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ON THE COMPLETE MEROMORPHIC INTEGRABILITY OF A THREE–DIMENSIONAL CORED GALACTIC HAMILTONIAN

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ABSTRACT. We characterize when the three–dimensional cored galactic Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\Big(p_x^2 + p_y^2 + \frac{p_z^2}{q}\Big) + \sqrt{1 + x^2 + y^2 + \frac{z^2}{q}},$$

is completely meromorphically integrable when $q \in [\sqrt{0.6}, 1]$. The key point for this characterization is to transform the non–polynomial cored Hamiltonian system into a polynomial one.

1. Introduction and statement of the main results

We consider the three-dimensional cored galactic Hamiltonian

$$H = \frac{1}{2}\Big(p_x^2 + p_y^2 + \frac{p_z^2}{q}\Big) + \sqrt{1 + x^2 + y^2 + \frac{z^2}{q}},$$

where $q > 0$. Its associated Hamiltonian system is

$$\begin{aligned} x' &= p_x,\\ y' &= p_y,\\ z' &= \frac{p_z}{q},\\ p_x' &= -\frac{x}{\sqrt{1 + x^2 + y^2 + z^2/q}},\\ p_y' &= -\frac{y}{\sqrt{1 + x^2 + y^2 + z^2/q}},\\ p_z' &= -\frac{z}{q\sqrt{1 + x^2 + y^2 + z^2/q}}, \end{aligned} \tag{1}$$

where the prime denotes derivative with respect to the time t. Note that this Hamiltonian system has three degrees of freedom.

The motivation for the choice of the potential

$$\sqrt{1 + x^2 + y^2 + z^2/q}$$

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comes from the interest of this potential in galactic dynamics, see for instance [2, 3, 5, 6, 7, 9, 12, 13, 14, 15, 16]. The parameter q gives the ellipticity of the potential, which ranges in the interval $\sqrt{0.6} \le q \le 1$. Lower values of q have no physical meaning and greater values of q are equivalent to reverse the role of the coordinate axes. So in this paper we consider $q \in [\sqrt{0.6}, 1]$. Note that the parameter q used here is the parameter denoted as q^2 in some other papers where there $q^2 \in [0.6, 1]$.

The main aim of this paper is to study the existence or non–existence of an additional meromorphic first integral F of the 3-dimensional cored galactic Hamiltonian system (1) *independent* of H, i.e. the gradients of F and H are linearly independent at any point of the phase space except perhaps in a zero Lebesgue measure set and such that $\{H, F\} = 0$. The existence of a such second independent and in involution first integral allows to simplify the study of the dynamics in two dimensions. Moreover, we will also study the existence of an additional third meromorphic first integral G of the 3-dimensional cored galactic Hamiltonian system independent with H and F and such that $\{H, G\} = \{F, G\} = 0$. Note that the existence of such two additional analytic first integrals independent and in involution will allow to describe completely the dynamics of a Hamiltonian system with three degrees of freedom, such as the 3-dimensional cored galactic Hamiltonian system (1) (see for more details [1]).

The Hamiltonian vector field X associated to system (1) is

$$X = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + \frac{p_z}{q}\frac{\partial}{\partial y} - \frac{x}{\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_x} \\ - \frac{y}{\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_y} - \frac{z}{q\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_y}.$$

Let U be an open and dense set in $\mathbb{R}^6$. We say that the non–locally constant function $F\colon U \to \mathbb{R}$ is a first integral of the vector field X on U, if $F(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))$ of X is defined in U. Clearly F is a *first integral* of X on U if and only if $XF = 0$ on U. An *meromorphic* first integral is a first integral F being F a meromorphic function. The Hamiltonian H is an analytic first integral of system (1) and thus it is a meromorphic first integral. By definition a Hamiltonian system with 3 degrees of freedom having 3 independent first integrals that are in *involution* is *completely integrable*, see again [1] for more details.

Proposition 1. *When* $q = 1$ *the* 3*-dimensional cored galactic Hamiltonian system* (1) *is completely integrable with the first integrals* H, $F = yp_x - xp_y$ *and* $G = zp_x - xp_z$.

Since $XF = 0$ and $XG = 0$ when $q = 1$, and clearly H, F and G are independent and in involution, the proposition follows. Hence, from now on we will restrict to the case $q \neq 1$, i.e. $q \in [\sqrt{0,6}, 1)$.

Note that the 3-dimensional cored galactic Hamiltonian system (1) is not a polynomial differential system, and consequently the Darboux theory of integrability (see for instance [4, 8]), which is very useful for finding first integrals, cannot be applied to system (1).

Our main result is the following one.

Theorem 2. *The 3-dimensional cored galactic Hamiltonian system* (1) *with* $q \in [\sqrt{0,6}, 1)$ *is not completely integrable with analytic first integrals.*

The proof of Theorem 2 is given in section 3.

2. Meromorphic first integrals of Hamiltonian systems with homogenous potential

During the last century many integrable natural Hamiltonian systems with Hamilton function of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \dots, q_n) \tag{2}$$

were found. Inside the class of Hamiltonian systems, the ones with homogenous polynomial potentials was investigated with a special care. Among lots of results we want to mention the work of Ziglin [18, 19] where the author developed an elegant theory which relates the integrability of Hamiltonian systems with properties of the monodromy group of variational equations along a particular solution and formulated the necessary conditions of integrability for complex Hamiltonian systems. Yoshida [17] used this theory to formulate a criterion of the non–existence of an additional first integral for homogeneous Hamiltonian systems. Applications of Ziglin theory are considerably restricted by the assumption about the existence of a nonresonant element in the monodromy group. Some of the restrictions and difficulties of Ziglin's theory can be overcome if instead of the monodromy group we investigate the differential Galois group. This approach was developed by Morales and Ramis [10, 11].

Let $c = (c_1, \dots, c_n) \neq (0, \dots, 0)$ be a solution of the non-linear system $V'(c) = c$ with $V(q_1, \dots, q_n)$ a homogeneous potential of degree k. Solutions of this system are called *the Darboux points*. Let $\lambda_1(c), \dots, \lambda_n(c) = k-1$ be the eigenvalues of the Hessian matrix $V''(c)$ of the potential calculated at a Darboux point. Then we have the following result due to Morales–Ramis (see [11]). We state it only in the case of completely meromorphic integrable Hamiltonian systems with Hamiltonian (2) and homogeneous potential V of degree 4.

Theorem 3. *If a Hamiltonian system with Hamiltonian of the form* (2) *with homogenous potential* $V(q_1, \dots, q_n)$ *of degree* 4 *is completely integrable with meromorphic first integrals, then each* $\lambda_i(c)$ *for* $1 \le i \le n-1$ *belongs to one of the following cases:*

$$j + 2j(j-1), \quad -\frac{1}{8} + \frac{2}{9}(1+3j)^2, \quad \frac{3}{8} + 2j(j+1),$$

where j *is an integer number, for all non–zero Darboux point* c *of* V.

3. Proof of Theorem 2

We introduce the change of variables

$$X = x, \quad Y = y, \quad Z = \frac{z}{\sqrt{q}}, \quad P_X = p_x, \quad P_Y = p_y, \quad P_Z = \sqrt{q}p_z$$

and the change of the independent variable from t to s given by

$$dt = \sqrt{1+X^2+Y^2+Z^2}\, ds,$$

the differential system (1) becomes

$$\begin{aligned}
\dot X &= P_X\sqrt{1+X^2+Y^2+Z^2},\\
\dot Y &= P_Y\sqrt{1+X^2+Y^2+Z^2},\\
\dot Z &= \frac{P_Z}{q^2}\sqrt{1+X^2+Y^2+Z^2},\\
\dot P_X &= -X,\\
\dot P_Y &= -Y,\\
\dot P_Z &= -Z.
\end{aligned} \tag{3}$$

Now we observe that in these variables the first integral H becomes

$$H = \frac{1}{2}\Big(P_X^2+P_Y^2+\frac{P_Z^2}{q^2}\Big)+\sqrt{1+X^2+Y^2+Z^2}.$$

Now we observe that since $\dot H = 0$ we can rewrite system (3) as the following system in the seven variables $(X, Y, Z, P_X, P_Y, P_Z, H)$:

$$\begin{aligned}
\dot X &= P_X\Big(H-\frac{1}{2}\Big(P_X^2+P_Y^2+\frac{P_Z^2}{q^2}\Big)\Big) = P_1(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot Y &= P_Y\Big(H-\frac{1}{2}\Big(P_X^2+P_Y^2+\frac{P_Z^2}{q^2}\Big)\Big) = P_2(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot Z &= \frac{P_Z}{q^2}\Big(H-\frac{1}{2}\Big(P_X^2+P_Y^2+\frac{P_Z^2}{q^2}\Big)\Big) = P_3(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot P_x &= -X = P_4(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot P_y &= -Y = P_5(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot P_z &= -Z = P_6(X,Y,Z,P_X,P_Y,P_Z,H),\\
\dot H &= 0 = P_7(X,Y,Z,P_x,P_y,P_z,H).
\end{aligned} \tag{4}$$

Note that the function $F(x, y, z, p_x, p_y, p_z)$ is a meromorphic first integral of system (1) if and only if $F(X, Y, \sqrt{q}Z, P_X, P_Y, P_Z/\sqrt{q})$ is a meromorphic first integral of system (3). Now substituting $\sqrt{1+X^2+Y^2+Z^2}$ by $H - (P_X^2 + P_Y^2 + P_Z^2/q^2)/2$ in the expression of the meromorphic first integral $F(X, Y, \sqrt{q}Z, P_X, P_Y, P_Z/\sqrt{q})$, if it appears, we obtain a meromorphic first integral of system (4). In short, every meromorphic first integral of system (1) produces a meromorphic first integral of system (4). The converse also holds easily taking into account that H is an analytic function in the variables (X, Y, Z, P_X, P_Y, P_Z). So in order to find a meromorphic first integral of system (1) it is sufficient to find a meromorphic first integral of the polynomial differential system (4).

We restrict the differential system (4) to $H = 0$. Then system (4) restricted to $H = 0$ becomes

$$
\begin{aligned}
\dot{X} &= -\frac{P_X}{2}\Big(P_X^2 + P_Y^2 + \frac{P_Z^2}{q^2}\Big),\\
\dot{Y} &= -\frac{P_Y}{2}\Big(P_X^2 + P_Y^2 + \frac{P_Z^2}{q^2}\Big),\\
\dot{Z} &= -\frac{P_Z}{2q^2}\Big(P_X^2 + P_Y^2 + \frac{P_Z^2}{q^2}\Big),\\
\dot{P}_X &= -X,\\
\dot{P}_Y &= -Y,\\
\dot{P}_Z &= -Z.
\end{aligned}
\tag{5}
$$

The differential system (5) is a Hamiltonian system with Hamiltonian

$$\frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{1}{8}\Big(P_X^2 + P_Y^2 + \frac{P_Z^2}{q^2}\Big)^2.$$

Now we use the theory summarized in section 2 to compute the values of the parameter q for which system (5) may be completely meromorphically integrable, i.e., there exist two functionally independent meromorphic first integrals in involution.

Setting $p_1 = X$, $p_2 = Y$, $p_3 = Z$, $q_1 = P_X$, $q_2 = P_Y$, $q_3 = P_Z$, system (5) can be written as a Hamiltonian system with Hamiltonian $H = (p_1^2+p_2^2+p_3^2)/2+V(q_1, q_2, q_3)$ with

$$V(q_1, q_2, q_3) = -\frac{1}{8}\Big(q_1^2 + q_2^2 + \frac{q_3^2}{q^2}\Big)^2, \tag{6}$$

being V a homogeneous potential of degree four. The Darboux points of the potential in (6) with $q \in [\sqrt{0.6}, 1)$ are

$$
\begin{aligned}
&c_1 = (-\sqrt{2 + q_2^2}i, q_2, 0),\ c_2 = (\sqrt{2 + q_2^2}i, q_2, 0),\ c_3 = (-\sqrt{2}i, 0, 0),\ c_4 = (\sqrt{2}i, 0, 0)\\
&c_5 = (0, -\sqrt{2}i, 0),\ c_6 = (0, \sqrt{2}i, 0),\ c_7 = (0, 0, -\sqrt{2}q^2i),\ c_8 = (0, 0, \sqrt{2}q^2i).
\end{aligned}
$$

Now the eigenvalues of $V''(c_k)$ are given in Table 1. By Theorem 3, taking into account Table 1, if system (5) is completely meromorphically integrable, the eigenvalues of $V''(c_k)$ must be one of the three possibilities given in Theorem 3.

If $q \in [\sqrt{0.6}, 1)$ then $1/q^2 \in (1, 1.66\cdots)$. First observe that $j+2j(j-1)$ is either 0, 1 or greater than or equal to 3. Moreover, $-\frac{1}{8} + \frac{2}{9}(1+3j)^2$ is either $\frac{7}{72} = 0.09722\cdots$, $\frac{55}{72} = 0.76388\cdots$ or greater than or equal to $\frac{199}{72} = 2.76388\cdots$. Finally, $\frac{3}{8} + 2j(j+1)$ is equal to $\frac{3}{8} = 0.375$ or greater than or equal to $\frac{35}{8} = 4.375$. Hence the eigenvalues $\{1, 1/q^2, 3\}$ do not belong to any of the three cases given in Theorem 3. This shows that when $q \in [\sqrt{0.6}, 1)$ system (5) is not completely meromorphically integrable and thus system (4) is not completely integrable with meromorphic first integrals. This completes the proof of Theorem 2.

Darbouxpoint	Eigenvalues
c_1	$\{1, 1/q^2, 3\}$
c_2	$\{1, 1/q^2, 3\}$
c_3	$\{1, 1/q^2, 3\}$
c_4	$\{1, 1/q^2, 3\}$
c_5	$\{1, 1/q^2, 3\}$
c_6	$\{1, 1/q^2, 3\}$
c_7	$\{q^2, q^2, 3\}$
c_8	$\{q^2, q^2, 3\}$

Table 1. Eigenvalues of $V''(c)$ of each Darboux point c.

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