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ON THE POLYNOMIAL INTEGRABILITY OF THE HOYER SYSTEMS

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ABSTRACT. The Hoyer polynomial differential systems depend on nine parameters. We provide necessary conditions in order that these systems have two functionally independent polynomial first integrals. We show that these conditions are not sufficient. Additionally, we illustrate how can be computed the polynomial first integrals of these systems using the Kovalevsky exponents.

1. Introduction and statement of the main results

Given a system of ordinary differential equations depending on parameters in general is very difficult to recognize for which values of the parameters the equations have first integrals because there are no satisfactory methods to answer this question.

In this paper we study the polynomial first integrals of the so-called Hoyer systems in $\mathbb{R}^3$ depending on nine parameters. These differential systems were introduced by P. Hoyer [6] in 1879 in his Ph.D Thesis. They have the form

$$\begin{aligned} \dot{x} &= ayz + bxz + cxy = P_1(x,y,z), \\ \dot{y} &= Ayz + Bxz + Cxy = P_2(x,y,z), \\ \dot{z} &= \alpha yz + \beta xz + \gamma xy = P_3(x,y,z), \end{aligned} \tag{1}$$

where $a, b, c, A, B, C, \alpha, \beta, \gamma \in \mathbb{R}$. These systems are the most general quadratic systems without self-interacting terms.

Among examples of Hoyer systems (1) there are the Euler systems describing the motion of a free rigid body, the (A,B,C) Lotka-Volterra systems and the Halphen systems [5]. Papers of Moullin-Ollagnier about polynomial first integrals [10] and rational first integrals of degree zero [11] for the three dimensional homogeneous Lotka-Volterra systems show that searching for those parameter values for which the system possess a first integral of a specified class is a very hard problem. The Hoyer systems admitting a quadratic polynomial first integral and a Poisson structure has been studied by Maciejewski and Przybylska in [9].

Given U an open set of $\mathbb{R}^3$, we say that a real non-constant function $H\colon U \to \mathbb{R}$ is a *first integral* if it is constant on every solution of system (1) contained in U, i.e., H satisfies

$$\frac{\partial H}{\partial x}P_1(x,y,z) + \frac{\partial H}{\partial y}P_2(x,y,z) + \frac{\partial H}{\partial z}P_3(x,y,z) = 0$$

on the points on U.

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The first integrals H_1, H_2 are *functionally independent* if the 2×3 matrix

$$\begin{pmatrix} \partial H_1/\partial x & \partial H_1/\partial y & \partial H_1/\partial z \\ \partial H_2/\partial x & \partial H_1/\partial y & \partial H_2/\partial z \end{pmatrix} (x, y, z)$$

has rank 2 at all points $(x, y, z) \in \mathbb{R}^3$ where they are defined with the exception (perhaps) of a zero Lebesgue measure set.

By definition a Hoyer system is *completely integrable* in an open set U if it has two first integrals functionally independent in U.

The first main result of this paper provides necessary conditions in order that the Hoyer system have two functionally independent polynomial first integrals.

Theorem 1. *In order that the Hoyer systems have two functionally independent polynomial first integrals one of the following eight conditions must hold:*

(2)

$$b = -A, \quad c = -\alpha, \quad C = -\beta;$$

$$b = -A, \quad c = -\alpha, \quad \beta C - B\gamma = \frac{pq}{(p+q)^2}(C + \beta)^2;$$

$$b = -A, \quad C = -\beta, \quad \alpha c - a\gamma = \frac{kl}{(k+l)^2}(\alpha + c)^2;$$

$$b = -A, \quad \alpha c - a\gamma = \frac{kl}{(k+l)^2}(\alpha + c)^2, \quad \beta C - B\gamma = \frac{pq}{(p+q)^2}(C + \beta)^2;$$

$$Ab - aB = \frac{mn}{(m+n)^2}(A + b)^2, \quad c = -\alpha, \quad C = -\beta;$$

$$Ab - aB = \frac{mn}{(m+n)^2}(A + b)^2, \quad c = -\alpha, \quad \beta C - B\gamma = \frac{pq}{(p+q)^2}(C + \beta)^2;$$

$$Ab - aB = \frac{mn}{(m+n)^2}(A + b)^2, \quad C = -\beta, \quad \alpha c - a\gamma = \frac{kl}{(k+l)^2}(\alpha + c)^2;$$

$$Ab - aB = \frac{mn}{(m+n)^2}(A + b)^2, \quad \alpha c - a\gamma = \frac{kl}{(k+l)^2}(\alpha + c)^2, \quad \beta C - B\gamma = \frac{pq}{(p+q)^2}(C + \beta)^2;$$

for some $p, q \in \mathbb{Z}$ coprime, $k, l \in \mathbb{Z}$ coprime and $m, n \in \mathbb{Z}$ coprime.

Theorem 1 is proved in section 2.

The eight conditions in order that the Hoyer systems can have two functionally independent polynomial first integrals given in Theorem 1 are necessary but not sufficient. Thus we shall provide Hoyer systems satisfying one of the eight conditions of Theorem 1 but having 0, 1 and 2 functionally independent polynomial first integrals.

Theorem 2. *All the following Hoyer systems satisfy the first condition of Theorem 1.*

(a) *The Hoyer system*

$$\dot{x} = -xy - xz + yz, \quad \dot{y} = -xy + xz + yz, \quad \dot{z} = xy + xz + yz, \tag{3}$$

has no polynomial first integrals.

(b) *The Hoyer system*

$$\dot{x} = -xy - xz + yz, \quad \dot{y} = -xy + xz + yz, \quad \dot{z} = xz + yz, \tag{4}$$

has no polynomial first integrals.

(c) *The Hoyer systems*

$$\dot{x} = ayz, \quad \dot{y} = -\beta xy + Bxz, \quad \dot{z} = \gamma xy + \beta xz, \tag{5}$$

with $aB\beta\gamma \neq 0$ have only one functionally independent polynomial first integral.

(d) *The Hoyer systems*

$$\dot{x} = ayz, \quad \dot{y} = Bxz, \quad \dot{z} = \gamma xy, \tag{6}$$

with $aB\gamma \neq 0$ have two functionally independent polynomial first integrals.

Theorem 2 is proved in section 4. We shall see in the proof of Theorem 2 that the ways of proving that the Hoyer systems in statements (a) and (b) have no polynomial first integrals are completely different.

Note that in the Hoyer systems the parameter space is 9–dimensional and it is not covered by the set of eight conditions of Theorem 1. Hence, it is possible to find Hoyer systems having only one functionally independent polynomial first integral which do not satisfy any of the eight conditions of Theorem 1. The next result shows this.

Theorem 3. *The Hoyer systems*

$$\dot{x} = yz, \quad \dot{y} = -xz + yz, \quad \dot{z} = \gamma xy, \tag{7}$$

with $\gamma \neq 0$ do not satisfy any of the eight conditions of Theorem 1, and it has only one functionally independent polynomial first integral.

Theorem 3 is proved in section 4.

The proofs of Theorems 1 and 3 rely on the fact that the Hoyer systems are homogeneous polynomial differential systems of degree 2 (and consequently they are quasi-homogeneous polynomial differential systems), and use the Yoshida's results for studying the polynomial first integrals of quasi-homogeneous polynomial differential systems. Hence we have summarized in section 3 Yoshida's results.

2. Proof of Theorem 1

As usual we denote by $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of non–negative integers, positive integers, real and complex numbers, respectively; and $\mathbb{C}[x_1, \cdots, x_n]$ denotes the polynomial ring over $\mathbb{C}$ in the variables $x_1, \cdots, x_n$. Here t can be real or complex. The following result, due to Zhang [16], will be used in a strong way in the proof of Theorem 1.

Theorem 4. *For an analytic vector field $\mathcal{X}$ defined in a neighborhood of the origin in $\mathbb{R}^n$ associated to system*

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n, \quad \mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \cdots, P_n(\mathbf{x})), \tag{8}$$

$P_i \in \mathbb{C}[x_1, \cdots, x_n]$ for $i = 1, \cdots, n$ and with $\mathbf{P}(\mathbf{0}) = \mathbf{0}$, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $D\mathbf{P}(\mathbf{0})$. Set

$$G = \left\{ (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n : \sum_{i=1}^{n} k_i \lambda_i = 0, \sum_{i=1}^{n} k_i > 0 \right\}.$$

Assume that system (8) has $r < n$ functionally independent analytic first integrals $\Phi_1(\mathbf{x}), \ldots, \Phi_r(\mathbf{x})$ in a neighborhood of the origin. If the $\mathbb{Z}$-linear space generated by G has dimension r, then any nontrivial analytic first integral of system (8) in a neighborhood of the origin is an analytic function of $\Phi_1(\mathbf{x}), \ldots, \Phi_r(\mathbf{x})$.

Extensions of Theorem 4 can be found in [1, 7].

We call each element $(k_1, \ldots, k_n) \in G$ a *resonant lattice* of the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Direct calculations show that the Hoyer systems (1) have three planes of singularities, $S_1 = (x, 0, 0)$, $S_2 = (0, y, 0)$ and $S_3 = (0, 0, z)$.

At the singularity $S_1 = (x, 0, 0)$, the 3-tuple of eigenvalues $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of the linear part of the Hoyer systems (1) are

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) = \left(0, \frac{(A_1 - \sqrt{B_1})x}{2}, \frac{(A_1 + \sqrt{B_1})x}{2}\right), \tag{9}$$

where

$$A_1 = \beta + C \quad \text{and} \quad B_1 = A_3^2 - 4\Delta_1, \quad \Delta_1 = \beta C - B\gamma.$$

From Theorem 4 we know that the number of functionally independent analytic first integrals of the Hoyer systems (1) in a neighborhood of the singularities S_1 is at most the number of linearly independent elements of the set

$$G_1 = \left\{(k_1, k_2, k_3) \in (\mathbb{Z}_+)^3 : \sum_{i=1}^{3} k_i\lambda_i = 0, \sum_{i=1}^{3} k_i > 0\right\}.$$

Consequently, the number of the functionally independent polynomial first integrals of the Hoyer systems (1) are at most the number of the linearly independent elements of G_1.

According to the eigenvalues (9) the resonant lattices satisfy

$$(A_1 - \sqrt{B_1})k_2 + (A_1 + \sqrt{B_1})k_3 = 0. \tag{10}$$

This last equation has the linearly independent non–negative solution $(k_1, k_2, k_3) = (1, 0, 0)$. In order that equation (10) has other linearly independent non–negative integer solutions different from the $(1, 0, 0)$, we must have

(i) either $(A_1 - \sqrt{B_1})(A_1 + \sqrt{B_1}) = 0$;
(ii) or $(A_1 - \sqrt{B_1})(A_1 + \sqrt{B_1}) \neq 0$ and $(A_1 - \sqrt{B_1})/(A_1 + \sqrt{B_1})$ is a rational number. Then $\Delta_1 \neq 0$. Assume that $A_1 \neq 0$ and set

$$(A_1 - \sqrt{B_1})/(A_3 + \sqrt{B_1}) = m_1/n_1, \quad m_1, n_1 \in \mathbb{Z} \setminus \{0\} \text{ coprime}.$$

This last equality can be written in an equivalent way as

$$\frac{\Delta_1}{A_1^2} = \frac{m_1 n_1}{(m_1 + n_1)^2}, \tag{11}$$

where we have used the fact that $B_1 = A_1^2 - 4\Delta_1$.

If $A_1 = 0$ then $(A_1 - \sqrt{B_1})/(A_1 + \sqrt{B_1}) = -1$.

In case (i) we obtain the independent solution $\beta C - B\gamma = 0$.

In case (ii) if $A_1 \neq 0$ then from (11) and the expressions of Δ_1 and A_1 it follows that

$$\beta C - B\gamma = \frac{m_1 n_1}{(m_1 + n_1)^2}(\beta + C)^2.$$

If $A_1 = 0$ then $\beta = -C$.

In short, by Theorem 4 the Hoyer systems can have two functionally independent polynomial first integrals if and only if cases (i) or (ii) hold.

At the singularity $S_2 = (0, y, 0)$, the 3-tuple of eigenvalues $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of the linear part of the Hoyer systems (1) are

$$\lambda = \left(0, \frac{(A_2 - \sqrt{B_2})y}{2}, \frac{(A_2 + \sqrt{B_2})y}{2}\right), \tag{12}$$

where

$$A_2 = \alpha + c \quad \text{and} \quad B_2 = A_2^2 - 4\Delta_2, \quad \Delta_2 = \alpha c - a\gamma.$$

According to the eigenvalues (12) the resonant lattices satisfy

$$(A_2 - \sqrt{B_2})k_2 + (A_2 + \sqrt{B_2})k_3 = 0. \tag{13}$$

This last equation has the linearly independent non–negative solution $(k_1, k_2, k_3) = (1, 0, 0)$. In order that equation (13) has other linearly independent non–negative integer solutions different from the $(1, 0, 0)$, we must have

(iii) either $(A_2 - \sqrt{B_2})(A_2 + \sqrt{B_2}) = 0$;

(iv) or $(A_2 - \sqrt{B_2})(A_2 + \sqrt{B_2}) \neq 0$ and $(A_2 - \sqrt{B_2})/(A_2 + \sqrt{B_2})$ is a rational number. Then $\Delta_2 \neq 0$. Assume that $A_2 \neq 0$ and set

$$(A_2 - \sqrt{B_2})/(A_2 + \sqrt{B_2}) = m_2/n_2, \quad m_2, n_2 \in \mathbb{Z} \setminus \{0\} \text{ coprime}.$$

This last equality can be written in an equivalent way as

$$\frac{\Delta_2}{A_2^2} = \frac{m_2 n_2}{(m_2 + n_2)^2}, \tag{14}$$

where we have used the fact that $B_2 = A_2^2 - 4\Delta_2$.

If $A_2 = 0$ then $(A_2 - \sqrt{B_2})/(A_2 + \sqrt{B_2}) = -1$.

In case (iii) we obtain the independent solution $\alpha c - a\gamma = 0$.

In case (iv) if $A_2 \neq 0$ then from (14) and the expressions of Δ_2 and A_2 we have

$$\alpha c - a\gamma = \frac{m_2 n_2}{(m_2 + n_2)^2}(\alpha + c)^2.$$

If $A_2 = 0$ then $c = -\alpha$.

In short, by Theorem 4 the Hoyer systems can have two functionally independent polynomial first integrals if and only if cases (iii) or (iv) hold.

At the singularity $S_3 = (0, 0, z)$, the 3-tuple of eigenvalues $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of the linear part of the Hoyer systems (1) are

$$\lambda = \left(0, \frac{(A_3 - \sqrt{B_3})z}{2}, \frac{(A_3 + \sqrt{B_3})z}{2}\right), \tag{15}$$

where

$$A_3 = A + b \quad \text{and} \quad B_3 = A_3^2 - 4\Delta_3, \quad \Delta_3 = Ab - aB.$$

According to the eigenvalues (15) the resonant lattices satisfy

$$(A_3 - \sqrt{B_3})k_2 + (A_3 + \sqrt{B_3})k_3 = 0. \tag{16}$$

This last equation has the linearly independent non–negative solution $(k_1, k_2, k_3) = (1, 0, 0)$. In order that equation (16) has other linearly independent non–negative integer solutions different from the $(1, 0, 0)$, we must have

(v) either $(A_3 - \sqrt{B_3})(A_3 + \sqrt{B_3}) = 0$;

(vi) or $(A_3 - \sqrt{B_3})(A_3 + \sqrt{B_3}) \neq 0$ and $(A_3 - \sqrt{B_3})/(A_3 + \sqrt{B_3})$ is a rational number. Then $\Delta_3 \neq 0$. Assume that $A_3 \neq 0$ and set

$$(A_3 - \sqrt{B_3})/(A_3 + \sqrt{B_3}) = m_3/n_3, \quad m_3, n_3 \in \mathbb{Z} \setminus \{0\} \text{ coprime}.$$

This last equality can be written in an equivalent way as

$$\frac{\Delta_3}{A_3^2} = \frac{m_3 n_3}{(m_3 + n_3)^2}, \tag{17}$$

where we have used the fact that $B_1 = A_3^2 - 4\Delta_3$.

If $A_3 = 0$ then $(A_3 - \sqrt{B_3})/(A_3 + \sqrt{B_3}) = -1$.

In case (v) we obtain the independent solution $Ab - aB = 0$.

In case (vi) if $A_3 \neq 0$ then from (17) and the expressions of Δ_1 and A_1 we get

$$Ab - aB = \frac{m_3 n_3}{(m_3 + n_3)^2}(A + b)^2.$$

If $A_3 = 0$ then $b = -A$.

In short, by Theorem 4 the Hoyer systems can have two functionally independent polynomial first integrals if and only if cases (v) or (vi) hold.

Putting together the six necessary conditions (i)–(vi) we get the eight necessary conditions described in the statement of Theorem 1.

3. Quasi-homogeneous polynomial differential systems

In this section we summarize some basic results on the analytic and polynomial integrability of the polynomial differential systems as in (8).

The polynomial differential system (8) is *quasi–homogeneous* if there exist $\mathbf{s} = (s_1, \cdots, s_n) \in \mathbb{N}^n$ and $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$,

$$P_i(\alpha^{s_1} x_1, \cdots, \alpha^{s_n} x_n) = \alpha^{s_i - 1 + d} P_i(x_1, \cdots, x_n),$$

for $i = 1, \ldots, n$. We call $\mathbf{s} = (s_1, \cdots, s_n)$ the *weight exponent* of system (8), and d the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case that $\mathbf{s} = (1, \cdots, 1)$ system (8) is the classical *homogeneous polynomial differential system of degree d.*

Note that if the polynomial differential system (8) is quasi–homogeneous with weight exponent $\mathbf{s}$ and weight degree $d > 1$, then the system is invariant under the change of variables $x_i \to \alpha^{w_i} x_i$, $t \to \alpha^{-1} t$, where $w_i = s_i/(d-1)$.

Recently the integrability of quasi–homogeneous polynomial differential systems have been investigated by several authors. Probably the best results have been provided by Yoshida [13, 14, 15], Furta [2] and Goriely [4], see also Tsygvintsev [12] and Llibre and Zhang [8].

A non–constant function $F(x_1, \ldots, x_n)$ is a *first integral* of system (8) if it is constant on all solution curves $(x_1(t), \cdots, x_n(t))$ of system (8); i.e. $F(x_1(t), \cdots, x_n(t)) =$ constant for all values of t for which the solution $(x_1(t), \cdots, x_n(t))$ is defined. If F is C^1, then F is a first integral of system (8) if and only if

$$\sum_{i=1}^{n} P_i \frac{\partial F}{\partial x_i} \equiv 0.$$

The function $F(x_1, \ldots, x_n)$ is *quasi–homogeneous of weight degree* m *with respect to the weight exponent* $\mathbf{s}$ if it satisfies

$$F(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n) = \alpha^m F(x_1, \ldots, x_n),$$

for all $\alpha \in \mathbb{R}^+$.

Given an analytic function F we can expand it in the form $F = \sum_i F^i$, where F^i is a quasi–homogeneous polynomial of weight degree i with respect to the weight exponent $\mathbf{s}$; i.e. $F^i(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n) = \alpha^i F^i(x_1, \ldots, x_n)$. The following result is well known, see for instance Proposition 1 of [8].

Proposition 5. *Let F be an analytic function and let $F = \sum_i F^i$ be its decomposition into weight–homogeneous polynomials of weight degree i with respect to the weight exponent $\mathbf{s}$. Then F is an analytic first integral of the weight–homogeneous polynomial differential system* (8) *with weight exponent $\mathbf{s}$ if and only if each weight homogeneous part F^i is a first integral of system* (8) *for all i.*

Suppose that system (8) is a quasi–homogeneous polynomial differential system of weight degree d with respect to the weight exponent $\mathbf{s}$. Then we define $\mathbf{w} = \mathbf{s}/(d-1)$. The interest for the quasi–homogeneous polynomial differential systems is based in the existence of the particular solutions of the form

$$(x_1(t), \ldots, x_n(t)) = \left(c_1 t^{-w_1}, \ldots, c_n t^{-w_n}\right),$$

where the coefficients $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ are given by the algebraic system of equations

$$P_i(c_1, \ldots, c_n) + w_i c_i = 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{18}$$

For a given $(w_1, \ldots, w_n)$ there may exist different $\mathbf{c}$'s, called the *balances.*

For each balance $\mathbf{c}$ we introduce a matrix

$$K(\mathbf{c}) = D\mathbf{P}(\mathbf{c}) + \mathrm{diag}(w_1, \ldots, w_n), \tag{19}$$

where as usual $D\mathbf{P}(\mathbf{c})$ denotes the differential of $\mathbf{P}$ evaluated at $\mathbf{c}$, and $\mathrm{diag}(w_1, \ldots, w_n)$ denotes the matrix whose diagonal is equal to $(w_1, \ldots, w_n)$ and zeros in the rest.

The eigenvalues of $K(\mathbf{c})$ are called the *Kowalevsky exponents* of the balance $\mathbf{c}$. Sophia Kowalevskaya was the first to introduce the matrix K to compute the Laurent series solutions of the rigid body motion. It can be shown that there always exists a Kowalevsky exponent equal to -1 related to the arbitrariness of the origin of the parametrization of the solution by the time. The eigenvector associated to the eigenvalue -1 is $(w_1 c_1, \ldots, w_n c_n)$, for more details see [13] or [2].

Probably the best results in order to know if a weight homogeneous polynomial of weight degree m with respect to the weight exponent $\mathbf{s}$ is a first integral of a quasi–homogeneous polynomial differential system (8) with weight degree d with respect to the weight exponent $\mathbf{s}$ is essentially due to Yoshida [13], and are the following two theorems.

Theorem 6. *Let $F(x_1, \ldots, x_n)$ be a weighted homogeneous polynomial first integral of weight degree m with respect to the weight exponent $\mathbf{s}$ of the quasi–homogeneous polynomial differential system* (8) *with weight degree d with respect to the weight exponent $\mathbf{s}$. Suppose the gradient of F evaluated at a balance $\mathbf{c}$ is finite and not identically zero. Then $m/(d-1)$ is a Kowalevsky exponent of the balance $\mathbf{c}$.*

The proof of Theorem 6 is based on using the variational equation along convenient particular solutions.

Theorem 7. *Let r be a positive integer such that $1 < r < n$, and let $F_k(x_1, \ldots, x_n)$ for $k = 1, \ldots, r$ be weighted homogeneous polynomial first integrals of weight degree m with respect to the weight exponent $\mathbf{s}$ of the quasi–homogeneous polynomial differential system (8) with weight degree d with respect to the weight exponent $\mathbf{s}$.*

Suppose that the gradients of F_k for $k = 1, \ldots, r$ evaluated at a balance $\mathbf{c}$ are finite, not identically zero and functionally independent. Then $m/(d-1)$ is a Kowalevsky exponent of the balance $\mathbf{c}$ with multiplicity at least r.

In fact Yoshida in [13] published Theorems 6 and 7 with m instead of $m/(d-1)$. Later on this was corrected, see [2, 3, 14].

4. Proof of Theorems 2 and 3

In this section we prove Theorems 2 and 3.

Proof of Theorem 2. The Hoyer system (3) satisfies the first condition of Theorem 1. It has four balances one of which is $\mathbf{c} = (1, -1, 1)$. The corresponding Kowalevsky exponents are $(2 - i\sqrt{3}, 2 + i\sqrt{3}, -1)$. In view of Theorem 6, system (3) has no polynomial first integrals. This concludes the proof of statement (a).

The Hoyer system (4) satisfies the first condition of Theorem 1. It has four balances one of which is $\mathbf{c} = (1, 1, 0)$. The corresponding Kowalevsky exponents are $(3, 1, -1)$. In view of Theorem 6, system (4) can have at most two functionally independent polynomial first integrals of degrees 1 and 3, and any other polynomial first integral must be functionally dependent on them. Furthermore such a polynomial first integrals must be homogeneous. We will see that such first integrals do not exist. Let H be a homogeneous polynomial of degree one. We write it as $H = d_1x + d_2y + d_3z$. Imposing that H is a first integral we get that $d_1 = d_2 = d_3 = 0$, i.e., $H = 0$ in contradiction with the fact that H is a first integral. Analogously, if H is a homogeneous polynomial of degree three, we write it as $H = d_1x^3 + d_2x^2y + d_3x^2z + d_4xy^2 + d_5xyz + d_6xz^2 + d_7y^3 + d_8y^2z + d_9yz^2 + d_{10}z^3$. Imposing that H is a first integral we get that $d_1 = d_2 = \cdots = d_{10} = 0$, i.e. $H = 0$ in contradiction with the fact that H is a first integral. This concludes the proof of statement (b).

The Hoyer system (5) satisfies the first condition of Theorem 1. It has four balances one of which is

$$\mathbf{c} = \Big(\frac{1}{S}, -\frac{\sqrt{S^2 - \beta S}}{\gamma\sqrt{aBS^2}}(S^2 + \beta S), \frac{\sqrt{S^2 - \beta S}}{\sqrt{aBS}}\Big)$$

with $S = \sqrt{\beta^2 + B\gamma}$. The corresponding Kowalevsky exponents are $(2, 2, -1)$. In view of Theorem 7, system (5) can have at most two functionally independent polynomial first integrals of degree 2, and any other polynomial first integral must be functionally dependent on them. Furthermore such a polynomial first integrals must be homogeneous. Let H be a first integral of degree two. We write it as $H = d_1x^2 + d_2xy + d_3xz + d_4y^2 + d_5yz + d_6z^2$. Imposing that H is a first integral we get that $d_1 = d_2 = d_3 = 0$, $d_4 = -\gamma d_6/B$ and $d_5 = -2\beta d_6/B$. This implies that system (6) has at most one functionally independent polynomial first integral, and a polynomial first integral is $H = -\gamma y^2 - 2\beta yz + Bz^2$. This concludes the proof of statement (c).

The Hoyer system (6) satisfies the first condition of Theorem 1. It has four balances one of which is

$$\mathbf{c} = \Big(-\frac{1}{\sqrt{B\gamma}}, -\frac{1}{\sqrt{a\gamma}}, -\frac{1}{\sqrt{aB}} \Big).$$

The corresponding Kowalevsky exponents are $(2, 2, -1)$. In view of Theorem 7, system (6) can have at most two functionally independent polynomial first integrals of degree 2, and any other polynomial first integral must be functionally dependent on them. Direct computations show that

$$H_1 = Bx^2 - ay^2 \quad \text{and} \quad H_2 = \gamma y^2 - Bz^2$$

are two functionally independent polynomial first integrals. This concludes the proof of statement (d). $\square$

Proof of Theorem 3. It is easy to check that the Hoyer systems (7) do not satisfy any of the eight conditions of Theorem 1. It has four balances one of which is

$$\mathbf{c} = \Big(-\frac{1+i\sqrt{3}}{2\sqrt{\gamma}}, -\frac{1}{\sqrt{\gamma}}, -\frac{1+i\sqrt{3}}{2} \Big).$$

The corresponding Kowalevsky exponents are $(2, (3 - i\sqrt{3})/2, -1)$. In view of Theorem 6, systems (7) can have at most one functionally independent polynomial first integral of degree 2, and any other polynomial first integral must be functionally independent of it. Furthermore such a polynomial first integral must be homogeneous. Let H be a first integral of degree two. We write it as $H = d_1x^2 + d_2xy + d_3xz + d_4y^2 + d_5yz + d_6z^2$. Imposing that H is a first integral we get that $d_2 = d_3 = d_4 = d_5 = 0$ and $d_1 = -\gamma d_6$. Then a polynomial first integral is $H = -\gamma x^2 + z^2$. This concludes the proof of the theorem. $\square$

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$$\dot{x}_2 = a_2x_2x_3 + b_2x_3x_1 + c_2x_1x_2,$$
$$\dot{x}_3 = a_3x_2x_3 + b_3x_3x_1 + c_3x_1x_2,$$

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