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On the Loewner Conjecture

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Abstract

We construct a series of examples of vectorfields relevant to the conjectures of Loewner and Caratheodory.

1 Introduction

We will note a one-dimensional foliations on the plane by $\mathcal{F}(\mathbb{R}^2)$. In many situations we don't have a unique foliation, but the union of n distinct foliations. We will denote an n-foliation by $\mathcal{F}^n(\mathbb{R}^2)$ and by $\mathcal{F}_i^n(\mathbb{R}^2)$ each individual foliation of $\mathcal{F}^n(\mathbb{R}^2)$.

The following definition is taken from [7], with small modifications:

Definition. A smooth one-dimensional foliation $\mathcal{F}^2(\mathcal{D}^2)$ with an isolated singularity at o is called a singular *Hessian foliation* if there exists a smooth real-valued function W on $\mathcal{D}^2$ whose Hessian

$$Hess(W) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{xy} & W_{yy} \end{pmatrix}$$

has the following properties:

1) *Hess(W)* is not a multiple of the identity for any $p \in \mathcal{D}^2 - o$.

2) The eigenspace corresponding to the large (small) eigenvalue of *Hess(W)* is tangent to $\mathcal{F}_1^2(\mathcal{D}^2)$, ($\mathcal{F}_2^2(\mathcal{D}^2)$) for each $p \in \mathcal{D}^2 - o$.

The Hessian foliations appears in the study of stagnation points in hydrodynamics and in differential geometry. In fact, (see for instance [5]), the directions of the *lines of curvature* in the Bonnet coordinates correspond to the leaves of the Hessian foliation of the Bonnet function. Let (dx, dy) be a principal direction corresponding to the eigenvalue λ:

$$W_{xx}dx + W_{xy}dy = \lambda dx,\ W_{xy}dx + W_{yy}dy = \lambda dy \tag{1.1}$$

Therefore:

$$(W_{xx} - W_{yy})dxdy + W_{xy}(dy^2 - dx^2) = 0 \tag{1.2}$$

Consider two complex numbers $dx + \imath dy$ and $\chi_1 + \imath\chi_2$ with:

$$\chi_1 = W_{xx} - W_{yy},\ \chi_2 = 2W_{xy} \tag{1.3}$$

The equation 1.2 shows that the radius-vector of $(dx + \imath dy)^2$ is orthogonal to the vector $\chi_2 - \imath\chi_1$ and so it is collinear to the vector $\chi_1 + \imath\chi_2$. The argument of $(dx+\imath dy)^2$ is the double of the argument of the complex number $(dx+\imath dy)^2$. Therefore, the index of the field of principal directions is half of the index of the field (χ_1, χ_2).

The Loewner conjecture about the index of an umbilic point will be proved if the index of the equilibrium point of:

$$\frac{dx}{dt} = F_1(x,y) = W_{xx} - W_{yy},$$
$$\frac{dy}{dt} = F_2(x,y) = 2W_{xy}. \tag{1.4}$$

is less or equal than two.

A vectorfield of the form 5.26 will be called a Loewner vector field.

This equation can be seen from a different point of view according to [7] and [9]. Consider the Cauchy-Riemann operator:

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right)$$

Then:

$$\frac{\partial^2}{\partial \overline{z}^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2\imath \frac{\partial^2}{\partial x \partial y}\right)$$

A Loewner vectorfield can be identified with the square of the Cauchy-Riemann operator. In section 3 we state a generalization of this vectorfield.

A necessary (but not sufficient) criterium for a system to be of this kind is:

$$\frac{\partial^3}{\partial x^3}\left(\frac{1}{2}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \frac{\partial^3}{\partial y^3}\left(\frac{1}{2}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x}\right) \tag{1.5}$$

since:

$$\frac{1}{2}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = W_{yyy} \tag{1.6}$$
$$\frac{1}{2}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x} = W_{xxx} \tag{1.7}$$

Another criterium is:

$$\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial y^2} = 2\frac{\partial^2 F_1}{\partial x \partial y} \tag{1.8}$$

To prove it, try to obtain W from the system. Then:

$$W_{xy} = \frac{F_2}{2} \Rightarrow W = \frac{1}{2}\int\int F_2 \, dx dy + \alpha(x) + \beta(y)$$

Substituting this expression of W into:

$$W_{xx} - W_{yy} = F_1$$

we arrive at:

$$\int \frac{\partial F_2}{\partial x} dy - \int \frac{\partial F_2}{\partial y} dx = 2(F_1 + \beta'' - \alpha'')$$

If we derive with respect x and y this expression, we obtain 1.8.

A system in the form (5.26) will be called a *basic system*.

In ([4]) it is proved the following:

Let $r = 3, 4, \dots, \infty, \omega$. The next conjectures are equivalents.

C^r-Loewner's Conjecture

The index of an umbilic, of a surface C^r embedded in $\mathbb{R}^3$, is at most one.

C^r-Loewner's Conjecture*

Let $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a map of class C^r defined in a neighborhood U of $(0,0) \in \mathbb{R}^2$. If $(0,0)$ is an isolated singularity of the vector field

$$X : (x,y) \to (\beta_{xx} - \beta_{yy}, 2\beta_{xy}),$$

then the index of X at $(0,0)$ is less or equal than 2.

2 Parcial results on Carathéodory conjecture

The conjecture seems true for tha analytic case with some proofs with more or less credibility. See [5]

Parcial results are the following:

From [9].

Let:

B be the open unit ball in R^2 centered ay 0,

$T = \partial B$,

$f \in C^2(\overline{B}, R$ be C^3 near T,

$\lambda, \mu, \lambda > \mu$ the eigenvalues of *Hess(W)* $f = H_f$

$\Sigma_\lambda, \Sigma_\mu$ the eigenspaces associated to the eigenvalues, $\Sigma = \Sigma_\lambda \bigcup \Sigma_\mu$

$\frac{\partial}{\partial r}$ the radial derivative.

Assume that the function $\lambda - \mu - \frac{\partial \mu}{\partial r}$ (resp. $\lambda - \mu - \frac{\partial \lambda}{\partial r}$) has no zeros on Σ_λ (resp. Σ_μ)

Then:

Σ is finite and Ind $\left(\frac{\partial^2 f}{\partial \overline{z}^2}, 0\right)$ is equal to:

$$\begin{aligned} 2 \quad &+ \quad \sharp(p \in T, H_f(p)p = \lambda(p)p, \lambda - \mu - \frac{\partial \mu}{\partial r}(p) < 0) \\ &- \quad \sharp(p \in T, H_f(p)p = \lambda(p)p, \lambda - \mu - \frac{\partial \mu}{\partial r}(p) > 0) \end{aligned}$$

If $\lambda - \mu - \frac{\partial \mu}{\partial r}(p) > 0$ on T , or $\lambda - \mu - \frac{\partial \lambda}{\partial r}(p) < 0$ on T, then:

$$\text{Ind}\left(\frac{\partial^2 f}{\partial \overline{z}^2}, 0\right) \leq 2.$$

In the same direction, in [8], the point 0 is a strong H_f umbilic if, for some $\delta > 0$ and all $r \in (0, \delta)$, the eigenvalues λ, μ satisfy on $|z| = r$ that $\min \lambda(z) > \max \mu(z)$ the, the index of the umbilic is at most 1.

Form [1]

" Let S be a surface in R^3. Then It is known that if S is a surface with constant mean curvature or special Weingarten, then the index of an isolated point on S is negative. In the paper we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed $\frac{1}{2}$.

We say that S is a Willmore surface is it is a stationary surface of the Wilmore functional W, where the Willmore functional is define by the integral of the square of the mean curvature. "

3 A generalization of Loewner's vectorfields.

3.1 Definition of $L_n(f)$

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A natural generalization of a Loewner vectorfield is the vectorfield, $L_n(f)$ defined as follows:

$$L_n(f) = (2^n)\left(\mathrm{Re}\frac{\partial^n}{\partial \overline{z}^n}\,,\,\mathrm{Im}\frac{\partial^n}{\partial \overline{z}^n}\right)(f) \tag{3.1}$$

$$= \left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right)^n (f) \tag{3.2}$$

If $n = 1$ we get a gradient vectorfield, if $n = 2$ a Loewner vectorfield. For other values of n:

$$L_3(f) = (f_{xxx} - 3f_{xyy}\,,\,3f_{xxy} - f_{yyy}).$$

$$L_4(f) = (f_{xxxx} - 6f_{xxyy} + f_{yyyy}\,,\,4f_{xxxy} - 4f_{xyyy}).$$

The first component of $L_n(f)$ is:

$$\sum_{k=0}^{k\leq\frac{n}{2}}(-1)^k\binom{n}{2k}(\frac{\partial}{\partial x})^{n-2k}(\frac{\partial}{\partial y})^{2k}$$

and the second one:

$$\sum_{k=0}^{k\leq\frac{n-1}{2}}(-1)^k\binom{n}{2k+1}(\frac{\partial}{\partial x})^{n-2k-1}(\frac{\partial}{\partial y})^{2k+1}$$

We can define also $L_0(f)$ as the vectorfield:

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = 0$$

It is interesting to remark that, $L_1(\imath f)$ is the canonical system associated to the hamiltonian $-f(x,y)$ and $L_2(\imath f)$ is the Loewner vectorfield $\frac{\pi}{2}$-rotated.

In the definition of L_n we can avoid the use of complex derivation. Suppose that we have a vectorfield whose components are $(f(x,y), g(x,y))$ then we can generalize the previous definition saying that $L_n(f,g)$ is a new vectorfield. The components are the real and imaginary parts of the Cauchy-Riemann operator applied to $f + \imath g$.

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(f+\imath g) &= \frac{1}{2}(f_x - g_y + \imath(g_x + f_y)) \\ L_1(f,g) &= (f_x - g_y, g_x + f_y).\end{aligned}$$

With this definition, $L_1(f)$ corresponds to $L_1(f,0)$.

We will use the notation:

$$\text{grad} f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$\text{sgrad} f = \begin{pmatrix} f_y \\ -f_x \end{pmatrix}$$

Then:

$$L(f,g) = \text{grad} f - \text{sgrad} f. \tag{3.3}$$

This characterization of L make possible to consider higher-dimensional cases and phase space other than the plane.

3.2 Some properties

1.- From 3.3 we get a characterization of those maps h that preserves the structure of a Loewner vectorfield i.e:

$$h^* L(f,g) = L(f \circ h, g \circ h)$$

The map h must be a canonical transformation and an area preserving map ($Dh = \pm 1$). Then:

$$h^* L(f,g) = h^*(\text{grad} f - \text{sgrad} f) = h^* \text{grad} f - h^* \text{sgrad} f = L(f \circ h, g \circ h)$$

2.- If in the expression of $L_n f$ we substitute $a f_{x^u y^v}$ by $a(\cos(\theta)^u (\sin(\theta)^v$ we obtain the trigonometric expansion of $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. Let us prove this fact:

For shortness we call $c = \cos(\theta)$ and $s = \sin(\theta)$. Then, the expression of:

$$L_n(f) = \left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right)^n f$$

becomes:

$$(\text{Re}(c + \imath s)^n, \text{Im}(c + \imath s)^n)$$

But this expression is equal to:

$$\left(\text{Re}(e^{in\theta}), \text{Im}(e^{in\theta})\right) = (\cos(n\theta), \sin(n\theta))$$

Let B be the open unit ball in R^n centered at O, we state the following conjecture:

2.- **The Generalized Loewner Conjecture**. *If f is $C^n(B)$ and $L_n(f)$ has an isolated equilibrium point on O, then:*

$$\text{Ind}(L_n(f), O) \le n \quad n \ge 0$$

It is easy to prove that there exists $f(x,y)$ such that $\mathrm{Ind}(L_n(f),O) = n$. One must take:

$$f = (x^2+y^2)^n$$

Then:

$$L_n(f) = r^n(\cos\theta, \sin\theta)$$

The equilibrium point of this vectorfield has $2n-2$ sectors, all of which are elliptic.

If $n=0$ the lines $y=C$ are invariant, therefore $\mathrm{Ind}(L_0(f),O) = 0$.

If $n = 1$, we have a gradient vectorfield. Since f is increasing on all non equilibrium trajectories it is not possible any elliptic sector. Therefore $(L_1(f),O) \leq 1$.

For $n > 2$ the numerical test where f is an homogeneous polynomial confirms the conjecture. An homogeneous polynomial has invariant rays. By means of a linear change of coordinates we can fix two rays on the axis, and by means of a scale we fix another ray as $y=x$. If the vectorfield is quadratic there are at most this three rays. The maximum number of elliptic sectors will be six. The test (see for instance in subsection 6.1one of the Mathematica programs) prove that after a blow-up some of the new equilibrium points are of saddle type. Therefore the conjecture is not contradicted.

4 Index and derivability

To study the dependence of the index with respect the differentiability of the vectorfield we study the bifurcation of the Loewner vectorfield corresponding to:

$$f(x,y) = \left(\frac{x^4+y^4}{x^2+y^2}\right)^a$$

The Loewner vectorfield can be regularized by a temporal scaling:

$$\frac{d\tau}{dt} = \frac{4af}{(x^2+y^2)^2(x^4+y^4)^2}$$

It becomes:

$$\begin{aligned}\frac{dx}{dt} &= (x^2-y^2)\left(ax^8+(4a-6)x^6y^2+(10a-8)x^4y^4+(4a-6)x^2y^6+ay^8\right)\\ \frac{dy}{dt} &= -2xy\left((a-1)x^8+4x^6y^2+6(1-a)x^4y^4+4x^2y^6+(a-1)y^8\right)\end{aligned}$$

In the next Figures (1, 2, 3, 4) we see the cases of
C^0-differentiability : $a < \frac{1}{2}$
and
C^1-differentiability : $a < \frac{3}{2}$
If the function is differentiable it doesn't exist elliptic sectors.

For surfaces it is easy to find examples of C^0 differentiability and with umbilic points of index greater than two. See [3], [2].

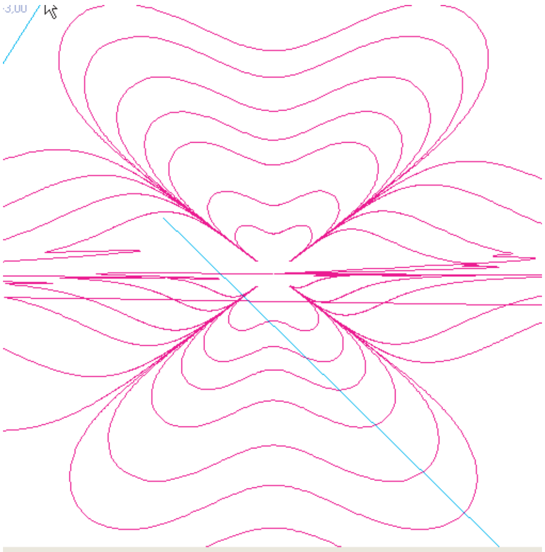

Figure 1: a=-3

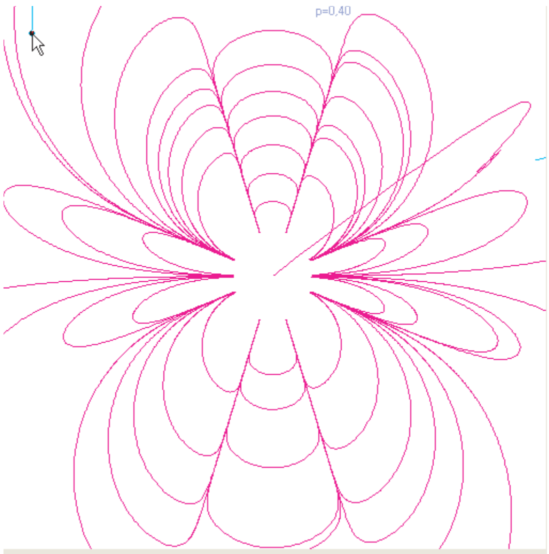

Figure 2: a=0.4

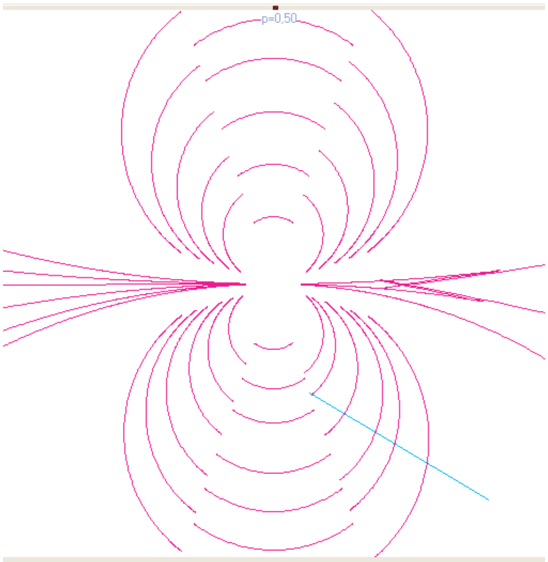

Figure 3: a=0.5

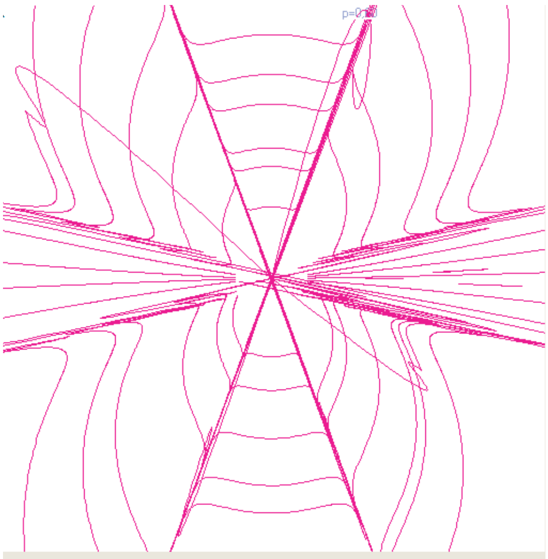

Figure 4: a=0.6

5 First integrals

In this section we want to study basic systems (5.26) with a given first integral.

5.1 Hamiltonian case

If we assume that the system 5.26 is hamiltonian, putting the condition:

$$F_1 = \frac{\partial H}{\partial y}, \quad F_2 = -\frac{\partial H}{\partial x}$$

into the condition (1.6) one arrives to

$$\frac{\partial^4 H}{\partial x^4} + \frac{\partial^4 H}{\partial x^2 \partial y^2} + \frac{\partial^4 H}{\partial y^4} = G(y) \tag{5.1}$$

5.2 General case

Assume now that $K(x, y)$ is a first integral. Then:

$$K_x(W_{xx} - W_{yy}) + 2K_y W_{xy} = 0 \tag{5.2}$$

Or

$$K_x W_{xx} + 2K_y W_{xy} - K_x W_{yy} = 0 \tag{5.3}$$

If we solve this equation, we find a Hessian foliation which leaves are the level sets of $K(x, y)$. If the first integral is, for instance:

$$\frac{-2x(x^2 - 3y^2)}{3(x^2 + y^2)^3} \tag{5.4}$$

whose level sets are in the Figure (5), we have a counterexample of the Loewner Conjecture.

In ([2]) it is constructed a closed surface with a single topological umbilic of index two. But the differentiability conditions do not seem clear.

As a guide to solve (5.3) we follow ([6]). We want to find $W(x, y)$ from the given first integral $K(x, y)$. The equation (5.2) is a particular case of a second order linear partial differential equation:

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} = 0 \tag{5.5}$$

To write (5.3) in normal form we consider, as usual, the equation of the characteristics:

$$A\left(\frac{\partial u}{\partial x}\right)^2 + 2B\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + C\left(\frac{\partial u}{\partial y}\right)^2 = 0 \tag{5.6}$$

In our particular case:

$$K_x\left(\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right) + 2K_y\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0 \tag{5.7}$$

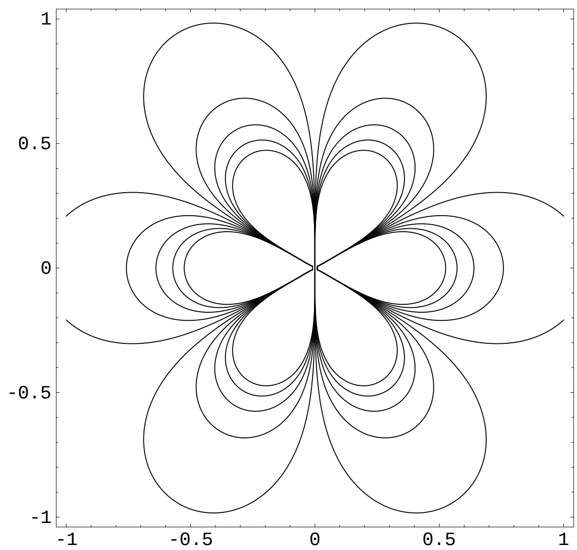

Figure 5: Level sets of (5.4)

The discriminant of (5.7):

$$K_y^2 + K_x^2$$

is positive. Therefore we have an equation of hyperbolic type.

The equation of (5.7) can be broken into two equations with real coefficients:

$$\frac{\partial u}{\partial x} - \alpha_1 \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} - \alpha_2 \frac{\partial u}{\partial y} = 0 \tag{5.8}$$

where α_1, α_2 are roots of the equation

$$K_x \alpha^2 + 2K_y \alpha - K_x = 0 \tag{5.9}$$

That is to say:

$$\alpha = \frac{-K_y \pm \sqrt{K_x^2 + K_y^2}}{K_x} \tag{5.10}$$

Assuming that $K_x(x, y) \neq 0$ we are lead to solve the equation:

$$K_x \frac{\partial u}{\partial x} + \left(K_y \pm \sqrt{K_x^2 + K_y^2} \right) \frac{\partial u}{\partial y} = 0 \tag{5.11}$$

Suppose that u_1, u_2 are independent solutions of (5.11). Then, the new variables:

$$u_1(x, y) = \xi, \quad u_2(x, y) = \eta \tag{5.12}$$

transform the equation (5.5) into its canonical form:

$$2B^* \frac{\partial^2 u}{\partial x \partial y} + D^* \frac{\partial u}{\partial x} + E^* \frac{\partial u}{\partial y} = 0 \tag{5.13}$$

where:

$$\begin{aligned} B^* &= A\chi_x\eta_x + B(\chi_x\eta_y + \chi_y\eta_x) + C\chi_y\eta_y \\ D^* &= L(\chi) \\ E^* &= L(\eta) \end{aligned}$$

Let's apply this process to the function (5.4).

$$\frac{2\left(x^4 - 6x^2y^2 + y^4\right)}{\left(x^2+y^2\right)^4}(W_{xx} - W_{yy}) + 2\frac{8x(x-y)y(x+y)}{\left(x^2+y^2\right)^4}W_{xy} = 0 \qquad (5.14)$$

Or equivalently:

$$\left(x^4 - 6x^2y^2 + y^4\right)(W_{xx} - W_{yy}) + 8x(x-y)y(x+y)W_{xy} = 0 \qquad (5.15)$$

Then, α_1, α_2 in (5.8) are:

$$\begin{aligned} \alpha_1 &= -1 - \frac{4xy}{x^2 - 2xy - y^2} \\ \alpha_2 &= 1 - \frac{4xy}{x^2 + 2xy - y^2} \end{aligned} \qquad (5.16)$$

We must solve the associated partial differential equation:

$$\begin{aligned} u_x + (1 + \frac{4xy}{x^2 - 2xy - y^2})u_y &= 0 \\ u_x + (-1 + \frac{4xy}{x^2 + 2xy - y^2})u_y &= 0 \end{aligned} \qquad (5.17)$$

They admit the two independent solutions:

$$\begin{aligned} u_1 &= \frac{x^2+y^2}{-x+y} \\ u_2 &= \frac{x^2+y^2}{x+y} \end{aligned} \qquad (5.18)$$

After some simplifications, the new variables:

$$\begin{aligned} x &= \frac{\eta\chi(-\eta+\chi)}{\eta^2+\chi^2} \\ y &= \frac{\eta(\eta+\chi)\chi}{\eta^2+\chi^2} \end{aligned} \qquad (5.19)$$

convert the equation (5.15) in:

$$\chi\eta\left(\eta^2+\chi^2\right)V_{\chi\eta} = 2V_\eta\eta^3 + 2V_\chi\chi^3 \qquad (5.20)$$

This equation has the solution:

$$\frac{1}{\chi^2} - \frac{1}{\eta^2} \qquad (5.21)$$

In the original variables:

$$W(x,y) = -\frac{4xy}{(x^2+y^2)^2} \qquad (5.22)$$

The differential equation is:

$$\begin{aligned} \frac{dx}{dt} &= 96xy\frac{y^2-x^2}{(x^2+y^2)^4}, \\ \frac{dy}{dt} &= 24\frac{x^4-6x^2y^2+y^4}{(x^2+y^2)^4}. \end{aligned} \tag{5.23}$$

This is not a counterexample to the Loewner Conjecture since W is not continuous at the origin.

Another solution is:

$$\chi^2\eta^2 \tag{5.24}$$

For this solution:

$$W(x,y) = \frac{(x^2+y^2)^4}{(x^2-y^2)^2} \tag{5.25}$$

The differential equation is:

$$\begin{aligned} \frac{dx}{dt} &= -192x^2y^2\frac{(y^2+x^2)^2}{(x^2-y^2)^3}, \\ \frac{dy}{dt} &= 48xy\frac{(y^2+x^2)^2(x^4-6x^2y^2+y^4)}{(x^2-y^2)^4}. \end{aligned} \tag{5.26}$$

Since the basic differential equation is linear a linear combination of the solutions is again a solution. Therefore:

$$\frac{a(x^2+y^2)^6+b(x^2-y^2)^2}{(x^2-y^2)^2(x^2+y^2)^2} \tag{5.27}$$

is a family of solutions with the same problems with the continuity.

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6 Appendixes

6.1 $L_3(f)$, f homogeneous polynomial

Cas homogeni, grau cinc amb 3 rectes invariants.

a14 $= -\frac{1}{192}\left(-60a^2 - b^2\right)$;
a50 $= -\frac{1}{192}\left(-108a^2 - 5b^2\right)$; **a41** $= -\frac{1}{192}\left(-b^2 - 12c^2\right)$;
a32 = 6a14;
a23 = 6a41;
a05 = 5a14 + 5a41 − a50;
f[x_, y_] = a50 ∗ x^5 + a41 ∗ x^4 ∗ y + a32 ∗ x^3 ∗ y^2 + a23 ∗ x^2 ∗ y^3 + a14 ∗ x ∗ y^4 + a05 ∗ y^5

$\frac{1}{192}\left(108a^2+5b^2\right)x^5+\frac{1}{192}\left(b^2+12c^2\right)x^4y+\frac{1}{32}\left(60a^2+b^2\right)x^3y^2+\frac{1}{32}\left(b^2+12c^2\right)x^2y^3+$

$\frac{1}{192}\left(60a^2+b^2\right)xy^4+\left(\frac{1}{192}\left(-108a^2-5b^2\right)+\frac{5}{192}\left(60a^2+b^2\right)+\frac{5}{192}\left(b^2+12c^2\right)\right)y^5$

xp = FullSimplify[D[D[D[f[x, y], x], x], x] − 3 ∗ D[D[D[f[x, y], y], y], x]]
yp = FullSimplify[3 ∗ D[D[D[f[x, y], x], y], x] − D[D[D[f[x, y], y], y], y]]

$x\left(b^2(x-y)-12c^2y\right)$

$y\left(60a^2(x-y)+b^2(x-y)-12c^2y\right)$

Solve[−6 ∗ a14 + a32 == 0, a32]
Solve[−6 ∗ a41 + a23 == 0, a23]

$\{\{\mathrm{a32}\to 6\mathrm{a14}\}\}$

$\{\{\mathrm{a23}\to 6\mathrm{a41}\}\}$

tn = FullSimplify[(yp/xp)/.y → m ∗ x]

$\frac{m(-16\mathrm{a14}+25\mathrm{a14}m+16\mathrm{a41}m-5\mathrm{a50}m)}{9\mathrm{a14}-5\mathrm{a50}+16\mathrm{a41}m}$

pol = Normal[Series[Denominator[tn] ∗ m − Numerator[tn], {m, 0, 6}]]

$(25\mathrm{a14}-5\mathrm{a50})m+(-25\mathrm{a14}+5\mathrm{a50})m^2$

vpol = pol/.m → 1

0

Solve[vpol == 0, a05]

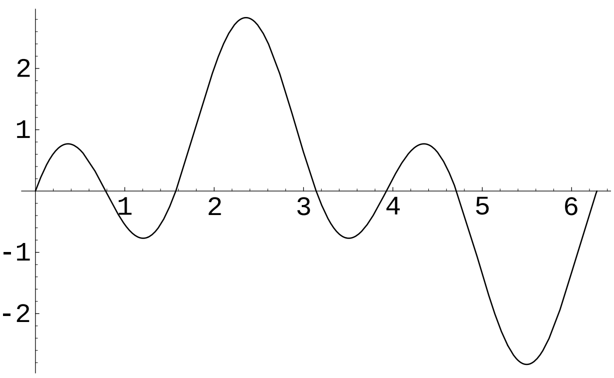
$\{\{\mathrm{a05}\to 5\mathrm{a14}+5\mathrm{a41}-\mathrm{a50}\}\}$

FullSimplify[Solve[pol == 0, m]]

$\{\{m\to 0\},\{m\to 1\}\}$

f1 = xp;
f2 = yp;

$\mathbf{f1p = Normal[FullSimplify[((x * f1 + y * f2)/.\{x \to r * Cos[a], y \to r * Sin[a]\})/r]]}$
$r^2\left(\left(15a^2+b^2\right)\mathrm{Cos}[a]-15a^2\mathrm{Cos}[3a]-\left(45a^2+b^2+12c^2\right)\mathrm{Sin}[a]+15a^2\mathrm{Sin}[3a]\right)$
$\mathbf{f2p = Normal[FullSimplify[((x * f2 - y * f1)/.\{x \to r * Cos[a], y \to r * Sin[a]\})/r\hat{\ }2]]}$
$15a^2r(-\mathrm{Cos}[a]+\mathrm{Cos}[3a]+\mathrm{Sin}[a]+\mathrm{Sin}[3a])$
$\mathbf{Series[f2p, \{r, 0, 3\}]}$
$15a^2(-\mathrm{Cos}[a]+\mathrm{Cos}[3a]+\mathrm{Sin}[a]+\mathrm{Sin}[3a])r+O[r]^4$
$\mathbf{g1 = FullSimplify[f1p/r]}$
$\mathbf{g2 = FullSimplify[f2p/r]}$
$r\left(\left(15a^2+b^2\right)\mathrm{Cos}[a]-15a^2\mathrm{Cos}[3a]-\left(45a^2+b^2+12c^2\right)\mathrm{Sin}[a]+15a^2\mathrm{Sin}[3a]\right)$
$15a^2(-\mathrm{Cos}[a]+\mathrm{Cos}[3a]+\mathrm{Sin}[a]+\mathrm{Sin}[3a])$
$\mathbf{sg2 = FullSimplify[g2/.r \to 0]}$
$\mathbf{ss = Solve[sg2 == 0, a]}$
$15a^2(-\mathrm{Cos}[a]+\mathrm{Cos}[3a]+\mathrm{Sin}[a]+\mathrm{Sin}[3a])$
Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solu
$\left\{\{a\to 0\},\{a\to -\pi\},\left\{a\to -\frac{3\pi}{4}\right\},\left\{a\to -\frac{\pi}{2}\right\},\left\{a\to \frac{\pi}{4}\right\},\left\{a\to \frac{\pi}{2}\right\},\{a\to \pi\}\right\}$
$\mathbf{Plot[-Cos[a] + Cos[3a] + Sin[a] + Sin[3a], \{a, 0, 2 * Pi\}]}$



−Graphics−
$\mathbf{m11 = FullSimplify[D[g1, r]];}$
$\mathbf{m12 = FullSimplify[D[g1, a]];}$
$\mathbf{m21 = FullSimplify[D[g2, r]];}$
$\mathbf{m22 = FullSimplify[D[g2, a]];}$
$\mathbf{A = \{\{m11, m12\}, \{m21, m22\}\};}$
$\mathbf{A0 = FullSimplify[A/.r \to 0];}$
$\mathbf{MatrixForm[A0]}$
$\Bigg(\ \left(15a^2+b^2\right)\mathrm{Cos}[a]-15a^2\mathrm{Cos}[3a]-\left(45a^2+b^2+12c^2\right)\mathrm{Sin}[a]+15a^2\mathrm{Sin}[3a]$
$0 \qquad 15a((-2+a)\mathrm{Cos}[a]+(2+3a$
$\mathbf{FullSimplify[Eigenvalues[A0/.a \to 0]]}$
$\left\{0, b^2\right\}$
$\mathbf{FullSimplify[Eigenvalues[A0/.a \to Pi/4]]}$
$\left\{-\frac{15\pi^2}{4\sqrt{2}}, -6\sqrt{2}c^2\right\}$
$\mathbf{FullSimplify[Eigenvalues[A0/.a \to Pi/2]]}$
$\left\{15\pi^2, -b^2-12c^2-15\pi^2\right\}$
$\mathbf{FullSimplify\left[Eigenvalues\left[A0/.a \to -\frac{3\pi}{4}\right]\right]}$
$\left\{\frac{135\pi^2}{4\sqrt{2}}, 6\sqrt{2}c^2\right\}$
$\mathbf{FullSimplify[Eigenvalues[A0/.a \to -Pi/2]]}$

$\left\{-15\pi^2, b^2+12c^2+15\pi^2\right\}$

FullSimplify[Eigenvalues[A0/.a → Pi]]

$\left\{-b^2, -60\pi^2\right\}$

Solve[{5a14 − a50 == a^2, −108a14 + 60a50 == b^2, 9a14 + 16a41 − 5a50 == c^2}, {a14, a50, a41}]

$\left\{\left\{\text{a14} \to -\frac{1}{192}\left(-60a^2-b^2\right), \text{a50} \to -\frac{1}{192}\left(-108a^2-5b^2\right), \text{a41} \to -\frac{1}{192}\left(-b^2-12c^2\right)\right\}\right\}$