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DYNAMICS OF THE POLYNOMIAL DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS NONLINEARITIES AND A STAR NODE

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Abstract. We consider the class of polynomial differential equations $\dot{x} = \lambda x + P^n(x, y)$, $\dot{y} = \lambda y + Q^n(x, y)$, in $\mathbb{R}^2$ where $P^n(x, y)$ and $Q^n(x, y)$ are homogeneous polynomials of degree $n > 1$ and $\lambda \neq 0$, i.e. the class of polynomial differential systems with homogeneous nonlinearities with a star node at the origin.

We prove that these systems are Darboux integrable. Moreover, for these systems we study the existence and non–existence of limit cycles surrounding the equilibrium point located at the origin.

1. Introduction and statement of the main results

By definition a two dimensional *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y, and the independent one (the time) t are real, and $P(x, y)$ and $Q(x, y)$ are polynomials in the variables x and y with real coefficients. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let U be a non–empty open and dense subset of $\mathbb{R}^2$. We say that a non–locally constant C^1 function $H : U \to \mathbb{R}$ is a *first integral* of the polynomial differential system (1) in U if H is constant on the trajectories of the polynomial differential system (1) contained in U, i.e. if

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} P + \frac{\partial H}{\partial y} Q = 0,$$

in the points of U.

It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait. Thus for polynomial differential systems a natural question arises: *Given a polynomial differential system in* $\mathbb{R}^2$*, how to recognize if it has a first integral*?

The easiest planar differential systems having a first integral are the Hamiltonian ones. The integrable planar differential systems which are not Hamiltonian are, in general, very difficult to detect. For non–Hamiltonian differential systems many different methods have been used for studying the existence of a first integral. These methods are based on: Noether symmetries [3], the Darbouxian theory of integrability [9], the Lie symmetries [27], the Painlevé analysis [2], the use of Lax

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pairs [19], the direct method [14] and [17], the linear compatibility analysis method [30], the Carlemann embedding procedure [6] and [1], the quasimonomial formalism [2], ... For a general introduction to the integrability of planar differential systems see the book of Goriely [16].

The problem of the integrability of the planar polynomial differential systems is very classical, but there is another classical problem related with the planar polynomial differential systems, the second part of the 16–th Hilbert problem. This problem essentially consists in finding a uniform upper bound for the maximum number of limit cycles that a planar polynomial differential systems of a given degree can exhibit, see for more details the surveys [18] and [22].

Roughly speaking an *elementary function* is a function which is composition of polynomials, exponentials, logarithmic and algebraic functions. Again roughly speaking a *Liouvillian function* is a function that can be expressed by quadratures of elementary functions. Note that the class of all elementary functions is a particular subclass of the class of all Liouvillian functions. A polynomial differential system is *Darboux integrable* if it has a first integral which is a Liouvillian function. For precise definitions of elementary and Liouvillian functions, and Darboux integrability see the works of Prelle and Singer [28] and of Singer [29], respectively.

Let $F = F(x, y)$ be a real polynomial not identically zero. The algebraic curve $F(x, y) = 0$ is an *invariant algebraic curve* of the polynomial differential system (1) if for some polynomial $K = K(x, y)$ we have

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = KF. \tag{2}$$

On the points of the algebraic curve $F = 0$ the gradient $(\partial F/\partial x, \partial F/\partial y)$ of F is orthogonal to the vector field (P, Q) (see (2)). Hence at every point of $F = 0$ the vector field (P, Q) is tangent to the curve $F = 0$, so the curve $F = 0$ is formed by trajectories of the vector field (P, Q). This justifies the name "invariant algebraic curve" because it is invariant under the flow of system (1). There is a strong relation between the invariant algebraic curves and the Darboux theory of integrability, see for instance the Chapter 8 of [10].

We consider the polynomial differential systems of the form

$$\dot{x} = \lambda x + P^n(x, y), \qquad \dot{y} = \lambda y + Q^n(x, y), \tag{3}$$

defined in $\mathbb{R}^2$ where $\lambda \neq 0$, $n > 1$, and $P^n(x, y)$ and $Q^n(x, y)$ are homogeneous polynomials of degree n.

Note that the polynomial differential systems (3) have a star node at the origin (i.e. a node with equal eigenvalues), and that those systems have homogeneous nonlinearities. These systems for $n = 2$ have been completely studied in the book of Ye Yanqian et al. [31], where they proved that they are Darboux integrable and have no periodic solutions, and consequently no limit cycles. Recall that a *limit cycle* of a system (3) is an isolated periodic solution in the set of all periodic solutions of system (3).

Our first result shows that the polynomial differential systems (3) for all $n > 1$ always are Darboux integrable, and we provide an explicit Liouvillian first integral for them and an invariant algebraic curve.

Theorem 1. *Consider a polynomial differential system with homogeneous nonlinearities* (3) *with* $\lambda \neq 0$ *and* $n > 1$.

(a) *The curve* $xQ^n(x,y) - yP^n(x,y) = 0$ *is an invariant algebraic curve of system* (3).

(b) *System* (3) *is Darboux integrable with the Liouvillian first integral*

$$H = (x^2+y^2)^{\frac{n-1}{2}} e^{(1-n)\int^{\arctan\frac{y}{x}} \frac{f(\theta)}{g(\theta)}d\theta} + (1-n)\lambda \int^{\arctan\frac{y}{x}} \frac{e^{(1-n)\int^{\theta}\frac{f(\mu)}{g(\mu)}d\mu}}{g(\theta)} d\theta,$$

where

$$\begin{aligned} f(\theta) &= \cos\theta P_n(\cos\theta,\sin\theta) + \sin\theta Q_n(\cos\theta,\sin\theta),\\ g(\theta) &= \cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta P_n(\cos\theta,\sin\theta). \end{aligned}$$

Theorem 1 is proved in section 2.

Our second result is about the periodic solutions of the differential systems (3) surrounding the origin.

Theorem 2. *Consider a polynomial differential system with homogeneous nonlinearities* (3) *with* $\lambda \neq 0$ *and* $n > 1$.

(a) *If* n *is even, then system* (3) *has no periodic solutions surrounding the origin.*

(b) *If* n *is odd and* $g(\theta)$ *vanishes for some* $\theta \in [0, 2\pi)$, *then system* (3) *has no periodic solutions surrounding the origin.*

(c) *If* n *is odd,* $g(\theta) \neq 0$ *for all* $\theta \in [0, 2\pi)$, *the origin is the unique equilibrium point of system* (3), *and*

$$\frac{\lambda}{g(\theta)} \int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta < 0,$$

then system (3) *has at least one limit cycle surrounding the origin.*

(d) *System* (3) *has at most one limit cycle surrounding the origin.*

(e) *The system*

$$\dot{x} = -x + x^3 - x^2y + xy^2 - y^3, \qquad \dot{y} = -y + x^3 + x^2y + xy^2 + y^3,$$

satisfies all the assumptions of statement (c), *and it has a unique cycle* $x^2+y^2 = 1$. *Moreover, this system has the first integral* $H = (x^2+y^2-1)e^{-2\arctan(y/x)}$.

Theorem 2 is proved in section 2.

On the other hand, the polynomial differential systems

$$\dot{x} = -y + P^n(x,y), \qquad \dot{y} = x + Q^n(x,y),$$

defined in $\mathbb{R}^2$ where $n > 1$ and $P^n(x,y)$ and $Q^n(x,y)$ are homogeneous polynomials of degree n with a linear center at the origin (instead of a star node) have been intensively studied for their limit cycles, centers and integrability, see for instance [4, 5, 7, 8, 12, 13, 15, 20, 21, 24, 26]. Other work close to the one here studied is the paper [11] where the authors studied the dynamics of the systems of the form

$$\dot{x} = P_n(x,y) + xR_m(x,y), \qquad \dot{y} = Q_n(x,y) + yR_m(x,y),$$

where P_n, Q_n and R_m are homogeneous polynomials of degrees n, n and $m \geq n$, respectively.

2. Proof of Theorems 1 and 2

In what follows F_x denotes the partial derivative of the function $F = F(x, y)$ with respect to the variable x, and similar for F_y.

In the proofs of statements (a) and (b) of Theorem 1 we shall use the following notations $F = xQ^n(x, y) - yP^n(x, y)$, $P(x, y) = \lambda x + P^n(x, y)$ and $Q(x, y) = \lambda y + Q^n(x, y)$.

Proof of statement (*a*) *of Theorem 1.* We must prove that $F = 0$ is an invariant algebraic curve of the differential system (3). Indeed, we have

$$\frac{\partial F}{\partial x}P + \frac{\partial F}{\partial y}Q = \frac{\partial F}{\partial x}\lambda x + \frac{\partial F}{\partial y}\lambda y + \frac{\partial F}{\partial x}P^n + \frac{\partial F}{\partial y}Q^n.$$

Then, taking into account that

$$xP^n_x + yP^n_y = nP^n \quad \text{and} \quad xQ^n_x + yQ^n_y = nQ^n$$

due to the Euler's theorem for homogeneous functions, we obtain

$$\begin{aligned}\frac{\partial F}{\partial x}\lambda x + \frac{\partial F}{\partial y}\lambda y = & \quad (Q^n + xQ^n_x - yP^n_x)\lambda x + (xQ^n_y - P^n - yP^n_y)\lambda y \\ = & \quad \lambda\big(x(Q^n + xQ^n_x + yQ^n_y) - y(P^n + xP^n_x + yP_y)\big) \\ = & \quad (n+1)\lambda(xQ^n - yP^n) = (n+1)\lambda F.\end{aligned}$$

On the other hand, substituting

$$xQ^n_x = nQ^n - yQ^n_y \quad \text{and} \quad yP^n_y = nP^n - xP^n_x,$$

in what follows, we get

$$\begin{aligned}\frac{\partial F}{\partial x}P^n + \frac{\partial F}{\partial y}Q^n = & \quad (Q^n + xQ^n_x - yP^n_x)P^n + (xQ^n_y - P^n - yP^n_y)Q^n \\ = & \quad \big(xP^nQ^n_x - yP^nP^n_x + xQ^nQ^n_y - yQ^nP^n_y\big) \\ = & \quad \big(-yP^nQ^n_y - yP^nP^n_x + xQ^nQ^n_y + xQ^nP^n_x\big) \\ = & \quad (P^n_x + Q^n_y)(xQ^n - yP^n) = (P^n_x + Q^n_y)F.\end{aligned}$$

In short, we have

$$\frac{\partial F}{\partial x}P + \frac{\partial F}{\partial y}Q = \big((n+1)\lambda + P^n_x + Q^n_y\big)F.$$

Therefore, $F = 0$ is an invariant algebraic curve of the polynomial differential system (3). Hence, statement (a) is proved. □

Proof of statement (*b*) *of Theorem 1.* In polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, system (3) becomes

$$\dot{r} = \lambda r + f(\theta)r^n, \qquad \dot{\theta} = g(\theta)r^{n-1}, \tag{4}$$

being $f(\theta)$ and $g(\theta)$ the functions defined in statement (b) of Theorem 1. Note that $f(\theta)$ and $g(\theta)$ are homogeneous trigonometric polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$.

The differential system (4) where $g(\theta) \neq 0$ can be written as the equivalent the differential equation

$$\frac{dr}{d\theta} = \frac{\lambda}{g(\theta)}\frac{1}{r^{n-2}} + \frac{f(\theta)}{g(\theta)}r. \tag{5}$$

Now we change the variable r by the new variable $\rho = r^{n-1}$, and the differential equation (5) becomes the linear differential equation

$$\frac{d\rho}{d\theta} = (n-1)\frac{\lambda}{g(\theta)} + (n-1)\frac{f(\theta)}{g(\theta)}\rho. \tag{6}$$

Solving it we find the first integral

$$H = \rho e^{(1-n)\int \frac{f(\theta)}{g(\theta)}\,d\theta} + (1-n)\lambda \int \frac{e^{(1-n)\int^{\theta} \frac{f(s)}{g(s)}\,ds}}{g(\theta)}\,d\theta.$$

Going back through the changes of variables we obtain the first integral of the statement (b) of Theorem 1. Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (4) is Darboux integrable. □

Proof of statement (*a*) *of Theorem 2.* By statement (a) of Theorem 1 we know that $F = xQ^n(x,y) - yP^n(x,y) = 0$ is an invariant algebraic curve of system (3). Since F is a homogeneous polynomial of degree $n+1$ odd, because for assumptions n is even, we have that F has at least a linear factor of the form $ax + by \not\equiv 0$.

It is well known (see for instance Proposition 8.4 of [10]) that if $F = 0$ is an invariant algebraic curve, then any factor of F is also an invariant algebraic curve. So the straight line $ax + by = 0$ through the origin of coordinates is invariant, i.e. formed by solutions of system (3). Therefore, it cannot be periodic solutions surrounding the origin. This completes the proof of statement (a). □

Proof of statement (*b*) *of Theorem 2.* If $\theta^* \in [0, 2\pi)$ is a zero of $g(\theta) = 0$, then $\sin\theta\, x - \cos\theta\, y$ is a factor of $xQ^n(x,y) - yP^n(x,y)$, and consequently by statement (a) of Theorem 1 the straight line $\sin\theta\, x - \cos\theta\, y = 0$ is invariant. Therefore, there are no periodic solutions surrounding the origin. Hence, statement (b) is proved. □

Proof of statement (*c*) *of Theorem 2.* Since there are no zeros $\theta^* \in [0, 2\pi)$ of $g(\theta) = 0$, there are no real linear factors of the homogeneous polynomial $xQ^n(x,y) - yP^n(x,y)$. Therefore, the Poincaré compactified vector field $p(\mathcal{X})$ of $\mathcal{X} = (\lambda x + P^n(x,y), \lambda y + Q^n(x,y))$ has a periodic orbit at infinity, see for details the Appendix.

Now we shall use the notation and the expressions of the proof of statement (b) of Theorem 1. System (3) in coordinates (ρ, θ) can be written as the differential equation (6). From (6) it follows that the stability of the origin $\rho = 0$ is controlled by the dominant term in the right hand part of (6) when $\rho > 0$ is sufficiently small, i.e. by the sign of $\lambda/g(\theta)$, recall that either $g(\theta) > 0$, or $g(\theta) < 0$, for all $\theta \in [0, 2\pi)$. So, the origin of the differential equation (6) is stable if $\lambda/g(\theta) < 0$ and unstable if $\lambda/g(\theta) > 0$.

Note that for arriving to the differential equation (6) from the system (3) we have changed the independent variable t by θ doing $d\theta = \rho g(\theta)dt$, so if $g(\theta) < 0$ we have changed the orientation of all the orbits, and in particular the stability of the origin and the stability of the periodic orbit at infinity.

The infinity of system (6) corresponds to $\rho = \infty$. Doing the change of variables $R = 1/\rho$ the infinity pass to the origin of the differential equation

$$\frac{dR}{d\theta} = (1-n)\frac{Rf(\theta)}{g(\theta)} + (1-n)\frac{\lambda R^2}{g(\theta)}. \tag{7}$$

We can think this differential equation as the differential system

$$
\begin{aligned}
\frac{dR}{d\theta} &= (1-n)\frac{Rf(\theta)}{g(\theta)} + (1-n)\frac{\lambda R^2}{g(\theta)} = f_R,\\
\frac{d\theta}{d\theta} &= 1 = f_\theta,
\end{aligned}
\tag{8}
$$

on the cylinder $(R,\theta) \in \mathbb{R} \times \mathbb{S}^1$ restricted to $R \geq 0$. The periodic orbit of the infinity corresponds now to the periodic orbit $R = 0$ on the cylinder. The kind of stability of the periodic orbit $R = 0$ is determined by the sign of the integral I between 0 and 2π of the divergence of the differential system (8) evaluated at $R = 0$ if this sign is different from zero, if $I < 0$ the periodic orbit is stable, and if $I > 0$ it is unstable, for more details see for instance Theorem 1.23 of [10]. In polar coordinates (R,θ) the expression of the divergence of system (8) is

$$\frac{1}{R}\frac{\partial(Rf_R)}{\partial R} + \frac{1}{R}\frac{\partial f_\theta}{\partial \theta}.$$

Consequently

$$I = \text{sign}\left(2\int_0^{2\pi}(1-n)\frac{f(\theta)}{g(\theta)}d\theta\right) = \text{sign}\left(-\int_0^{2\pi}\frac{f(\theta)}{g(\theta)}d\theta\right),$$

The stability of the origin is given by the sign of $(n-1)\lambda/g(\theta)$ (see (6)), i.e. if this sign is negative the origin is stable, and if it is positive the origin is unstable. In short, if

$$\text{sign}\left(\frac{\lambda}{g(\theta)}\int_0^{2\pi}\frac{f(\theta)}{g(\theta)}d\theta\right) < 0,$$

then the origin of system (3) and its periodic orbit at infinity have the same stability. Since by assumption the unique equilibrium point is the origin, then by the Poincaré–Bendixson theorem (see for instance Theorem 1.25 and Corollary 1.30 of [10]) it follows that there is a periodic orbit γ surrounding the origin, recall that any periodic orbit of a planar differential system must surround at least one equilibrium point (see for instance Theorem 1.31 of [10]).

Now we shall prove that the periodic orbit γ is a limit cycle. We denote by $\rho^*(\theta)$ the periodic solution of system (8) corresponding to the periodic orbit γ, and we denote by $\rho(\theta,\rho_0)$ the solution of system (8) such that $\rho(0,\rho_0) = \rho_0$. Since $d\theta/d\theta = 1$, it follows that the Poincaré map $\Pi : (0,\infty) \to (0,\infty)$ given by $\Pi(\rho_0) = \rho(2\pi,\rho_0)$ is well defined. Due to the fact that the differential equation (8) is analytic, the map Π is analytic. Since $\rho^*(\theta)$ is a periodic solution $\rho^*(0)$ is a fixed point of the map Π. If this fixed point is not isolated in the set of all fixed points of the map Π, we have that Π is the identity because Π is analytic, but Π cannot be the identity because the origin is a node, otherwise it will be a center. Hence, the fixed point $\rho^*(0)$ of the map Π is isolated, and consequently the periodic orbit γ is a limit cycle. This completes the proof of statement (c). □

Proof of statement (d) *of Theorem 2.* Our polynomial differential system (3) when it can have periodic orbits surrounding the origin it can be written as the Riccati differential equation (7), see the proof of the statement (c) of Theorem 2. It is well known that a Riccati differential equation either has a continuum of periodic orbits, or it has at most two periodic orbits, see for instance [23, 25]. Note that our Riccati equation (7) already has a periodic the orbit $R = 0$, which corresponds to the infinity. So it has at most one periodic orbit without taking into account the one at infinity.

In short system (3) has a continuum of periodic orbits or at most one periodic orbit surrounding the origin, but since it has a node at the origin it cannot have a continuum of periodic orbits surrounding it. Hence the statement (d) is proved. □

Proof of statement (e) of Theorem 2. For the differential system of the statement (d) we have $n = 3$, $\lambda = -1$, $f(\theta) = g(\theta) = 1$, and consequently

$$\frac{\lambda}{g(\theta)} \int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta = -2\pi < 0.$$

Of course, the origin is an equilibrium point for that system. Since the system in polar coordinates becomes

$$\dot{r} = r(r^2 - 1), \quad \dot{\theta} = r^2.$$

It is clear that the unique equilibrium point of the system is the origin. Note that from the system in polar coordinates it follows that the limit cycle is $r = 1$. From statement (b) of Theorem 1 we get immediately the first integral H described in the statement (d). This completes the proof of statement (d). □

Appendix: The Poincaré compactification

Let $\mathcal{X} = (P, Q)$ be any planar vector field of degree n. The *Poincaré compactified vector field* $p(\mathcal{X})$ corresponding to $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows. Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the Poincaré sphere) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is polynomial $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \backslash \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbb{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ (the infinity of $\mathcal{X}$) is invariant under the flow of $p(\mathcal{X})$.

We denote by P^n and Q^n the homogeneous part of degree n of the polynomials P and Q which define the vector field $\mathcal{X}$ of degree n. It is known that the real linear factors of $xQ^n(x, y) - yP^n(x, y)$ provide the equilibrium points of the compactified vector field $p(\mathcal{X})$ at infinity, see for more details Chapter 5 of [10]. Hence, since the infinity $\mathbb{S}^1$ is invariant, if $xQ^n(x, y) - yP^n(x, y)$ has no real factors, then the infinity is a periodic orbit.

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