On the periodic orbits of perturbed Hooke Hamiltonian systems with three degrees of freedom

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Dedicated to a common friend Jorge Sotomayor on the occasion of his 70th birthday

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Abstract

We study periodic orbits of Hamiltonian differential systems with three degrees of freedom using the averaging theory. We have chosen the classical integrable Hamiltonian system with the Hooke potential and we study periodic orbits which bifurcate from the periodic orbits of the integrable system perturbed with a non–autonomous potential.

Key words: Hamiltonian system, Hooke potential, periodic orbit, averaging theory
2010 MSC: 37G15, 34C29, 34C25

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1. Introduction

In this paper we study the spatial motion of a particle of unitary mass under the action of a central force with Hamiltonian given by

$$H_0(x, y, z, p_x, p_y, p_z) = \frac{1}{2} \left(p_x^2 + p_y^2 + p_z^2 \right) + V_0 \left(\sqrt{x^2 + y^2 + z^2} \right),$$

perturbed by the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z, t) = H_0(x, y, z, p_x, p_y, p_z) + \varepsilon V(t, x, y, z), \qquad (1)$$

where ε is a small parameter and $V(t, x, y, z)$ is a perturbation of the potential eventually depending on the time t.

We consider a central force derived from a potential of the form

$$V_0 \left(\sqrt{x^2 + y^2 + z^2} \right) = \pm (x^2 + y^2 + z^2)^{\alpha/2}, \qquad (2)$$

with α an integer. The Hamilton equations associated to Hamiltonian (1) are

$$\dot{x} = p_x,$$
$$\dot{y} = p_y,$$
$$\dot{z} = p_z,$$
$$\dot{p_x} = -\partial \left(V_0 \left(\sqrt{x^2 + y^2 + z^2} \right) + \varepsilon V(t, x, y, z) \right) / \partial x, \qquad (3)$$
$$\dot{p_y} = -\partial \left(V_0 \left(\sqrt{x^2 + y^2 + z^2} \right) + \varepsilon V(t, x, y, z) \right) / \partial y,$$
$$\dot{p_z} = -\partial \left(V_0 \left(\sqrt{x^2 + y^2 + z^2} \right) + \varepsilon V(t, x, y, z) \right) / \partial z,$$

where the dot denotes derivative with respect to the time t. We shall apply the averaging theory for studying the periodic orbits of the Hamiltonian system (3).

The averaging method (see for instance [11]) gives a quantitative relation between the solutions of some non–autonomous periodic differential system and the solutions of its autonomous averaged differential system, and in particular allows to study the periodic orbits of the non–autonomous periodic differential system in function of the periodic orbits of the averaged one, see for more details [1, 3, 8, 9, 11, 12] and mainly section 2. Our aim is to apply the *averaging theory* to one class of Hamiltonian systems (3) for studying their periodic solutions. But the tools that we shall use are very general and can be applied to other classes of Hamiltonian or differential systems.

Of course there is a long tradition in studying the periodic orbits of the differential systems using the averaging method, see chapter 4 of [6], the book [11], the chapter 11 of [12], the paper [5], and many others. In the paper [5] they use the averaging method of second order for studying the periodic orbits, but in that paper they studied a 2–dimensional non–autonomous

Hamiltonian system using the transformation to action–angle coordinates, and here we study 6–dimensional non–autonomous Hamiltonian systems. On the other hand, this paper extends to Hamiltonian systems with three degrees of freedom some results of [7] where the authors studied periodic orbits of Hamiltonian systems with two degrees of freedom. We note that for applying the averaging method to Hamiltonian systems in general it is not necessary to write the unperturbed integrable Hamiltonian in action–angle coordinates.

The unique central forces coming from the central potentials of the form (2) for which all bounded orbits are periodic are the Hooke's force and the Kepler's force, that correspond to the potentials

$$ k(x^2 + y^2 + z^2), \quad \text{and} \quad -\frac{k}{\sqrt{x^2 + y^2 + z^2}} \quad \text{with } k > 0, \qquad (4) $$

respectively. This result was proved by J. Bertrand in 1873, see [4].

We will apply the averaging theory to Hamiltonian systems (3) with the Hooke potential $V_0(\sqrt{x^2 + y^2 + z^2}) = k(x^2 + y^2 + z^2)$ for studying which periodic orbits of system (3) with $\varepsilon = 0$ can be continued to periodic orbits of the same system with $\varepsilon \neq 0$ sufficiently small. The periodic orbits for perturbed Kepler Hamiltonian systems with three degrees of freedom will be studied in a future article.

Since generically the periodic orbits of Hamiltonian systems leave on cylinders fulfilled of periodic orbits, and every one of the periodic orbits of one of these cylinders belongs to a different level of the Hamiltonian (for more details see [2, 10]), we shall apply the averaging theory described in section 2 to Hamiltonian systems (3) with potential $V_0(\sqrt{x^2 + y^2 + z^2}) = k(x^2 + y^2 + z^2)$ restricted to a fixed level of the Hamiltonian. To work in a fixed level of the Hamiltonian is necessary in order to apply the averaging theory for studying periodic orbits (see Theorem 4), because the periodic orbits provided by the averaging must be isolated in the set of all periodic orbits.

Consider the spatial motion of a particle of unitary mass under the action of the Hooke potential $V_0(\sqrt{x^2 + y^2 + z^2}) = (x^2 + y^2 + z^2)/2$ perturbed by a non–autonomous potential $\varepsilon V(t, x, y, z)$ where ε is a small parameter and $V(t, x, y, z)$ is 2π–periodic in the variable t. In cartesian coordinates the

Hamiltonian governing this motion is

$$H(x, y, z, p_x, p_y, p_z, t) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) +$$

$$\frac{1}{2}\left(x^2 + y^2 + z^2\right) + \varepsilon V(t, x, y, z). \quad (5)$$

The corresponding Hamiltonian equations are

$$\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{z} &= p_z, \\
\dot{p}_x &= -x - \varepsilon \partial V(t, x, y, z)/\partial x, \\
\dot{p}_y &= -y - \varepsilon \partial V(t, x, y, z)/\partial y, \\
\dot{p}_z &= -z - \varepsilon \partial V(t, x, y, z)/\partial z.
\end{aligned} \quad (6)$$

The main result on the periodic orbits of the Hamiltonian system (6) is the following one.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small, $h > 0$ and every zero $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of $f_k = f_k(x_0, y_0, z_0, p_{x_0}, p_{z_0})$ for $k = 1, 2, 3, 4, 5$, where*

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} -B\sin t \\ \pm \dfrac{Ap_{y_0} + D\sin t}{p_{y_0}\cos t - y_0 \sin t} \\ -C\sin t \\ B\cos t \\ C\cos t \end{bmatrix} \Bigg|_{\substack{x = p_{x_0}\sin t + x_0 \cos t \\ y = \pm(p_{y_0}\sin t + y_0 \cos t) \\ z = p_{z_0}\sin t + z_0 \cos t \\ p_x = p_{x_0}\cos t - x_0 \sin t \\ p_z = p_{z_0}\cos t - z_0 \sin t \\ p_{y_0} = \pm\sqrt{2h - p_{x_0}^2 - p_{z_0}^2 - x_0^2 - y_0^2 - z_0^2}}} dt,$$

with

$$\begin{aligned}
A &= -V(t, x, y, z)/(p_{y_0}\cos t - y_0 \sin t), \\
B &= -\partial V(t, x, y, z)/\partial x, \\
C &= -\partial V(t, x, y, z)/\partial z, \\
D &= B(p_{x_0}\cos t - x_0 \sin t) + C(p_{z_0}\cos t - z_0 \sin t),
\end{aligned}$$

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satisfying that

$$J(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*) = \left. \frac{\partial(f_1, f_2, f_3, f_4, f_5)}{\partial(x_0, y_0, z_0, p_{x_0}, p_{z_0})} \right|_{\substack{x_0 = x_0^*, y_0 = y_0^*, z_0 = z_0^*, \\ p_{x_0} = p_{x_0}^*, p_{z_0} = p_{z_0}^*}} \neq 0 \quad (7)$$

and

$$p_{y_0}^* = \pm\sqrt{2h - p_{x_0}^{*2} - p_{z_0}^{*2} - x_0^{*2} - y_0^{*2} - z_0^{*2}} \neq 0, \qquad (8)$$

there exists a 2π–periodic solution $\varphi(t; x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*, \varepsilon)$ of the Hamiltonian system (6) such that $\varphi(0; x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*, \varepsilon) \to (x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{y_0}^*, p_{z_0}^*)$ when $\varepsilon \to 0$.

Remark 2. *In Theorem 1 the signs $+$ or $-$ in $\pm$ must be used in the same order than the sign of the determination of the square root of p_y. See equation (13).*

If we take $z = p_z = 0$ and consider $V(t, x, y) = V(t, x, y, 0)$ then Theorem 1 extends some results of [7].

For example an application of Theorem 1 is the next one.

Corollary 3. *Consider the Hamiltonian system (6) with $V(t, x, y, z) = (x - x^3 + z - z^3)\cos t$. Then for $\varepsilon \neq 0$ sufficiently small the following statements hold.*

(a) *For every $h \in (4/9, 8/9]$ system (6) has at least 8 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level $H = h$ for $\varepsilon = 0$.*

(b) *For every $h \in (8/9, 4/3]$ system (6) has at least 40 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level $H = h$ for $\varepsilon = 0$.*

(c) *For every $h \in (4/3, +\infty)$ system (6) has at least 52 periodic orbits bifurcating from the periodic orbits of the Hamiltonian level $H = h$ for $\varepsilon = 0$.*

The study of the periodic orbits for the perturbed Hooke Hamiltonian systems is done in section 3. More precisely in that section it is proved Theorem 1 and Corollary 3.

2. Basic results

In this section we present the basic results from the averaging theory that we shall need for proving the main result of this paper. The key tool for proving the algorithm is the averaging theory. For a general introduction to the averaging theory and related topics see the books [1, 9, 11, 12]. But the results that we shall use are presented in what follows.

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{9}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ appearing in (9), are $\mathcal{C}^2$ in their variables and T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}), \tag{10}$$

has an open subset of Ω fulfilled of periodic solutions. A solution of this problem is given in the following using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (9) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system (10) along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{11}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (11).

We assume that there exists an open set W with $\mathrm{Cl}(W) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(W)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic. The set $\mathrm{Cl}(W)$ is *isochronous* for the system (10), that is it is a set formed only by periodic orbits, all of them having the same period T. Then an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $\mathrm{Cl}(W)$ is given in the next result.

Theorem 4. (Perturbations of an isochronous set) *We assume that there exists an open and bounded set W with $\mathrm{Cl}(W) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(W)$, the solution $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(W) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}, 0) F_1(t, \mathbf{x}(t, \mathbf{z})) dt. \tag{12}$$

If there exists $a \in W$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (9) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

For a proof of Theorem 4 see Corollary 1 of [3].

3. Periodic orbits of the perturbed spatial Hooke Hamiltonian

In this section we shall prove Theorem 1 and Corollary 3.

Proof of Theorem 1. We want to apply Theorem 4 to the Hamiltonian system (6). Since Theorem 4 needs that the periodic orbits of system (6) are isolated in the set of all periodic orbits of the system, we must restrict our study to every fixed Hamiltonian level, otherwise the set of periodic orbits would not be isolated. Thus we consider a fixed Hamiltonian level h and we restrict the Hamiltonian system (6) to this Hamiltonian level.

We isolate p_y in the Hamiltonian level

$$\frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(x^2 + y^2 + z^2) + \varepsilon V(t, x, y, z) = h,$$

and expand the obtained expression of p_y in power series of ε obtaining

$$p_y = \pm\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}\mp$$
$$\varepsilon\frac{V(t, x, y, z)}{\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}} + O(\varepsilon^2). \quad (13)$$

Therefore system (6) restricted to the Hamiltonian level $H(x, y, z, p_x, p_y, p_z, t) = h$ becomes

$$\dot{x} = p_x,$$
$$\dot{y} = \pm\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}\mp$$
$$\varepsilon\frac{V(t, x, y, z)}{\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}} + O(\varepsilon^2),$$
$$\dot{z} = p_z, \quad (14)$$
$$\dot{p}_x = -x - \varepsilon\frac{\partial V(t, x, y, z)}{\partial x},$$
$$\dot{p}_z = -z - \varepsilon\frac{\partial V(t, x, y, z)}{\partial z}.$$

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We will apply the averaging theory described in Section 2 for studying the periodic orbits of system (14). More precisely we shall analyze which periodic orbits of system (14) with $\varepsilon = 0$ can be continued to periodic orbits of system (14) with $\varepsilon \neq 0$ sufficiently small.

For simplicity, in what follows we develop the proof of Theorem 1 only for the positive determination of the square root of p_y, that is for

$$p_y = \sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2} -$$
$$\varepsilon \frac{V(t, x, y, z)}{\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}} + O(\varepsilon^2).$$

The same arguments can be applied to the negative determination.

The general solution of system (14) with $\varepsilon = 0$ in the Hamiltonian level $H(x, y, z, \, p_x, p_y, p_z) = h$ and with initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $p_x(0) = p_{x_0}$ and $p_z(0) = p_{z_0}$ is

$$x(t) = p_{x_0} \sin t + x_0 \cos t,$$

$$y(t) = p_{y_0} \sin t + y_0 \cos t,$$

$$z(t) = p_{z_0} \sin t + z_0 \cos t, \tag{15}$$

$$p_x(t) = p_{x_0} \cos t - x_0 \sin t,$$

$$p_z(t) = p_{z_0} \cos t - z_0 \sin t,$$

where

$$p_{y_0} = \sqrt{2h - p_{x_0}^2 - p_{z_0}^2 - x_0^2 - y_0^2 - z_0^2}. \tag{16}$$

All these periodic solutions form a five dimensional open set of periodic solutions with period 2π in the Hamiltonian level $H(x, y, z, p_x, p_y, p_z) = h$. So system (14) with ε sufficiently small satisfies the assumptions of Theorem 4.

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We write system (14) into the form (9)

$$\dot{x} = F_{0,1}(x,y,z,p_x,p_z) + \varepsilon F_{1,1}(x,y,z,p_x,p_z) + O(\varepsilon^2),$$
$$\dot{y} = F_{0,2}(x,y,z,p_x,p_z) + \varepsilon F_{1,2}(x,y,z,p_x,p_z) + O(\varepsilon^2),$$
$$\dot{z} = F_{0,3}(x,y,z,p_x,p_z) + \varepsilon F_{1,3}(x,y,z,p_x,p_z) + O(\varepsilon^2),$$
$$\dot{p}_x = F_{0,4}(x,y,z,p_x,p_z) + \varepsilon F_{1,4}(x,y,z,p_x,p_z) + O(\varepsilon^2),$$
$$\dot{p}_z = F_{0,5}(x,y,z,p_x,p_z) + \varepsilon F_{1,5}(x,y,z,p_x,p_z) + O(\varepsilon^2),$$

where $F_0 = (F_{0,1}, F_{0,2}, F_{0,3}, F_{0,4}, F_{0,5})$ is

$$F_{0,1} = p_x,$$
$$F_{0,2} = \sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2},$$
$$F_{0,3} = p_z,$$
$$F_{0,4} = -x,$$
$$F_{0,5} = -z,$$

and $F_1 = (F_{1,1}, F_{1,2}, F_{1,3}, F_{1,4}, F_{1,5})$ is

$$F_{1,1} = 0,$$
$$F_{1,2} = -\frac{V(t,x,y,z)}{\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}},$$
$$F_{1,3} = 0, \tag{17}$$
$$F_{1,4} = -\frac{\partial V(t,x,y,z)}{\partial x},$$
$$F_{1,5} = -\frac{\partial V(t,x,y,z)}{\partial z}.$$

The periodic solution $\mathbf{x}(t,\mathbf{z},0)$ of system (9) with $\varepsilon = 0$ now is the periodic solution $(x(t), y(t), z(t), p_x(t), p_z(t))$ given by (15) of system (14) with initial conditions $\mathbf{z} = (x_0, y_0, z_0, p_{x_0}, p_{z_0})$. After an easy but long computation the fundamental matrix $M_{\mathbf{z}}(t)$ of the differential system (11) such that $M_{\mathbf{z}}(0)$ is the identity matrix of $\mathbb{R}^5$ is

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$$M_{\mathbf{z}}(t) = \begin{pmatrix} \cos t & 0 & 0 & \sin t & 0 \\[2mm] -\dfrac{x_0 \sin t}{p_{y_0}} & \dfrac{p_{y_0}\cos t - y_0 \sin t}{p_{y_0}} & -\dfrac{z_0 \sin t}{p_{y_0}} & -\dfrac{p_{x_0}\sin t}{p_{y_0}} & -\dfrac{p_{z_0}\sin t}{p_{y_0}} \\[2mm] 0 & 0 & \cos t & 0 & \sin t \\[2mm] -\sin t & 0 & 0 & \cos t & 0 \\[2mm] 0 & 0 & -\sin t & 0 & \cos t \end{pmatrix}. \quad (18)$$

We note that p_{y_0} must be different from zero in order that the matrix $M_{\mathbf{z}}(t)$ be well defined. This is the reason that in the statement of Theorem 1 we have the assumption $p_{y_0}^* \neq 0$. In fact this is a technical restriction that would be able to be avoided working with another fundamental matrix, but here we do not take care of this.

By Theorem 4 we must study the zeros $(x_0, y_0, z_0, p_{x_0}, p_{z_0})$ in the set

$$W = \{(x_0, y_0, z_0, p_{x_0}, p_{z_0}) \in \mathbb{R}^5 : 0 < x_0^2 + y_0^2 + z_0^2 + p_{x_0}^2 + p_{z_0}^2 < R\},$$

where $R > 0$ is an arbitrary constant, of the system

$$\mathcal{F}(x_0, y_0, z_0, p_{x_0}, p_{z_0}) = (f_1, f_2, f_3, f_4, f_5)(x_0, y_0, z_0, p_{x_0}, p_{z_0}) = 0,$$

where according to (12) if we denote $f_k = f_k(x_0, y_0, z_0, p_{x_0}, p_{z_0})$ we have

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \int_0^{2\pi} M_{\mathbf{z}}^{-1}(t) \begin{bmatrix} F_{1,1} \\ F_{1,2} \\ F_{1,3} \\ F_{1,4} \\ F_{1,5} \end{bmatrix} \Bigg|_{\substack{x = p_{x_0}\sin t + x_0 \cos t \\ y = p_{y_0}\sin t + y_0 \cos t \\ z = p_{z_0}\sin t + z_0 \cos t \\ p_x = p_{x_0}\cos t - x_0 \sin t \\ p_z = p_{z_0}\cos t - z_0 \sin t \\ p_{y_0} = \sqrt{2h - p_{x_0}^2 - p_{z_0}^2 - x_0^2 - y_0^2 - z_0^2}}} dt, \quad (19)$$

where $F_{1,k} = F_{1,k}(x, y, z, p_x, p_z)$ for $k = 1, 2, 3, 4, 5$ are as in (17).

Note that in (19) only $F_{1,2}(x, y, z, p_x, p_z)$ depends on the variables p_x and p_z through the expression $\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2}$. But when we restrict the integrant function of (19) on the periodic solution of system (14) with $\varepsilon = 0$, that is on $x = p_{x_0}\sin t + x_0 \cos t$, $y = p_{y_0}\sin t + y_0 \cos t$,

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$z = p_{z_0} \sin t + z_0 \cos t$, $p_x = p_{x_0} \cos t - x_0 \sin t$ and $p_z = p_{z_0} \cos t - z_0 \sin t$, we get that $\sqrt{2h - p_x^2 - p_z^2 - x^2 - y^2 - z^2} = p_{y_0} \cos t - y_0 \sin t$. Now doing an easy computation, using (18) and (19), we get the expression of $(f_1, f_2, f_3, f_4, f_5)$ given in the statement of Theorem 1. Hence Theorem 4 completes the proof of Theorem 1. $\qquad\square$

Proof of Corollary 3. We apply Theorem 1 to our Hamiltonian system (6) with

$$V(t, x, y, z) = (x - x^3 + z - z^3) \cos t.$$

Again, for simplicity, we work only with the positive determination of the square root of p_y. The same arguments can be applied to the negative determination.

Since

$$A = \frac{(x^3 - x + z^3 - z) \cos t}{p_{y_0} \cos t - y_0 \sin t}, \quad B = (3x^2 - 1) \cos t, \quad C = (3z^2 - 1) \cos t$$

we obtain after some long computations that the functions $f_k = f_k(x_0, y_0, z_0, p_{x_0}, p_{z_0})$ are

$$f_1 = -\frac{3\pi}{2} p_{x_0} x_0,$$

$$\begin{aligned}
f_2 = {} & \frac{\pi}{4 \left(p_{y_0}^2 + y_0^2\right)^3} \Big(\left(x_0^3 + \left(9p_{x_0}^2 - 4\right) x_0 + z_0 \left(9p_{z_0}^2 + z_0^2 - 4\right)\right) p_{y_0}^5 + \\
& 3y_0 \left(-5p_{x_0}^3 + \left(3x_0^2 + 4\right) p_{x_0} + p_{z_0} \left(-5p_{z_0}^2 + 3z_0^2 + 4\right)\right) p_{y_0}^4 + \\
& 6y_0^2 \left(x_0^3 - 3p_{x_0}^2 x_0 + z_0^3 - 3p_{z_0}^2 z_0\right) p_{y_0}^3 - \\
& 2y_0^3 \left(5p_{x_0}^3 + \left(9x_0^2 - 8\right) p_{x_0} + p_{z_0} \left(5p_{z_0}^2 + 9z_0^2 - 8\right)\right) p_{y_0}^2 - \\
& y_0^4 \left(3x_0^3 + \left(3p_{x_0}^2 - 4\right) x_0 + z_0 \left(3p_{z_0}^2 + 3z_0^2 - 4\right)\right) p_{y_0} - \\
& y_0^5 \left(3p_{x_0}^3 + \left(3x_0^2 - 4\right) p_{x_0} + p_{z_0} \left(3p_{z_0}^2 + 3z_0^2 - 4\right)\right) \Big),
\end{aligned}$$

$$f_3 = -\frac{3\pi}{2} p_{z_0} z_0,$$

$$f_4 = \frac{\pi}{4} \left(3p_{x_0}^2 + 9x_0^2 - 4\right),$$

$$f_5 = \frac{\pi}{4} \left(3p_{z_0}^2 + 9z_0^2 - 4\right),$$

(20)

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where $p_{y_0} = \sqrt{2h - p_{x_0}^2 - p_{z_0}^2 - x_0^2 - y_0^2 - z_0^2}$. The determinant (7) is written as

$$
\begin{aligned}
J = \ & \frac{81\pi^5 \left(p_{x_0}^2 - 3x_0^2\right)\left(p_{z_0}^2 - 3z_0^2\right)}{64\left(p_{y_0}^2 + y_0^2\right)^4}\Big(3p_{z_0}\left(5p_{z_0}^2 - 3z_0^2 - 4\right)p_{y_0}^6 - \\
& 6y_0\left(x_0^3 + 4x_0 + z_0\left(-15p_{z_0}^2 + z_0^2 + 4\right)\right)p_{y_0}^5 + 3p_{z_0}y_0^2 \\
& \left(-15p_{z_0}^2 + 33z_0^2 + 4\right)p_{y_0}^4 - 4y_0^3\big(-9x_0^3 + 4x_0 + \\
& z_0\left(15p_{z_0}^2 - 9z_0^2 + 4\right)\big)p_{y_0}^3 + p_{z_0}y_0^4\big(-15p_{z_0}^2 - \\
& 39z_0^2 + 28\big)p_{y_0}^2 + 6p_{x_0}^2 x_0 y_0\left(15p_{y_0}^4 - 10y_0^2 p_{y_0}^2 - y_0^4\right)p_{y_0} - \\
& 2y_0^5\left(3x_0^3 - 4x_0 + z_0\left(3p_{z_0}^2 + 3z_0^2 - 4\right)\right)p_{y_0} + 3p_{x_0}^3\big(5p_{y_0}^6 - \\
& 15y_0^2 p_{y_0}^4 - 5y_0^4 p_{y_0}^2 - y_0^6\big) + p_{x_0}\big(-3\left(3x_0^2 + 4\right)p_{y_0}^6 + \\
& 3\left(33x_0^2 + 4\right)y_0^2 p_{y_0}^4 + \left(28 - 39x_0^2\right)y_0^4 p_{y_0}^2 + \left(4 - 3x_0^2\right)y_0^6\big) + \\
& p_{z_0}y_0^6\left(-3p_{z_0}^2 - 3z_0^2 + 4\right)\Big),
\end{aligned}
$$ (21)

where again $p_{y_0} = \sqrt{2h - p_{x_0}^2 - p_{z_0}^2 - x_0^2 - y_0^2 - z_0^2}$.

We say that a zero $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 2, 3, 4, 5$, is *admissible* if it satisfies (7), or equivalently $J \neq 0$ and $p_{y_0^*} \neq 0$, see (21) and (8), respectively.

Note that the functions f_1, f_3, f_4 and f_5 in (20) are independent of y_0. Thus in order to obtain the zeros $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 2, 3, 4, 5$, we compute the zeros of f_1, f_3, f_4 and f_5 and later the zeros of f_2. By a simple calculation we have the following zeros $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 3, 4, 5$:

$$
\left(0, 0, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right), \ \left(0, 0, -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right), \ \left(\frac{2}{3}, -\frac{2}{3}, 0, 0\right), \ \left(-\frac{2}{3}, \frac{2}{3}, 0, 0\right), \ \text{(22)}
$$

$$
\left(-\frac{2}{3}, -\frac{2}{3}, 0, 0\right), \ \left(\frac{2}{3}, \frac{2}{3}, 0, 0\right), \ \text{(23)}
$$

$$\left(0, -\frac{2}{3}, -\frac{2}{\sqrt{3}}, 0\right), \ \left(0, \frac{2}{3}, \frac{2}{\sqrt{3}}, 0\right), \ \left(-\frac{2}{3}, 0, 0, -\frac{2}{\sqrt{3}}\right), \ \left(\frac{2}{3}, 0, 0, \frac{2}{\sqrt{3}}\right), \quad (24)$$

$$\left(0, \frac{2}{3}, -\frac{2}{\sqrt{3}}, 0\right), \ \left(0, -\frac{2}{3}, \frac{2}{\sqrt{3}}, 0\right), \ \left(-\frac{2}{3}, 0, 0, \frac{2}{\sqrt{3}}\right), \ \left(\frac{2}{3}, 0, 0, -\frac{2}{\sqrt{3}}\right), \quad (25)$$

$$\left(0, 0, -\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right), \ \left(0, 0, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right). \quad (26)$$

In what follows we analyze the above five sets of zeros separately.

Case 1. Substituting the values of $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ obtained in (22) into the expression of f_2 in (20) and taking into account (16) we have $f_2(y_0, h) = 0$ for all values of y_0. Thus the non vanishing condition (7) is not satisfied.

Case 2. Substituting the values of $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ obtained in (23) into the expression of f_2 in (20) and taking into account (16) we have

$$f_2(y_0, h) = \pm \frac{2\pi\sqrt{-9y_0^2 + 18h - 8}\left(162y_0^4 - 99(9h-4)y_0^2 + 8(4-9h)^2\right)}{9(9h-4)^3}.$$

In order that the function $f_2(y_0, h)$ be real we need that $h > 4/9$. From the above expression of f_2 we obtain the following 4 admissible zeros $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 2, 3, 4, 5$

$$s_{1,2} = \left(\mp\frac{2}{3}, -\frac{\sqrt{\left(11 - \sqrt{57}\right)\left(9h - 4\right)}}{6}, \mp\frac{2}{3}, 0, 0\right),$$

$$\quad (27)$$

$$s_{3,4} = \left(\mp\frac{2}{3}, \frac{\sqrt{\left(11 - \sqrt{57}\right)\left(9h - 4\right)}}{6}, \mp\frac{2}{3}, 0, 0\right).$$

For the above zeros s_i the determinant (21) is given by

$$J(s_i) = \pm \frac{3\pi^5\sqrt{114\left(7\sqrt{57} - 45\right)}}{9h - 4},$$

while (8) is given by

$$p_{y_0}^*(s_i) = \frac{\sqrt{\left(\sqrt{57} - 3\right)\left(9h - 4\right)}}{6},$$

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$i = 1, 2, 3, 4$. Applying Theorem 1 and taking into account the negative determination of the square root of p_y the proof of item (a) of Corollary 3 is completed.

Case 3. Substituting the values of $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ obtained in (24) into the expression of f_2 in (20) and taking into account (16) we have

$$f_2(y_0, h) = \pm \frac{\pi\sqrt{-9y_0^2 + 18h - 16}}{9(9h - 8)^3} \left[\left(162y_0^4 - 99\,(9h - 8) + 8\,(9h - 8)^2 \right) + \right.$$

$$\left. 27\sqrt{3}y_0 \left(\sqrt{-9y_0^2 + 18h - 16}\,(-6y_0^2 + 9h - 8) \right) \right].$$

Again in order that the function $f_2(y_0, h)$ be real we need that $h > 8/9$. From the above expression of f_2 we obtain 8 admissible zeros $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 2, 3, 4, 5$, denoted by

$$s_{5,6} = \left(0, y_{01}^*, \mp\frac{2}{3}, \mp\frac{2}{\sqrt{3}}, 0 \right), \quad s_{7,8} = \left(\mp\frac{2}{3}, y_{01}^*, 0, 0, \mp\frac{2}{\sqrt{3}} \right),$$

$$s_{9,10} = \left(0, y_{02}^*, \mp\frac{2}{3}, \mp\frac{2}{\sqrt{3}}, 0 \right), \quad s_{11,12} = \left(\mp\frac{2}{3}, y_{02}^*, 0, 0, \mp\frac{2}{\sqrt{3}} \right),$$

$$(28)$$

where y_{01}^* and y_{02}^* are given by

$$y_{01}^* = -\sqrt{\frac{73(9h - 8)}{504} - \frac{\lambda(h)}{2} - \frac{1}{2}\sqrt{\frac{\eta(h)}{\lambda(h)} - \gamma(h) + 2\,\sigma(h)}}, \qquad (29)$$

$$y_{02}^* = \sqrt{\frac{73(9h - 8)}{504} - \frac{\lambda(h)}{2} + \frac{1}{2}\sqrt{\frac{\eta(h)}{\lambda(h)} - \gamma(h) + 2\,\sigma(h)}}, \qquad (30)$$

with

$$\lambda(h) = \sqrt{\gamma(h) + \sigma(h)}, \quad \sigma(h) = \frac{457(9h - 8)^2}{63504}, \quad \gamma(h) = \frac{\alpha(h)}{1102248\sqrt[3]{2}} + \frac{\beta(h)}{\alpha(h)},$$

$$\beta(h) = 24\sqrt[3]{2}(9h - 8)^4, \quad \alpha(h) = 972(9h - 8)^2\sqrt[3]{1375 + 27\sqrt{2473}},$$

$$\eta(h) = \frac{545(9h - 8)^3}{296352}.$$

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For the zeros in (28) the determinant (21) is given by

$$J(s_{5,6,7,8}) = \mp \frac{243\pi^5}{2(9h-8)^3\sqrt{\nu(h)}}\Big(-3(\mu(h))^{5/2} + 6\sqrt{3\nu(h)}(\mu(h))^2 +$$

$$3\nu(h)(\mu(h))^{3/2} - 45\sqrt{3}(\nu(h))^{3/2}\mu(h) + 26(\nu(h))^2\sqrt{\mu(h)} +$$

$$9\sqrt{3}(\nu(h))^{5/2}\Big), \quad (31)$$

$$\mu(h) = \frac{73(9h-8)}{504} - \frac{\lambda(h) + \tau(h)}{2}, \quad \nu(h) = \frac{13(9h-8)}{168} + \frac{\lambda(h) + \tau(h)}{2},$$

$$\tau(h) = \sqrt{\sigma(h) - \gamma(h) - \frac{\eta(h)}{4\,\lambda(h)}},$$

$$J(s_{9,10,11,12}) = \mp \frac{243\pi^5}{2(9h-8)^3\sqrt{\psi(h)}}\Big(3(\chi(h))^{5/2} + 6\sqrt{3\psi(h)}(\chi(h))^2 -$$

$$3\psi(h)(\chi(h))^{3/2} - 45\sqrt{3}(\psi(h))^{3/2}\chi(h) - 26(\psi(h))^2\sqrt{\chi(h)} +$$

$$9\sqrt{3}(\psi(h))^{5/2}\Big), \quad (32)$$

$$\chi(h) = \frac{73(9h-8)}{504} + \frac{\xi(h) - \lambda(h)}{2}, \quad \psi(h) = \frac{13(9h-8)}{168} + \frac{\lambda(h) - \xi(h)}{2},$$

$$\xi(h) = \sqrt{2\sigma(h) - \gamma(h) - \frac{\eta(h)}{4\,\lambda(h)}},$$

while the values of p_{y0}^* in (8) are given by

$$p_{y0}^*(s_{5,6,7,8}) = \sqrt{\frac{13(9h-8)}{168} + \frac{\lambda(h)}{2} + \frac{1}{2}\sqrt{-\frac{\eta(h)}{4\lambda(h)} + 2\sigma(h) - \gamma(h)}}, \quad (33)$$

$$p_{y0}^*(s_{9,10,11,12}) = \sqrt{\frac{13(9h-8)}{168} + \frac{\lambda(h)}{2} - \frac{1}{2}\sqrt{-\frac{\eta(h)}{4\lambda(h)} + 2\sigma(h) - \gamma(h)}}, \quad (34)$$

and λ, σ, γ, η are as above.

Case 4. Substituting the values of $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ obtained in (25) into the expression of f_2 in (20) and taking into account (16) we have

$$f_2(y_0, h) = \pm \frac{\pi \sqrt{-9y_0^2 + 18h - 16}}{9(9h-8)^3} \left[\left(162y_0^4 - 99\,(9h-8) + 8\,(9h-8)^2\right) + \right.$$
$$\left. 27\sqrt{3}y_0 \left(\sqrt{-9y_0^2 + 18h - 16}\,(6y_0^2 - 9h + 8)\right) \right].$$

Again we also need that $h > 8/9$. From the above expression of f_2 we obtain 8 admissible zeros $(x_0^*, y_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ of f_k for $k = 1, 2, 3, 4, 5$, denoted by

$$s_{13,14} = \left(0, y_{03}^*, \mp\frac{2}{3}, \pm\frac{2}{\sqrt{3}}, 0\right), \quad s_{15,16} = \left(\pm\frac{2}{3}, y_{03}^*, 0, 0, \mp\frac{2}{\sqrt{3}}\right),$$

$$s_{17,18} = \left(0, y_{04}^*, \mp\frac{2}{3}, \pm\frac{2}{\sqrt{3}}, 0\right), \quad s_{19,20} = \left(\pm\frac{2}{3}, y_{04}^*, 0, 0, \mp\frac{2}{\sqrt{3}}\right),$$

(35)

where $y_{03}^* = -y_{01}^*$ and $y_{04}^* = -y_{02}^*$, y_{01}^* and y_{02}^* are given in (29) and (30), respectively. For the zeros in (35) the determinant (21) is given by

$$J(s_{13,14,15,16}) = J(s_{5,6,7,8}), \quad J(s_{17,18,19,20}) = J(s_{9,10,11,12}),$$

where $J(s_{5,6,7,8})$ and $J(s_{9,10,11,12})$ are given in (31) and (32), respectively. The values of $p_{y_0}^*$ in (8) are given by

$$p_{y_0}^*(s_{13,14,15,16}) = p_{y_0}^*(s_{5,6,7,8}), \quad p_{y_0}^*(s_{17,18,19,20}) = p_{y_0}^*(s_{9,10,11,12}),$$

where $p_{y_0}^*(s_{5,6,7,8})$ and $p_{y_0}^*(s_{9,10,11,12})$ are given in (33) and (34), respectively.

From the cases 3 and 4, applying Theorem 1 and taking into account the negative determination of the square root of p_y the proof of item (b) of Corollary 3 is completed.

Case 5. Substituting the values of $(x_0^*, z_0^*, p_{x_0}^*, p_{z_0}^*)$ obtained in (26) into the expression of f_2 in (20) and taking into account (16) we have

$$f_2(y_0, h) = \pm \frac{2\sqrt{3}\pi y_0 \left(6y_0^4 - 7(3h-4)y_0^2 + 2(4-3h)^2\right)}{(3h-4)^3}.$$

The existence of $p_{y_0}^*$ implies that $h > 4/3$. By a simple calculation we have the following 6 admissible solutions

$$s_{21,22} = \left(0, 0, 0, \mp\frac{2}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}\right),$$

$$s_{23,24} = \left(0, -\sqrt{\frac{3h-4}{2}}, 0, \mp\frac{2}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}\right), \tag{36}$$

$$s_{25,26} = \left(0, \sqrt{\frac{3h-4}{2}}, 0, \mp\frac{2}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}\right).$$

For these zeros the determinant (21) is given by

$$J(s_{21,22}) = \pm\frac{36\pi^5\sqrt{3}}{3h-4}, \quad J(s_{23,24,25,26}) = \pm\frac{18\pi^5\sqrt{3}}{3h-4},$$

while (8) can be written as

$$p_{y_0}^*(s_{21,22}) = \sqrt{\frac{2(3h-4)}{3}}, \quad p_{y_0}^*(s_{23,24,25,26}) = \sqrt{\frac{3h-4}{6}}.$$

Applying Theorem 1 and taking into account the negative determination of the square root of p_y the proof of item (c) of Corollary 3 is completed. $\quad\square$

Acknowledgements

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