\Delta_{33}<0,\ \Delta_{32}>0,\ \Delta_{31}>0,\ \Delta_{30}<0\ \Delta_{29}<0 \text{ and } \Delta_{28}>0.$$

In addition, $a_{27}(-\frac{1}{2})=\frac{27}{16}\sqrt{\frac{2}{e^{21}}}$. By continuity there exists $\varepsilon_0>0$ such that if $x\in(-\frac{1}{2},-\frac{1}{2}+\varepsilon_0)$, then $\Delta_i a_i(x)>0$ for $i=28,29,\dots,33$. By applying Lemma 4.5, if we define $\omega(x):=\big(\sup(x)+\frac{1}{2}\big)^{1/2}$, then it follows that $\frac{1}{\sqrt{2}}<u<\omega(x)$ for all $x\in(-\frac{1}{2},0)$. Thus, setting

$$\hat a_{27}:=a_{27}+\frac{a_{28}}{\sqrt{2}}+a_{29}\omega^2+a_{30}\omega^3+\frac{a_{31}}{4}+\frac{a_{32}}{4\sqrt{2}}+a_{33}\omega^6,$$

we can assert that

$$S(x,e^x,u)>\sum_{i=0}^{26}a_i(x)u^i+\hat a_{27}(x)u^{27} \tag{6}$$

for all $x\in(-\frac{1}{2},-\frac{1}{2}+\varepsilon_0)$. Moreover, from (5), $\hat a_{27}(x)$ is continuous at $x=-\frac{1}{2}$, and

$$\lim_{x\longrightarrow-\frac{1}{2}}\hat a_{27}(x)=a_{27}\left(-1/2\right)=\frac{27}{16}\sqrt{\frac{2}{e^{21}}}.$$

By continuity, there exist $\varepsilon_1>0$ such that if $x\in(-\frac{1}{2},-\frac{1}{2}+\varepsilon_1)$, then $\hat a_{27}(x)>m>0$. Consequently, for each $x\in(-\frac{1}{2},-\frac{1}{2}+\varepsilon_1)$, we have that $S(x,e^x,u)$ tends to $+\infty$ as $u\longrightarrow+\infty$, and so $S(x,e^x,u)>0$ for $u>R$ with R big enough. To quantify it we fix x and apply Lemma 4.8 to the polynomial in u on the right-hand side of the inequality (6). We first verify, using CAP techniques, that we can take $\varepsilon_0=\varepsilon_1=5\times10^{-4}$. Using CAP methods again we prove that

$$\max\left\{\left|\frac{a_{26}(x)}{\hat a_{27}(x)}\right|,\left|\frac{a_{25}(x)}{\hat a_{27}(x)}\right|^{\frac{1}{2}},\dots,\left|\frac{a_0(x)}{2\hat a_{27}(x)}\right|^{\frac{1}{27}}\right\}<2.6 \text{ for all } x\in(-\tfrac{1}{2},-\tfrac{1}{2}+\varepsilon_1).$$

We obtain in this way $R = 5.2$. From (4), taking $z = u^2 - \frac{1}{2}$ into account, this shows that

$$D\left(x, e^x, z, \sqrt{\frac{2z+1}{2x+1}}e^x\right) > 0 \text{ for } x \in (-\tfrac{1}{2}, -\tfrac{1}{2} + \varepsilon_1) \text{ and } z > 26.54.$$

Finally, since $z = \sigma(x)$, it suffices to find $\varepsilon_2 < 5 \times 10^{-4}$ such that if $x \in (-\frac{1}{2}, -\frac{1}{2} + \varepsilon_2)$, then $\sigma(x) > 26.54$. Taking $\varepsilon_2 = 10^{-23}$ the result follows from the fact that $A(-\frac{1}{2} + \varepsilon_2) > A(26.54)$, which has been checked by computing a rigorous enclosure using computer interval arithmetic. In short, $W[\ell_0, \ell_1, \ell_2]$ is nonvashing on $I_1 = (-\frac{1}{2}, -\frac{1}{2} + 10^{-23})$.

Finally let us prove that $W[\ell_0, \ell_1, \ell_2]$ is nonvashing on $I_3 = (-0.1, 0)$. To this end recall first that, on account of Lemma 4.7, we have $\sigma(I) \subset \mu(I)$ for every interval $I \subset I_3$, where $\mu(x) = -x + [1, \frac{5}{3}]x^2$. Since in addition $e^x \in 1 + x + \frac{1}{2}e^{I_3}x^2$ and $e^{\sigma(x)} \in e^{\mu(x)} \subset 1 + \mu(x) + \frac{1}{2}e^{\mu(I_3)}\mu(x)^2$ for all $x \in I_3$, from (3) it follows that it suffices to verify that the range of the interval-valued function

$$h(x) := D\left(x, 1 + x + \frac{1}{2}e^{I_3}x^2, \mu(x), 1 + \mu(x) + \frac{1}{2}e^{\mu(I_3)}\mu(x)^2\right)$$

does not contain the zero. Some computations show that $h(x) = x^3\hat{p}(x)$, where $\hat{p}$ is a polynomial of degree 66 with interval coefficients. By computing a rigorous enclosure of the range of $\hat{p}$ by means of computer interval arithmetic we can assert that $\hat{p}\big([-0.1, 0]\big) \subset [1.677298 \times 10^{-61}, 1.308564 \times 10^{15}]$, and this proves that $0 \notin h(I_3)$ as desired. This completes the proof of the result. ■

Concerning the proof of Lemma 4.9, let us note that the inclusion $\hat{p}\big([-0.1, 0]\big) \subset (0, +\infty)$ is obtained by the *branch and bound method*, i.e., given an initial interval, in our case $[-0.1, 0]$, we subdivide it in smaller subintervals (branching) and check that $\hat{p}$ is positive in each one (bound). This is done recursively (see [29] for details about this method).

Lemma 4.10. *$W[\ell_0, \ell_1, \ell_2]$ is a nonvanishing on $I_2 = [-\frac{1}{2} + 10^{-23}, -0.1]$.*

Proof. This proof strongly relies on the rigorous evaluation of $W[\ell_0, \ell_1, \ell_2]$ on I_2 using interval arithmetic in a computer. The procedure goes as follows:

First, given any $x_0 \in (-\frac{1}{2}, 0)$, the combination of Lemmas 4.5 and 4.6 enables us to compute a rigorous interval enclosure of $\sigma(x_0)$. Indeed, on account of Lemma 4.5, we obtain an initial approximation interval $L_0 = \big[\inf(x_0), \sup(x_0)\big]$ that contains $\sigma(x_0)$. Then, by applying Lemma 4.6 we perform an iterative bisection process starting with L_0 that consists in bisecting the previous approximation interval and checking the sign of $z \longmapsto f(z, x_0)$ at its midpoint to decide which half contains $\sigma(x_0)$.

Second, given an interval $[a, b] \subset (-\frac{1}{2}, 0)$, by using the fact that the involution σ is decreasing, we can compute a rigorous enclosure of $\sigma([a, b]) = [\sigma(b), \sigma(a)]$ by applying the previous paragraph.

Third, we split the interval I_2 in two subintervals, namely $I_{21} = [-\frac{1}{2} + 10^{-23}, -0.4999999]$ and $I_{22} = [-0.4999999 - 10^{-10}, -0.1]$. In the interval I_{21} we apply the branch and bound method while checking that the rigorous interval enclosures of $S\left(x, e^x, \sqrt{\sigma(x) + \frac{1}{2}}\right)$ in equation (4) do not contain the zero. In the interval I_{22} we do the same but, instead of evaluating S, we evaluate the entries of the matrix that defines the Wronskain $W[\ell_0, \ell_1, \ell_2]$, and next we compute its determinant. ■

In the proof of Lemma 4.10 we use two different (but equivalent) expressions for the evaluation of the Wronskain $W[\ell_0, \ell_1, \ell_2]$. This is so because mathematically equivalent expressions may lead to different results due to the fact that the computations are done using floating point interval arithmetic. To be more precise let us denote by E_1 the expression used in the interval I_{21}, and by E_2 the one in I_{22}. It occurs that the evaluation of E_2 is faster (in computational time) than the evaluation of E_1. Hence, ideally it would be better to use E_2 combined with the branch and bound method. However, since E_2 has bigger slope than E_1 on I_{21}, the branch and bound method needs to split I_{21} in tiny intervals, which leads to worse performance in computational time compared with E_2.

Remark 4.11 The computations for the proof of Lemma 4.10 take ca six and a half hours on a desktop computer with a 2.8GHz CPU. □

On account of Lemmas 4.9 and 4.10 we have the following result.

Proposition 4.12. *$W[\ell_0, \ell_1, \ell_2]$ is a nonvanishing on $\left(-\frac{1}{2}, 0\right)$.*

To show that the second order leading principal minor of the Wronskian does not vanish we follow exactly the same approach as in the proof of the third order one. Thus, for the sake of shortness, we do not include here the proof of the following two lemmas, that we have proved with the help of computer-assisted methods.

Lemma 4.13. *$W[\ell_0, \ell_1]$ is a nonvanishing on $I_1 = (-\frac{1}{2}, -\frac{1}{2} + 10^{-23})$ and $I_3 = (-0.1, 0)$.*

Lemma 4.14. *$W[\ell_0, \ell_1]$ is a nonvanishing on $I_2 = [-\frac{1}{2} + 10^{-23}, -0.1]$.*

By the combination of Lemmas 4.13 and 4.14 we obtain the following result.

Proposition 4.15. *$W[\ell_0, \ell_1]$ is a nonvanishing on $\left(-\frac{1}{2}, 0\right)$.*

We are now in position to finish the proof of Theorem A.

Proof of Theorem A. By applying Corollary 3.5 we can assert that there exists $\varepsilon > 0$ such that $\left(\bar{J}_0, \bar{J}_1, \bar{J}_2\right)$ is an ECT-system on $[0, \varepsilon]$. In order to prove it on $(0, \frac{1}{2})$, recall Remark 4.3, it suffices to show that none of the leading principal minors of the Wronskian of $\left(\ell_0, \ell_1, \ell_2\right)$ vanish on $\left(-\frac{1}{2}, 0\right)$. This follows from Propositions 4.4, 4.12 and 4.15. Accordingly, on account of the characterization in Lemma 3.3, we conclude that $\left(\bar{J}_0, \bar{J}_1, \bar{J}_2\right)$ is an ECT-system on $[0, \frac{1}{2})$ and this completes the proof of the result. ■

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