This is a preprint of: "Limit cycles for continuous and discontinuous perturbations of uniform isochronous cubic centers", Jaume Llibre, Jackson Itikawa, *J. Comput. Appl. Math.*, vol. 277, 171–191, 2015.
DOI: [10.1016/j.cam.2014.09.007]

LIMIT CYCLES FOR CONTINUOUS AND DISCONTINUOUS PERTURBATIONS OF UNIFORM ISOCHRONOUS CUBIC CENTERS

JAUME LLIBRE[1] AND JACKSON ITIKAWA[1]

Abstract. Let p be a uniform isochronous cubic polynomial center. We study the maximum number of small or medium limit cycles that bifurcate from p or from the periodic solutions surrounding p respectively, when they are perturbed, either inside the class of all continuous cubic polynomial differential systems, or inside the class of all discontinuous differential systems formed by two cubic differential systems separated by the straight line $y = 0$.

In the case of continuous perturbations using the averaging theory of order 6 we show that the maximum number of small limit cycles that can appear in a Hopf bifurcation at p is 3, and this number can be reached. For a subfamily of these systems using the averaging theory of first order we prove that at most 3 medium limit cycles can bifurcate from the periodic solutions surrounding p, and this number can be reached.

In the case of discontinuous perturbations using the averaging theory of order 6 we prove that the maximum number of small limit cycles that can appear in a Hopf bifurcation at p is 5, and this number can be reached. For a subfamily of these systems using the averaging method of first order we show that the maximum number of medium limit cycles that can bifurcate from the periodic solutions surrounding p is 7, and this number can be reached.

We also provide all the first integrals and the phase portraits in the Poincaré disc for the uniform isochronous cubic centers.

1. Introduction and Statement of the Main Results

One of the main open problems in the qualitative theory of planar differential systems is the investigation of the limit cycles that can bifurcate from such systems when we vary the parameters.

A classical way to investigate limit cycles is perturbing a differential system which has a center. In this case the perturbed system displays

2010 *Mathematics Subject Classification.* Primary 34A36, 34C07, 34C25, 34C29, 34M35, 37G15.

Key words and phrases. polynomial vector field, limit cycle, averaging method, periodic orbit, uniform isochronous center.

limit cycles that bifurcate, either from the center (having the so-called Hopf bifurcation), or from some of the periodic orbits around the center, see for instance Pontrjagin [18], the second part of the book [4], and the hundreds of references quoted there. The problem of studying the limit cycles bifurcating from a center, or from its periodic solutions has been exhaustively studied in the last century and is closely related to the Hilbert's 16^{th} problem. Nevertheless, in spite of all efforts, there is no general method to solve this problem.

In the last decades several works about the bifurcation of limit cycles in planar differential systems having a uniform isochronous center have been published see for instance [1, 9, 11]. Aside from its importance in physical applications, isochronicity is closely related to the uniqueness and existence of solutions for some boundary value, perturbation, or bifurcation problems. It is also important in stability theory, since a periodic solution of the central region is Liapunov stable if and only if the neighboring periodic solutions have the same period. For more details on these two last paragraphs see [5]. Moreover, the interest in this problem has also been revived due to the proliferation of powerful methods of computerized research, and special attention has been dedicated to polynomial differential systems, see [3, 7] and the bibliography therein.

Let $p \in \mathbb{R}^2$ be a center of a differential polynomial system in $\mathbb{R}^2$, without loss of generality we can assume that p is the origin of coordinates. We say that p is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that p is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, takes the form $\dot{r} = G(\theta, r)$, $\dot{\theta} = k$, $k \in \mathbb{R} \setminus \{0\}$, for more details see Conti [7]. The next result is well-known.

Proposition 1. *Assume that a planar differential polynomial system of degree n has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time the system can be written as*

$$\dot{x} = -y + x\, f(x, y), \quad \dot{y} = x + y\, f(x, y),$$

where $f(x, y)$ is a polynomial in x and y of degree $n-1$, and $f(0, 0) = 0$.

The following result due to Collins [6] in 1997, also obtained by Devlin, Lloyd and Pearson [8] in 1998, and by Gasull, Prohens and Torregrosa [11] in 2005 characterizes the uniform isochronous centers of cubic polynomial systems.

Theorem 2. *A planar cubic differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{1}$$

where $f(x,y) = a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2$*, and satisfies* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$.

In this article a *small limit cycle* is one which bifurcates from a center equilibrium point, and a *medium limit cycle* is one which bifurcates from a periodic orbit surrounding a center.

We study the largest number of small and medium limit cycles for the uniform isochronous cubic centers, when they are perturbed either inside the class of all continuous cubic polynomial differential systems, or inside the class of all discontinuous differential systems formed by two cubic differential systems separated by the straight line $y = 0$. The method is based on the averaging theory. For more details about the averaging theory see the book of Sanders, Verhulst and Murdock [19].

In order to study the bifurcation phenomenon in these systems we take into account the following result due to Collins [6].

Proposition 3. *The planar cubic differential system* (1) *can be reduced to either one of the following two forms.*

$$\dot{x} = -y + x^2y, \quad \dot{y} = x + xy^2, \tag{2}$$

$$\dot{x} = -y + x^2 + Ax^2y, \quad \dot{y} = x + xy + Axy^2, \tag{3}$$

where $A \in \mathbb{R}$.

For now on we shall call (2) and (3) as *Collins first form* and *Collins second form*, respectively.

We consider the following continuous systems

$$\begin{aligned} \dot{x} &= -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y), \\ \dot{y} &= \ \ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y), \end{aligned} \tag{4}$$

where $f(x,y)$ is as in Theorem 2, and the system

$$\dot{x} = -y + x^2y + \varepsilon p_K(x,y), \quad \dot{y} = \ \ x + xy^2 + \varepsilon q_K(x,y), \tag{5}$$

where

$p_j = \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2 y + \alpha_8^j xy^2 + \alpha_9^j y^3,$

$q_j = \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2 y + \beta_8^j xy^2 + \beta_9^j y^3,$

$p_K = \alpha_0 + p_1, \qquad q_K = \beta_0 + q_1.$

Moreover we consider the discontinuous systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0; \\ X_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{6}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{Y}(x,y) = \begin{cases} Y_1(x,y) & \text{if} \quad y > 0; \\ Y_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{7}$$

where

$$X_1(x,y) = \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y) \end{pmatrix},$$

$$X_2(x,y) = \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i u_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i v_i(x,y) \end{pmatrix},$$

$$Y_1(x,y) = \begin{pmatrix} -y + x^2 y + \varepsilon p_K(x,y) \\ x + xy^2 + \varepsilon q_K(x,y) \end{pmatrix},$$

$$Y_2(x,y) = \begin{pmatrix} -y + x^2 y + \varepsilon u_K(x,y) \\ x + xy^2 + \varepsilon v_K(x,y) \end{pmatrix},$$

$u_j = \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2 y + \gamma_8^j xy^2 + \gamma_9^j y^3,$

$v_j = \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2 y + \delta_8^j xy^2 + \delta_9^j y^3,$

$u_K = \gamma_0 + u_1, \qquad v_K = \delta_0 + v_1.$

In what follows we state our main results.

Theorem 4. *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the differential system* (4) *is 3 using the averaging theory of order 6, and this number can be reached.*

Theorem 4 is proved in section 3. For more details on the averaging theory see section 2.

Theorem 5. *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the differential system* (5) *is 3 using the first order averaging theory and this number can be reached.*

Theorem 5 is proved in section 4.

Theorem 6. *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the discontinuous differential system* (6) *is 5 using the averaging method of order 6 and this number can be reached.*

Theorem 6 is proved in section 5.

Theorem 7. *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the discontinuous differential system* (7) *is 7 using the averaging method of first order and this number can be reached.*

Theorem 7 is proved in section 6.

Theorems 4 and 5 extend previous results presented in [11]. In that work the authors studied some subfamilies of uniform isochronous cubic centers, proving the existence of one or two limit cycles. Moreover Theorem 7 extend the work done in [16] on the number of medium limit cycles which can bifurcate from a family of uniform isochronous quadratic centers perturbed by discontinuous differential systems with the straight line of discontinuity $y = 0$, to the uniform isochronous cubic centers given by the Collins first form.

In this work we also provide the phase portraits and the first integrals for the uniform isochronous cubic centers.

Theorem 8. *The first integrals H of system* (1) *in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are described in what follows.*

Case 1: $\mathbf{a_1^2 - a_2^2 \neq 0}$.

Subcase 1.1: $\mathbf{a_4 \neq 0}$.

Subcase 1.1.1: $\mathbf{4a_4 \neq a_1^2 - a_2^2}$.

$$H = e^{-2\arctan\left[\frac{R+2a_4 r(-a_2\cos\theta + a_1\sin\theta)}{RS}\right]}$$
$$\left[\frac{a_4 r^2}{R + r(a_2\cos\theta - a_1\sin\theta)(a_2 a_4 r\cos\theta - a_1 a_4\sin\theta - R)}\right]^S,$$

where $R = a_1^2 - a_2^2$, $S = \sqrt{4a_4/R - 1}$.

In case of a negative square root, we have a complex first integral and therefore both its real and imaginary parts are also first integrals, if not null.

Subcase 1.1.2: $\mathbf{4a_4 = a_1^2 - a_2^2}$.

$$H = \frac{r e^{\frac{2}{2 - a_2 r\cos\theta + a_1 r\sin\theta}}}{2 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

Subcase 1.2: $\mathbf{a_4 = 0}$.

$$H = \frac{r}{1 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

Case 2: $\mathbf{a_1^2 - a_2^2 = 0}$.

Subcase 2.1: $\mathbf{a_2 = a_1}$.

Subcase 2.1.1: $\mathbf{a_1 = 0}$.

$$H = \frac{r^2}{1 - a_4 r^2\cos^2\theta + a_3 r^2\sin(2\theta)}.$$

Subcase 2.1.2: $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

Subcase 2.1.2.1: $\mathbf{a_3(a_1^2 + 4a_3) \neq 0}$.

$$H = e^{-2\arctan\left[\frac{a_1+2a_3 r(\cos\theta-\sin\theta)}{a_1 R}\right]} \left[\frac{a_3 r^2(\sin(2\theta)-1)}{(\cos\theta-\sin\theta)^2[1+a_1 r(\sin\theta-\cos\theta)+a_3 r^2(\sin(2\theta)-1)]}\right]^R,$$

where $R = \sqrt{-1 - 4a_3/a_1^2}$.

Subcase 2.1.2.2: $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

Subcase 2.1.2.3: $\mathbf{a_3 = -a_1^2/4}$.

$$H = \frac{2re^{\frac{2}{2-a_1 r(\cos\theta-\sin\theta)}}}{2 - a_1 r(\cos\theta - \sin\theta)}.$$

Subcase 2.2: $\mathbf{a_2 = -a_1}$.

Subcase 2.2.1: $\mathbf{a_1 = 0}$. *This case becomes the subcase 2.1.1.*

Subcase 2.2.2: $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

Subcase 2.2.2.1: $\mathbf{a_3(4a_3 - a_1^2) \neq 0}$.

$$H = \frac{e^{\frac{1}{R}\left[-2\arctan\left(\frac{a_1+2a_3 r(\sin\theta+\cos\theta)}{a_1 R}\right)+R\,\mathrm{arctanh}(\tan\theta)\right]} a_3 r^2(\sec(2\theta)+\tan(2\theta))}{1 + a_1 r(\sin\theta+\cos\theta) + a_3 r^2(1+\sin(2\theta))},$$

where $R = \sqrt{4a_3/a_1^2 - 1}$.

Subcase 2.2.2.2: $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

Subcase 2.2.2.3: $\mathbf{a_3 = a_1^2/4}$.

$$H = \frac{r}{e^{1+\frac{1}{2}a_1 r(\cos\theta+\sin\theta)}\left(1+\frac{1}{2}a_1 r(\cos\theta+\sin\theta)\right)}.$$

Theorem 8 is proved in section 7.

We say that two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on the Poincaré sphere $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by the Poincaré compactified vector field of X into orbits of the flow induced by the the Poincaré compactified vector field of Y preserving or reversing simultaneously the sense of all orbits. For more details on the Poincaré compactification see Chapter 5 of [10].

Theorem 9. *The global phase portrait in the Poincaré disc of the differential system* (1) *is topologically equivalent to one of the three phase portraits presented in Figure 1.*

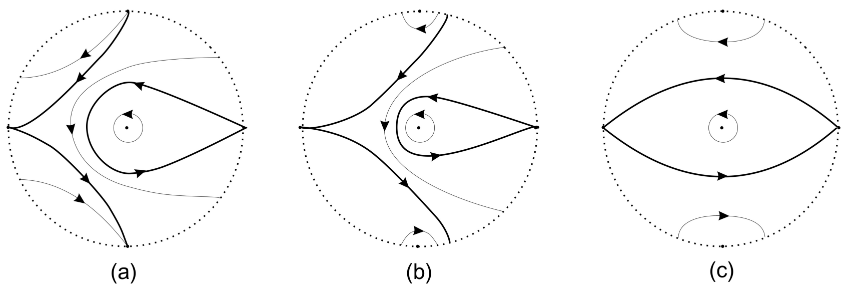

FIGURE 1. Phase portraits of cubic uniform isochronous centers

More precisely, the global phase portrait of (1) *is topologically equivalent to the phase portrait (a) of Figure 1 if one of the following conditions holds*

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 \leq (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $0 < a_3 \leq a_1^2/4$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $-a_1^2/4 \leq a_3 < 0$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $-a_2^2/4 \leq a_4 < 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $0 < a_4 \leq a_1^2/4$;

the phase portrait (b) if one of the following conditions holds

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 > (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 > a_1^2/4$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $a_3 < -a_1^2/4$, *and* $a_4 = 0$;

- $a_1 = 0$, *and* $a_2 \neq 0$ *and* $a_4 < -a_2^2/4$;
- $a_1 \neq 0$, *and* $a_2 = 0$ *and* $a_4 > a_1^2/4$;

the phase portrait (c) if one of the following conditions holds

- $a_1 a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) < 0$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 < 0$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $a_3 > 0$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $a_4 > 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $a_4 < 0$;
- $a_1 = a_2 = 0$.

The cases where $a_3 = a_4 = 0$ are omitted in Theorem 9 because in such cases system (1) is a quadratic polynomial differential system, which has already been exhaustively studied, see for instance system S_2 at p.38 of [3].

Theorem 9 is proved in section 8.

Collins [6] presented the phase portraits and first integrals for the uniform isochronous cubic centers, but he applied the forms (2) and (3) in order to obtain the results and therefore one needs to change the differential systems to such forms before getting the phase portraits and first integrals. Our results present the first integrals in terms of all the parameters of the uniform isochronous centers. Moreover, the phase portrait for the case $A = 1/4$, Figure 2-d, pp. 347 of [6] is not correct, because it presents two saddle-nodes at infinity which do not exist.

The rest of the paper is organized as follows. In section 2 we present some results on the averaging theory and technical propositions used in our study. The next four sections are dedicated to prove our main results. More precisely in those sections we present the proofs of Theorems 4, 5, 6 and 7, respectively. Finally, in sections 7 and 8 we respectively provide the proofs of Theorems 8 and 9. All calculations were performed with the assistance of the software *Mathematica*.

2. Preliminary results

In this section we introduce some preliminary results on the averaging theory that we shall use in our study of the uniform isochronous cubic centers.

The following result is due to Llibre, Novaes and Teixeira [15].

Consider the general differential system

$$\dot{x}(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) \ + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{8}$$

where $F_i : \mathbb{R}\times I \to \mathbb{R}^n$ for $i = 1,2,\ldots,k$ and $R : \mathbb{R}\times I\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are continuous functions and T-periodic in the first variable, I being an open subset of $\mathbb{R}^n$.

Moreover, let L be a positive integer, $x = (x_1, x_2, \ldots, x_n) \in I$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, y_{j2}, \ldots, y_{jn}) \in \mathbb{R}^n$, $j = 1,\ldots,L$. Given $F : \mathbb{R} \times I \to \mathbb{R}^n$ a sufficiently smooth function, for each $(t,x) \in \mathbb{R}\times I$ we denote by $\partial^L F(t,x)$ a symmetric L-multilinear map which is applied to a 'product' of L vectors of $\mathbb{R}^n$, which we denote as $\bigodot_{j=1}^{L} y_j \in \mathbb{R}^{nL}$. The definition of such L-multilinear map is

$$\partial^L F(t,x) \bigodot_{j=1}^{L} y_j = \sum_{i_1,\ldots,i_L=1}^{n} \frac{\partial^L F(t,x)}{\partial x_{i_1},\ldots,\partial x_{i_L}} y_{1i_1}\cdots y_{Li_L}. \tag{9}$$

We define ∂^0 as the identity functional. Given a positive integer b and a vector $y \in \mathbb{R}^n$ we also denote $y^b = \bigodot_{j=1}^{b} y \in \mathbb{R}^{nb}$.

Let $\varphi(\cdot,z) : [0,t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system $\dot{x}(t) = F_0(t,x)$ such that $\varphi(0,z) = z$. For $i = 1,\ldots,k$ we define the averaged function $f_i : I \to \mathbb{R}^n$ of order i as

$$f_i(z) = \frac{y_i(T,z)}{i!}, \tag{10}$$

where $y_i : \mathbb{R}\times I \to \mathbb{R}^n$, $i = 1,\ldots,k-1$ are defined recurrently by the following integral equation.

$$\begin{aligned} y_i(t,z) = i! \int_0^t \Bigg[& F_i(s,\varphi(s,z)) \\ & + \sum_{l=1}^{i}\sum_{S_l} \frac{1}{b_1!\,b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}} \partial^L F_{i-l}(s,\varphi(s,z)) \bigodot_{j=1}^{l} y_j(s,z)^{b_j} \Bigg] ds, \end{aligned} \tag{11}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \ldots, b_l)$ satisfying $b_1 + 2b_2 + \ldots lb_l = l$ and $L = b_1 + b_2 + \ldots + b_l$. Observe that if $F_0 = 0$ then $\varphi(t,z) = z$ for each $t \in \mathbb{R}$. Therefore $y_1(t,z) = \int_0^t F_1(s,z)ds$ and $f_1(t,z) = \int_0^T F_1(t,z)dt$ as usual in averaging theory.

Theorem 10. *Suppose $F_0 = 0$. In addition, for the functions of* (8) *we assume the following conditions.*

(i) For each $t \in \mathbb{R}$, $F_i(t,\cdot) \in \mathcal{C}^{k-i}$, $i = 1,\ldots,k$, $\partial^{k-i} F_i$ is locally Lipschitz in the second variable for $i = 1,\ldots,k$ and R is a continuous function locally Lipschitz in the second variable;

(ii) Assume that $f_i = 0,\ i = 1, \ldots, r-1$ *and* $f_r \neq 0,\ r \in \{1, \ldots, k\}$ *(here we are taking* $f_0 = 0$*). Moreover, suppose that for some* $a \in I$ *with* $f_r(a) = 0$ *there exists a neighborhood* $V \subset I$ *of* a *such that* $f_r(z) \neq 0, \forall z \in \bar{V} \setminus \{a\}$ *and* $d_B(f_r(z), V, 0) \neq 0$.

Then, for sufficiently small $|\varepsilon| > 0$ *there exists a T-periodic solution* $x(\cdot, \varepsilon)$ *of* (8) *such that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$.

The proof of this theorem can be found in section 3 of [15].

The next result provides a method to write a perturbed differential system under the form (8) for $k = 1$ and $F_0 = 0$ which can be used to apply the averaging theory of first order.

Theorem 11. *Consider the unperturbed system* $\dot{x} = P(x, y),\ \dot{y} = Q(x, y)$*, where* $P, Q : \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions, and assume that this system has a continuous family of period solutions* $\{\Gamma_h\} \subset \{(x, y) : \mathcal{H}(x, y) = h, h_1 < h < h_2\}$*, where* $\mathcal{H}$ *is a first integral of the system. For a given first integral* H *assume that* $xQ(x, y) - yP(x, y) \neq 0$ *for all* (x, y) *in the period annulus formed by the ovals* $\{\Gamma_h\}$*. Let* $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ *be a continuous function such that*

$$H(\rho(R, \theta) \cos\theta, \rho(R, \theta) \sin\theta) = R^2$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ *and all* $\theta \in [0, 2\pi)$*. Then the differential equation which describes the dependence between the square root of the energy* $R = \sqrt{h}$ *and the angle* θ *for the perturbed system* $\dot{x} = P(x, y) + \varepsilon p(x, y),\ \dot{y} = Q(x, y) + \varepsilon q(x, y)$*, where* $p, q : \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions is*

$$\frac{dR}{d\theta} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2) \tag{12}$$

where $\mu = \mu(x, y)$ *is the integrating factor corresponding to the first integral* H *of the unperturbed system and* $x = \rho(R, \theta) \cos\theta$*,* $y = \rho(R, \theta) \sin\theta$.

For more details see [2]. We also need the next results. The first one can be found in Proposition 1 of [17] and the latter in [12].

Proposition 12. *Let* $f_0, \ldots, f_n$ *be analytic functions defined on an open interval* $I \subset \mathbb{R}$*. If* $f_0, \ldots, f_n$ *are linearly independent then there exists* $s_1, \ldots, s_n \in I$ *and* $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ *such that for every* $j \in \{1, \ldots, n\}$ *we have* $\sum_{i=0}^{n} \lambda_i f_i(s_j) = 0$.

We say that the functions $(f_0, \ldots, f_n)$ defined on the interval I form an *Extendend Chebyshev system* or ET-system on I, if and only if, any nontrivial linear combination of these functions has at most n zeros counting their multiplicities and this number is reached. The functions $(f_0, \dots, f_n)$ are an *Extendend Complete Chebyshev system* or an ECT-system on I if and only if for any $k \in \{0, 1, \dots, n\}$, $(f_0, \dots, f_k)$ form an ET-system.

Theorem 13. *Let $f_0, \dots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then $(f_0, \dots, f_n)$ is an ECT-system on I if and only if for each $k \in \{0, 1, \dots, n\}$ and all $y \in I$ the Wronskian*

$$W(f_0, \dots, f_k)(y) = \begin{vmatrix} f_0(y) & f_1(y) & \cdots & f_k(y) \\ f_0'(y) & f_1'(y) & \cdots & f_k'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \cdots & f_k^{(k)}(y) \end{vmatrix}$$

is different from zero.

The next result follows easily from Lemma 2.13 of [13].

Proposition 14. *Consider $g(t) = f(t - \lambda) f(\lambda - t)$, for $\lambda \in \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^\infty$, defined by*

$$f(t) = \begin{cases} e^{-1/t} & \text{if } \; t > 0; \\ 0 & \text{if } \; t \le 0. \end{cases}$$

Clearly g is $\mathcal{C}^\infty$, nonnegative and $g(t) > 0 \Leftrightarrow t \in (-\lambda, \lambda)$. Then the function h defined by

$$h(t) = 2 \frac{\int_{-\infty}^{t} g(s) ds}{\int_{-\infty}^{\infty} g(s) ds} - 1$$

is $\mathcal{C}^\infty$ and $h(t) = -1$ for $t \le -\lambda$, $h(t) = 1$ for $t \ge \lambda$ and $-1 < h(t) < 1$ otherwise.

3. Proof of Theorem 4

We use the Collins first and second forms, respectively systems (2) and (3) in this article to prove Theorem 4. We were able to apply up to the averaging theory of order 6.

Collins first form

Consider system (4) with $f(x,y) = xy$, that is, the unperturbed system is the Collins first form.

$$
\begin{aligned}
\dot{x} &= -y + x^2 y + \sum_{i=1}^{6} \varepsilon^i p_i(x,y),\\
\dot{y} &= \ \ x + xy^2 + \sum_{i=1}^{6} \varepsilon^i q_i(x,y).
\end{aligned} \tag{13}
$$

In order to analyze the Hopf bifurcation for system (13), applying Theorem 10, we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$, and by doing a Taylor expansion truncated at the 6^{th} order in ε we obtain an expression for $dr/d\theta$ similar to (8) with $F_0 = 0, k = 6$. The explicit expression is quite large so we omit it.

System (13) is a polynomial system. The functions $F_i(\theta, r),\ i = 1,\ldots,6$ and $R(\theta, r, \varepsilon)$ of system (13) are analytic, and since the variable θ appears through sinus and cosinus, they are $2\pi-$periodic. Hence the assumptions of Theorem 10 are satisfied. We take I of Theorem 10 as $I = \{r : 0 < r < 1\}$ because the Collins first form has the period annulus of the center in the band $-1 < x < 1$.

Applying Theorem 10 we obtain the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1).$$

Clearly $f_1(r)$ has no solution in I. Thus there is no small limit cycle which bifurcates from the uniform isochronous center at the origin by the averaging method of first order. Setting $\beta_2^1 = -\alpha_1^1$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of second order using Theorem 10, obtaining the averaging function of second order.

$$f_2(r) = \pi r(\alpha_1^2 + \beta_2^2).$$

Since $f_2(r)$ has no solution in I, there is no small limit cycle which bifurcates from the uniform isochronous center at the origin applying the averaging method of second order. Doing $\beta_2^2 = -\alpha_1^2$ we get $f_2(r) = 0$, and then we can apply the averaging method of third order obtaining

$$f_3(r) = r(A_3 r^2 + A_1),$$

where

$$A_3 = \frac{\pi}{4}(4\alpha_1^1 + 3\alpha_6^1 + \alpha_8^1 + \beta_7^1 + 3\beta_9^1), \quad A_1 = \pi(\alpha_1^3 + \beta_2^3).$$

Thus $f_3(r)$ has one solution in I if $0 < -A_1/A_3 < 1$. Hence applying the averaging theory of third order it is proved that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached.

In order to apply the averaging method of fourth order, we need to have $f_3(r) = 0$ so we set $\beta_2^3 = -\alpha_1^3$ and $\beta_7^1 = -(4\alpha_1^1 + 3\alpha_6^1 + \alpha_8^1 + 3\beta_9^1)$. The resulting averaging function of fourth order is

$$f_4(r) = r(B_3 r^2 + B_1),$$

where

$$\begin{aligned} B_3 =& \frac{\pi}{4}(4\alpha_1^1\alpha_2^1 + 2\alpha_1^1\alpha_7^1 + 2\alpha_1^1\beta_8^1 + 3\beta_1^1\beta_9^1 + \alpha_2^1\alpha_8^1 + 3\alpha_2^1\beta_9^1 - 2\alpha_3^1\beta_3^1 \\ & + \alpha_3^1\alpha_4^1 - \beta_3^1\beta_4^1 + \alpha_4^1\alpha_5^1 - \beta_4^1\beta_5^1 + 2\alpha_5^1\beta_5^1 + \beta_1^1\alpha_8^1 + 4\alpha_1^2 + 3\alpha_6^2 \\ & + \beta_7^2 + \alpha_8^2 + 3\beta_9^2), \\ B_1 =& \pi(\alpha_1^4 + \beta_2^4). \end{aligned}$$

Then $f_4(r)$ has one solution in I if $0 < -B_1/B_3 < 1$. Hence we can show that at most 1 small limit cycle can bifurcate from the uniform isochronous center and this number can be reached. Solving $B_1 = 0$ for β_2^4 and $B_3 = 0$ for β_7^2, we obtain $f_4(r) = 0$ so we can apply the averaging theory of order 5, and its corresponding averaging function is

$$f_5(r) = r(C_5 r^4 + C_3 r^2 + C_1),$$

where

$$\begin{aligned} C_5 =& \frac{\pi}{4}(2\alpha_1^1 + 2\alpha_6^1 + \alpha_8^1 + \beta_9^1), \\ C_3 =& \frac{\pi}{4}(4\alpha_1^1(\alpha_2^1)^2 + 2\alpha_1^1\alpha_2^1\alpha_7^1 + 2\alpha_1^1\alpha_2^1\beta_8^1 + 2\alpha_1^1(\alpha_3^1)^2 - \alpha_1^1\alpha_3^1\beta_4^1 \\ & + \beta_1^1\beta_3^1\beta_4^1 + 2\alpha_1^1\alpha_3^1\alpha_5^1 - 2\alpha_1^1\beta_3^1\beta_5^1 + \alpha_1^1(\alpha_4^1)^2 - \alpha_1^1(\beta_4^1)^2 - \alpha_1^1\beta_3^1\alpha_4^1 \\ & + \alpha_1^1\alpha_4^1\beta_5^1 - 2\alpha_1^1(\beta_5^1)^2 + \alpha_1^1\beta_4^1\alpha_5^1 + 4\alpha_1^1\alpha_2^2 + 2\alpha_1^1\alpha_7^2 + 2\alpha_1^1\beta_8^2 \\ & + 3\beta_1^1\beta_9^2 + (\alpha_2^1)^2\alpha_8^1 + 3(\alpha_2^1)^2\beta_9^1 + 3\beta_1^1\alpha_2^1\beta_9^1 + \alpha_2^1\alpha_3^1\alpha_4^1 + 2\alpha_2^1\alpha_4^1\alpha_5^1 \\ & - \alpha_2^1\beta_4^1\beta_5^1 + 4\alpha_2^1\alpha_5^1\beta_5^1 + \beta_1^1\alpha_2^1\alpha_8^1 + 4\alpha_2^1\alpha_1^2 + \alpha_2^1\alpha_8^2 + 3\alpha_2^1\beta_9^2 \\ & + 2\beta_1^1\alpha_3^1\beta_3^1 - 2\alpha_3^1\beta_3^2 + \alpha_3^1\alpha_4^2 - \beta_3^1\beta_4^2 + \beta_1^1\alpha_4^1\alpha_5^1 + \alpha_4^1\alpha_3^2 - \beta_4^1\beta_3^2 \\ & + \alpha_4^1\alpha_5^2 - \beta_4^1\beta_5^2 + 2\beta_1^1\alpha_5^1\beta_5^2 + \alpha_5^1\alpha_4^2 - \beta_5^1\beta_4^2 + 2\alpha_5^1\beta_5^2 + 2\alpha_7^1\alpha_1^2 \\ & + \alpha_8^1\beta_1^2 + \alpha_8^1\alpha_2^2 + 3\beta_9^1\beta_1^2 + 2\beta_8^1\alpha_1^2 + 3\beta_9^1\alpha_2^2 - 2\beta_3^1\alpha_3^2 + 2\beta_5^1\alpha_5^2 \\ & + \beta_1^1\alpha_8^2 + 4\alpha_1^3 + 3\alpha_6^3 + \beta_7^3 + \alpha_8^3 + 3\beta_9^3), \\ C_1 =& \pi(\alpha_1^5 + \beta_2^5). \end{aligned}$$

The averaging function of fifth order $f_5(r)$ can have at most 2 solutions in I. Thus applying the averaging method of order 5 it is proved that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

In order to apply the averaging theory of order 6 we solve $C_1 = 0$ for β_2^5, $C_3 = 0$ for β_7^3 and C_5 for β_9^1, resulting that $f_5(r) = 0$. Calculating the averaging function of sixth order we have

$$f_6(r) = r(D_5 r^4 + D_4 r^3 + D_3 r^2 + D_1),$$

where

$$\begin{aligned}
D_5 = &-\frac{1}{384}\pi(45\alpha_1^1\alpha_2^1 + 192\alpha_6^1\alpha_2^1 - 112\alpha_3^1\alpha_4^1 - 112\alpha_4^1\alpha_5^1 - 192\alpha_1^1\alpha_7^1 \\
&+ 96\alpha_1^1\alpha_9^1 + 288\alpha_6^1\alpha_9^1 + 96\alpha_8^1\alpha_9^1 - 192\alpha_1^2 - 192\alpha_6^2 - 96\alpha_8^2 \\
&+ 192\alpha_6^1\beta_1^1 + 288\alpha_3^1\beta_3^1 + 64\alpha_5^1\beta_3^1 - 16\beta_3^1\beta_4^1 + 320\alpha_3^1\beta_5^1 + 96\alpha_5^1\beta_5^1 \\
&+ 237\alpha_1^1\beta_1^1 - 16\beta_4^1\beta_5^1 + 288\alpha_1^1\beta_6^1 + 288\alpha_6^1\beta_6^1 + 96\alpha_8^1\beta_6^1 - 96\beta_9^2), \\
D_4 = &-\frac{1}{8}\alpha_1^1\pi(\alpha_2^1 + \beta_1^1)(\alpha_4^1 + \beta_3^1 + 2\beta_5^1), \\
D_3 = &-\frac{1}{512}\pi(108\alpha_2^1(\alpha_1^1)^3 + 36\beta_1^1(\alpha_1^1)^3 - 384\alpha_3^1\alpha_4^1(\alpha_1^1)^2 \\
&+ 72\alpha_1^2(\alpha_1^1)^2 + 256\alpha_3^1\beta_3^1(\alpha_1^1)^2 + 128\beta_3^1\beta_4^1(\alpha_1^1)^2 - 256\alpha_5^1\beta_5^1(\alpha_1^1)^2 \\
&+ 384\beta_4^1\beta_5^1(\alpha_1^1)^2 + 319(\alpha_2^1)^3\alpha_1^1 - 27(\beta_1^1)^3\alpha_1^1 - 256\alpha_2^1(\alpha_3^1)^2\alpha_1^1 \\
&- 256\alpha_2^1(\alpha_4^1)^2\alpha_1^1 + 9\alpha_2^1(\beta_1^1)^2\alpha_1^1 + 128\alpha_2^1(\beta_4^1)^2\alpha_1^1 - 128\beta_1^1(\beta_4^1)^2\alpha_1^1 \\
&- 128\alpha_4^1\alpha_5^1(\alpha_1^1)^2 + 512\alpha_2^1(\beta_5^1)^2\alpha_1^1 - 512\alpha_2^1\alpha_3^1\alpha_5^1\alpha_1^1 - 256\alpha_5^1\alpha_3^2\alpha_1^1 \\
&+ 572\alpha_2^1\alpha_2^2\alpha_1^1 - 256\alpha_7^1\alpha_2^2\alpha_1^1 - 512\alpha_3^1\alpha_3^2\alpha_1^1 - 256(\alpha_2^1)^2\alpha_7^1\alpha_1^1 \\
&- 256\alpha_4^1\alpha_4^2\alpha_1^1 - 256\alpha_3^1\alpha_5^2\alpha_1^1 - 256\alpha_2^1\alpha_7^2\alpha_1^1 + 256\alpha_2^3\alpha_1^1 - 256\alpha_7^3\alpha_1^1 \\
&+ 867(\alpha_2^1)^2\beta_1^1\alpha_1^1 + 256(\alpha_3^1)^2\beta_1^1\alpha_1^1 + 828\alpha_2^2\beta_1^1\alpha_1^1 + 128\alpha_2^1\alpha_4^1\beta_3^1\alpha_1^1 \\
&+ 128\alpha_4^2\beta_3^1\alpha_1^1 - 128\alpha_4^1\beta_1^1\beta_3^1\alpha_1^1 + 128\alpha_2^1\alpha_3^1\beta_4^1\alpha_1^1 - 256\alpha_2^1\alpha_5^1\beta_4^1\alpha_1^1 \\
&+ 128\alpha_3^2\beta_4^1\alpha_1^1 - 128\alpha_5^2\beta_4^1\alpha_1^1 - 128\alpha_3^1\beta_1^1\beta_4^1\alpha_1^1 - 256\alpha_2^1\alpha_4^1\beta_5^1\alpha_1^1 \\
&- 128\alpha_4^2\beta_5^1\alpha_1^1 + 256\alpha_2^1\beta_3^1\beta_5^1\alpha_1^1 - 256\beta_1^1\beta_3^1\beta_5^1\alpha_1^1 - 256(\alpha_2^1)^2\beta_8^1\alpha_1^1 \\
&- 256\alpha_2^2\beta_8^1\alpha_1^1 + 828\alpha_2^1\beta_1^2\alpha_1^1 + 60\beta_1^1\beta_1^2\alpha_1^1 + 128\alpha_4^1\beta_3^2\alpha_1^1 \\
&+ 256\beta_5^1\beta_3^2\alpha_1^1 + 128\alpha_3^1\beta_4^2\alpha_1^1 - 128\alpha_5^1\beta_4^2\alpha_1^1 + 256\beta_4^1\beta_4^2\alpha_1^1 \\
&- 128\alpha_4^1\beta_5^2\alpha_1^1 + 256\beta_3^1\beta_5^2\alpha_1^1 + 512\beta_5^1\beta_5^2\alpha_1^1 - 256\alpha_2^1\beta_8^2\alpha_1^1 \\
&+ 768\beta_1^3\alpha_1^1 - 256\beta_8^3\alpha_1^1 + 30\alpha_1^2(\beta_1^1)^2 + 128\alpha_1^2(\beta_4^1)^2 \\
&+ 256\alpha_1^2(\beta_5^1)^2 - 128(\alpha_2^1)^2\alpha_3^1\alpha_4^1 - 384(\alpha_2^1)^2\alpha_4^1\alpha_5^1 + 768(\alpha_2^1)^3\alpha_6^1 \\
&+ 256(\alpha_2^1)^3\alpha_8^1 - 482(\alpha_2^1)^2\alpha_1^2 - 256(\alpha_3^1)^2\alpha_1^2 - 128(\alpha_4^1)^2\alpha_1^2 \\
&- 256\alpha_3^1\alpha_5^1\alpha_1^2 - 256\alpha_2^1\alpha_7^1\alpha_1^2 - 128\alpha_3^1\alpha_4^1\alpha_2^2 - 256\alpha_4^1\alpha_5^1\alpha_2^2
\end{aligned}$$

$$
\begin{aligned}
&+1536\alpha_2^1\alpha_6^1\alpha_2^2+512\alpha_2^1\alpha_8^1\alpha_2^2-512\alpha_1^2\alpha_2^2-128\alpha_2^1\alpha_4^1\alpha_3^2\\
&-128\alpha_2^1\alpha_3^1\alpha_4^2-256\alpha_2^1\alpha_5^1\alpha_4^2-128\alpha_3^2\alpha_4^2-256\alpha_2^1\alpha_4^1\alpha_5^2\\
&-128\alpha_4^2\alpha_5^2-256\alpha_1^2\alpha_7^2-128(\alpha_2^1)^2\alpha_8^2-128\alpha_2^2\alpha_8^2-512\alpha_2^1\alpha_1^3\\
&-256\alpha_7^1\alpha_1^3+768\alpha_6^1\alpha_2^3+256\alpha_8^1\alpha_2^3-128\alpha_4^1\alpha_3^3-128\alpha_3^1\alpha_4^3\\
&-128\alpha_5^1\alpha_4^3-128\alpha_4^1\alpha_5^3-128\alpha_2^1\alpha_8^3-512\alpha_1^4-384\alpha_6^4-128\alpha_8^4\\
&-256\alpha_2^1\alpha_4^1\alpha_5^1\beta_1^1+768(\alpha_2^1)^2\alpha_6^1\beta_1^1+256(\alpha_2^1)^2\alpha_8^1\beta_1^1+60\alpha_2^1\alpha_1^2\beta_1^1\\
&+768\alpha_6^1\alpha_2^2\beta_1^1+256\alpha_8^1\alpha_2^2\beta_1^1-128\alpha_5^1\alpha_4^2\beta_1^1-128\alpha_4^1\alpha_5^2\beta_1^1\\
&-128\alpha_2^1\alpha_8^2\beta_1^1-128\alpha_8^3\beta_1^1+256\alpha_3^1(\beta_1^1)^2\beta_3^1+128\alpha_4^1\alpha_1^2\beta_3^1\\
&+256\alpha_3^3\beta_3^1-256\alpha_3^2\beta_1^1\beta_3^1+128\alpha_3^1\alpha_1^2\beta_4^1-128\alpha_5^1\alpha_1^2\beta_4^1\\
&+128(\beta_1^1)^2\beta_3^1\beta_4^1-768(\alpha_2^1)^2\alpha_5^1\beta_5^1-128\alpha_4^1\alpha_1^2\beta_5^1-512\alpha_5^1\alpha_2^2\beta_5^1\\
&-512\alpha_2^1\alpha_5^2\beta_5^1-256\alpha_5^3\beta_5^1-512\alpha_2^1\alpha_5^1\beta_1^1\beta_5^1-256\alpha_5^2\beta_1^1\beta_5^1\\
&+256\alpha_1^2\beta_3^1\beta_5^1+128(\alpha_2^1)^2\beta_4^1\beta_5^1+128\alpha_2^2\beta_4^1\beta_5^1-256\alpha_2^1\alpha_1^2\beta_8^1\\
&-256\alpha_1^3\beta_8^1-128\alpha_4^1\alpha_5^1\beta_1^2+768\alpha_2^1\alpha_6^1\beta_1^2+256\alpha_2^1\alpha_8^1\beta_1^2-128\alpha_8^2\beta_1^2\\
&-256\alpha_3^1\beta_3^1\beta_1^2-128\beta_3^1\beta_4^1\beta_1^2-256\alpha_5^1\beta_5^1\beta_1^2+256\alpha_3^2\beta_3^2-256\alpha_3^1\beta_1^1\beta_3^2\\
&-128\beta_1^1\beta_4^1\beta_3^2-128\beta_1^1\beta_3^1\beta_4^2+128\alpha_2^1\beta_5^1\beta_4^2+128\beta_3^2\beta_4^2-512\alpha_2^1\alpha_5^1\beta_5^2\\
&-256\alpha_5^2\beta_5^2-256\alpha_5^1\beta_1^1\beta_5^2+128\alpha_2^1\beta_4^1\beta_5^2+128\beta_4^2\beta_5^2-256\alpha_1^2\beta_8^2\\
&-384(\alpha_2^1)^2\beta_9^2-384\alpha_2^2\beta_9^2-384\alpha_2^1\beta_1^1\beta_9^2-384\beta_1^2\beta_9^2+768\alpha_6^1\beta_1^3\\
&+256\alpha_8^1\beta_1^3+256\alpha_3^1\beta_3^3+128\beta_4^1\beta_3^3+128\beta_3^1\beta_4^3+128\beta_5^1\beta_4^3\\
&-256\alpha_5^1\beta_5^3+128\beta_4^1\beta_5^3-384\alpha_2^1\beta_9^3-384\beta_1^1\beta_9^3-128\beta_7^4-384\beta_9^4),\\
D_1=&\pi(\alpha_1^6+\beta_2^6).
\end{aligned}
$$

Therefore $f_6(r)$ can have 3 solutions in I according to Proposition 12. By Theorem 13 (r, r^3, r^4, r^5) is an ECT-system because $W_1(z) = z$, $W_2(z) = 2z^3$, $W_3(z) = 6z^5$, $W_4(z) = 48z^7$, where $W_j(z), j = 1, 2, 3$ denotes the Wronskian of the first j functions in (r, r^3, r^4, r^5), are nonzero in I. Moreover D_1, D_3, D_4 and D_5 are linearly independent functions. In fact only D_5 presents the coefficients α_9^1 and α_6^2, only D_3 has the coefficient α_2^2, and D_1 is the only one with the coefficients α_1^6 and β_2^6. We claim that D_4 is also linearly independent of the other coefficients. Suppose that this is false. Then there exist real numbers k, l, m not all zero such that $D_4 = kD_1+lD_3+mD_5$. But D_1 is the only one with the variables α_1^6 and β_2^6, so in order to D_4 does not present these variables we must set $k = 0$. Since the other two functions D_3 and D_5 also have variables which uniquely appears in their respective expressions, the same argument holds so $l = m = 0$. But then $D_4 \equiv 0$, which is a contradiction. Therefore D_1, D_3, D_4 and D_5 are linearly independent functions. Hence applying the averaging theory of order 6 we can show that at most 3 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached.

Now we perform similar calculations to the Collins second form.

Collins second form

Consider system (4) with $f(x,y) = x + Axy$.

$$\begin{aligned} \dot{x} &= -y + x^2 + Ax^2y + \sum_{i=1}^{6} \varepsilon^i p_i(x,y), \\ \dot{y} &= \quad x + xy + Axy^2 + \sum_{i=1}^{6} \varepsilon^i q_i(x,y), \end{aligned} \tag{14}$$

where $A \in \mathbb{R}\backslash\{0\}$, since for $A = 0$ system (14) is a quadratic system, which has been exhaustively studied.

Similarly to the previous procedures applied in the Collins first form, in order to analyze the Hopf bifurcation for system (14), applying Theorem 10, we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$, and by doing a Taylor expansion truncated at the 6^{th} order in ε we obtain an expression for $dr/d\theta$ similar to (8) with $F_0 = 0, k = 6$. Using the same arguments as in the proof of the Collins first form the differential equation $dr/d\theta = \ldots$ satisfies the assumptions of Theorem 10. We take $I = \{r : 0 < r < r_0 < 1\}$, where the unperturbed system has periodic solutions passing through the point $(r < r_0, \theta = 0)$.

Applying Theorem 10 we obtain the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1).$$

Clearly $f_1(r)$ has no solution in I. Setting $\beta_2^1 = -\alpha_1^1$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of order 2 using Theorem 10, obtaining

$$f_2(r) = \pi r(\alpha_1^2 + \beta_2^2)$$

Again $f_2(r)$ has no solution in I. Doing $\beta_2^2 = -\alpha_1^2$ we get $f_2(r) = 0$. Then we can apply the averaging method of third order

$$f_3(r) = r(A_3 r^2 + A_1),$$

where

$$A_3 = \frac{\pi}{4}(4A\alpha_1^1 + \alpha_4^1 + 3\alpha_6^1 + \alpha_8^1 - 3\beta_3^1 - \beta_5^1 + \beta_7^1 + 3\beta_9^1),$$

$$A_1 = \pi(\alpha_1^3 + \beta_2^3).$$

Thus $f_3(r)$ can have one solution in I if $0 < -A_1/A_3 < r_0$. In order to apply the averaging method of forth order, we need to have $f_3(r) = 0$. We set $\beta_2^3 = -\alpha_1^3$ and $\beta_7^1 = -(4A\alpha_1^1 + \alpha_4^1 + 3\alpha_6^1 + \alpha_8^1 - 3\beta_3^1 - \beta_5^1 + 3\beta_9^1)$. The resulting averaging function of fourth order is

$$f_4(r) = r(B_3 r^2 + B_1),$$

where

$$\begin{aligned} B_3 =& \frac{\pi}{4}(4A\alpha_1^1\alpha_2^1 + 4A\alpha_1^2 + 3\alpha_1^1\alpha_3^1 + 3\beta_1^1\beta_3^1 - 3\alpha_1^1\beta_4^1 + 3\alpha_1^1\alpha_5^1 + 2\alpha_1^1\alpha_7^1 \\ &+ 2\alpha_1^1\beta_8^1 + 3\beta_1^1\beta_9^1 + \alpha_2^1\alpha_4^1 - \alpha_2^1\beta_5^1 + \alpha_2^1\alpha_8^1 + 3\alpha_2^1\beta_9^1 - 2\alpha_3^1\beta_3^1 \\ &+ \alpha_3^1\alpha_4^1 - \beta_3^1\beta_4^1 + \alpha_4^1\alpha_5^1 - \beta_4^1\beta_5^1 + 2\alpha_5^1\beta_5^1 + \beta_1^1\alpha_8^1 - 3\beta_3^2 + \alpha_4^2 \\ &- \beta_5^2 + 3\alpha_6^2 + \beta_7^2 + \alpha_8^2 + 3\beta_9^2), \\ B_1 =& \pi(\alpha_1^4 + \beta_2^4). \end{aligned}$$

Then $f_4(r)$ has one solution in I if $0 < -B_1/B_3 < r_0$. Solving $B_1 = 0$ for β_2^4, and $B_3 = 0$ for β_7^2, we obtain $f_4(r) = 0$, and we can apply the averaging theory of order 5. Its corresponding averaging function is

$$f_5(r) = r(C_5 r^4 + C_3 r^2 + C_1),$$

where

$$\begin{aligned} C_5 =& \frac{\pi}{24}(12A^2\alpha_1^1 + 18A\alpha_1^1 - 17A\beta_3^1 + 7A\alpha_4^1 - 19A\beta_5^1 + 12A\alpha_6^1 + 6A\alpha_8^1 \\ &+ 6A\beta_9^1 - 12\beta_3^1 + 6\alpha_4^1 - 6\beta_5^1 + 18\alpha_6^1 + 12\alpha_8^1 + 18\beta_9^1), \\ C_3 =& \frac{\pi}{4}(4A\alpha_1^1(\alpha_2^1)^2 + 4A\alpha_1^1\alpha_2^2 + 4A\alpha_2^1\alpha_1^2 + 4A\alpha_1^3 - 3(\alpha_1^1)^2\beta_3^1 - 3(\beta_1^1)^2\beta_3^1 \\ &+ 3(\alpha_1^1)^2\alpha_4^1 - 3(\alpha_1^1)^2\beta_5^1 + 3\beta_1^1\alpha_1^1\beta_4^1 + 3\alpha_1^1\alpha_2^1\alpha_3^1 - 3\alpha_1^1\alpha_2^1\beta_4^1 + \beta_7^3 \\ &+ 6\alpha_1^1\alpha_2^1\alpha_5^1 + 2\alpha_1^1\alpha_2^1\alpha_7^1 + 2\alpha_1^1\alpha_2^1\beta_8^1 + 2\alpha_1^1(\alpha_3^1)^2 - 3\beta_1^1\alpha_1^1\alpha_3^1 + \alpha_8^3 \\ &- \alpha_1^1\alpha_3^1\beta_4^1 + \beta_1^1\beta_3^1\beta_4^1 + 2\alpha_1^1\alpha_3^1\alpha_5^1 - 2\alpha_1^1\beta_3^1\beta_5^1 + \alpha_1^1(\alpha_4^1)^2 - \alpha_1^1(\beta_4^1)^2 \\ &- \alpha_1^1\beta_3^1\alpha_4^1 + \alpha_1^1\alpha_4^1\beta_5^1 - 2\alpha_1^1(\beta_5^1)^2 + \alpha_1^1\beta_4^1\alpha_5^1 + \alpha_8^1\alpha_2^2 + 2\beta_5^1\alpha_5^2 + 3\beta_9^3 \\ &- 3\alpha_1^1\beta_4^2 + 3\alpha_1^1\alpha_5^2 + 2\alpha_1^1\alpha_7^2 + 2\alpha_1^1\beta_8^2 + 3\beta_1^1\beta_9^2 + (\alpha_2^1)^2\alpha_4^1 - (\alpha_2^1)^2\beta_5^1 \\ &+ (\alpha_2^1)^2\alpha_8^1 + 3(\alpha_2^1)^2\beta_9^1 + 3\beta_1^1\alpha_2^1\beta_9^1 + \alpha_2^1\alpha_3^1\alpha_4^1 + 2\alpha_2^1\alpha_4^1\alpha_5^1 + \beta_1^1\alpha_8^2 \\ &- \alpha_2^1\beta_4^1\beta_5^1 + 4\alpha_2^1\alpha_5^1\beta_5^1 + \beta_1^1\alpha_2^1\alpha_8^1 + \alpha_2^1\alpha_4^2 - \alpha_2^1\beta_5^2 + \alpha_2^1\alpha_8^2 + 3\alpha_2^1\beta_9^2 \\ &+ 2\beta_1^1\alpha_3^1\beta_3^1 + 3\alpha_3^1\alpha_1^2 + 3\beta_3^1\beta_1^2 - 2\alpha_3^1\beta_3^2 + \alpha_3^1\alpha_4^2 - \beta_3^1\beta_4^2 + \beta_1^1\alpha_4^1\alpha_5^1 \\ &+ \alpha_4^1\alpha_2^2 + \alpha_4^1\alpha_3^2 - \beta_4^1\beta_3^2 + \alpha_4^1\alpha_5^2 - \beta_4^1\beta_5^2 + 2\beta_1^1\alpha_5^1\beta_5^1 - 3\beta_3^3 + 3\beta_1^1\beta_3^2 \\ &+ 3\alpha_5^1\alpha_1^2 + \alpha_5^1\alpha_4^2 - \beta_5^1\beta_4^2 + 2\alpha_5^1\beta_5^2 + 2\alpha_7^1\alpha_1^2 + \alpha_8^1\beta_1^2 + 3\alpha_1^1\alpha_3^2 + \alpha_4^3 \\ &+ 3\beta_9^1\beta_1^2 - 3\beta_4^1\alpha_1^2 + 2\beta_8^1\alpha_1^2 - \beta_5^1\alpha_2^2 + 3\beta_9^1\alpha_2^2 - 2\beta_3^1\alpha_3^2 - \beta_5^3 + 3\alpha_6^3), \end{aligned}$$

$C_1 = \pi(\alpha_1^5 + \beta_2^5)$.

The averaging function $f_5(r)$ has at most 2 solutions in I. In order to apply the averaging method of order 6 we solve $C_1 = 0$ for β_2^5, $C_3 = 0$ for β_7^3, and $C_5 = 0$ for β_9^1, resulting $f_5(r) = 0$. We remark that these expressions only hold for $A \neq -3$. The results for $A = -3$ are presented later on. Calculating the averaging function of sixth order we obtain

$$f_6(r) = r(D_5 r^4 + D_3 r^2 + D_1).$$

where the expressions of D_i for $i = 1, 3, 5$ are very long and we do not give them here.

Therefore $f_6(r)$ has at most 2 solutions in I. Using the same arguments than in the proof of the Collins first form for $f_6(r)$ we can show that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

Now we analyze the bifurcation of small limit cycles for the center of (14) in the case $A = -3$. We remark that until the averaging method of order 5 the respective averaging functions for this special case can be obtained by plugging $A = -3$ in the equations of the general case, so we do not explicit them. Hence we solve $C_1 = 0$ for β_2^5, $C_3 = 0$ for β_7^3, and $C_5 = 0$ for α_8^1, and we get $f_5(r) = 0$ when $A = -3$. Calculating the averaging function of sixth order we obtain

$$f_6(r) = r(\mathcal{D}_5 r^4 + \mathcal{D}_4 r^3 + \mathcal{D}_3 r^2 + \mathcal{D}_1),$$

where again we do not provide the explicit expressions of $\mathcal{D}_j$ for $j = 1, 3, 4, 5$ because they are too much long.

Therefore $f_6(r)$ has at most 3 solutions in I. Using similar arguments as those applied in the proof of the Collins first form for $f_6(r)$ it is proved that at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

This completes the proof of Theorem 4.

4. Proof of Theorem 5

A first integral H and its corresponding integrating factor μ for system (2) are $H(x, y) = (x^2 + y^2)/(1 - x^2)$ and $\mu = -2/(x^2 - 1)^2$. When $h \in (0, 1)$ then $H(x, y) = h$ are periodic solutions around the center $(0, 0)$ contained in the open disc of radius 1 centered at the origin. For proving Theorem 5 we shall use Theorem 11. Therefore applying the notation of Theorem 11 we have $h_1 = 0$, $h_2 = 1$ and $\rho(R, \theta) = R/(R^2 \cos^2 \theta + 1)$ for all $0 < R < 1$ and $\theta \in [0, 2\pi)$. Then all the hypotheses of Theorem 11 are satisfied for system (2). Using

Theorem 11 we transform the perturbed differential system (5) into the form

$$\frac{dR}{d\theta} = \varepsilon \frac{\sum_{i=0}^{5} M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta} + O(\varepsilon^2) \tag{15}$$

where

$$\begin{aligned}
M_0(\theta,\alpha,\beta) =& -\sqrt{1+R^2\cos^2\theta}(\alpha_0\cos\theta+\beta_0\sin\theta),\\
M_1(\theta,\alpha,\beta) =& -\alpha_1\cos^2\theta-(\alpha_2+\beta_1)\cos\theta\sin\theta-\beta_2\sin^2\theta,\\
M_2(\theta,\alpha,\beta) =&(-1/4\sqrt{2})\sqrt{2+R^2+R^2\cos(2\theta)}((7\alpha_0+3\alpha_3+\alpha_5\\
&+\beta_4)\cos\theta+(\alpha_0+\alpha_3-\alpha_5-\beta_4)\cos(3\theta)+2(\alpha_4\\
&+\beta_0+\beta_3+\beta_5+(\alpha_4+\beta_0+\beta_3-\beta_5)\cos(2\theta))\sin\theta),\\
M_3(\theta,\alpha,\beta) =& -(2\alpha_1+\alpha_6)\cos^4\theta-(2\alpha_2+\alpha_7+\beta_1+\beta_6)\cos^3\theta\sin\theta\\
&-(\alpha_1+\alpha_8+\beta_2+\beta_7)\cos^2\theta\sin^2\theta-(\alpha_2+\alpha_9+\beta_8)\\
&\cos\theta\sin^3\theta-\beta_9\sin^4\theta,\\
M_4(\theta,\alpha,\beta) =&(-1/2\sqrt{2})\cos\theta\sqrt{2+R^2+R^2\cos(2\theta)}(\alpha_0+\alpha_3+\alpha_5\\
&+(\alpha_0+\alpha_3-\alpha_5)\cos(2\theta)+\alpha_4\sin(2\theta)),\\
M_5(\theta,\alpha,\beta) =&(-1/4)\cos\theta((3(\alpha_1+\alpha_6)+\alpha_8)\cos\theta+(\alpha_1+\alpha_6-\alpha_8)\\
&\cos 3\theta+2(\alpha_2+\alpha_7+\alpha_9+(\alpha_2+\alpha_7-\alpha_9)\cos 2\theta)\sin\theta),
\end{aligned}$$

where $\alpha=(\alpha_0,\ldots,\alpha_9)$ and $\beta=(\beta_0,\ldots,\beta_9)$.

We must study the zeros of the averaged function $f:(0,1)\to\mathbb{R}$ defined by

$$f(R)=\int_0^{2\pi}\frac{\sum_{i=0}^{5} M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta}d\theta.$$

By computing the previous integral, we obtain

$$\begin{aligned}
f(R) =& \pi(\alpha_6-\alpha_1-3\alpha_8-\beta_2-\beta_7+3\beta_9)g_0-\pi(\alpha_1+\alpha_6+\alpha_8)g_1\\
&+2\pi(\alpha_8-\beta_9)g_2+2\pi(\alpha_6-\alpha_8-\beta_7+\beta_9)g_3,
\end{aligned} \tag{16}$$

where

$$g_0=R,\quad g_1=R^3,\quad g_2=R\sqrt{1+R^2},\quad g_3=(1-\sqrt{1+R^2})/R.$$

In order to find the maximum number of simple zeros of the function f we need to prove that the four functions $g_i:(0,1)\to\mathbb{R}$, $i\in\{0,\ldots,3\}$ given in (16) are an ECT-system and according to Theorem 13 this is the case if each Wronskian $W_j(g_0,\ldots,g_j)\neq 0,\ j\in\{0,\ldots,3\}$. More precisely

$$W_0=R,\qquad W_1=2R^3,\qquad W_2=-2R^6/(1+R^2)^{3/2}$$

$$W_3 = 12R^2(8+12R^2+4R^4-8(1+R^2)^{3/2}-R^4\sqrt{1+R^2})/(1+R^2)^{7/2}.$$

For $R \in (0,1)$ we have that all the Wronskians above are nonzero. Moreover the rank of the Jacobian matrix of the coefficients of g_i, $i = 0,\dots,3$ in $f(R)$ in the variables $\alpha_1, \alpha_6, \alpha_8, \beta_2, \beta_7, \beta_9$ is 4. Thus applying the averaging theory of first order and Theorem 13 it is proved that at most 3 medium limit cycles can bifurcate from the periodic solutions surrounding the cubic uniform isochronous center of the Collins first form and this number can be reached. This completes the proof of Theorem 5.

5. Proof of Theorem 6

We use the Collins first and second forms to prove Theorem 6. We shall apply to them the averaging theory of order 6.

Collins first form

In order to analyze the Hopf bifurcation for system (6) with $f(x,y) = xy$ we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$ and by doing a Taylor expansion truncated at the 5^{th} order in ε we obtain an expression for $dr/d\theta$ similar to (8) with $F_0 = 0, k = 6$. The explicit expression is quite large so we omit it.

In addition, to fulfill the conditions of Theorem 10 we apply the regularization theory. For this purpose we take the function $h(\tau)$ and $\lambda > 0$ of Proposition 14 and transform system (6) with $f(x,y) = xy$ in the $\mathcal{C}^\infty$−system

$$\bar{\mathcal{X}} = \frac{X_1+X_2}{2} + h(\tau)\frac{X_1-X_2}{2},$$

where X_1 and X_2 are given in (6) with $f(x,y) = xy$. For $\tau < -\lambda$ this system is equal to X_2, for $\tau > \lambda$ it is X_1 and it is a smooth differential system otherwise. When $\lambda \to 0$ it tends to system (6) with $f(x,y) = xy$. We shall have I of Theorem 10 as $I = \{r : 0 < r < 1\}$. Now we have all the assumptions of Theorem 10 satisfied and applying it we obtain the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1 + \gamma_1^1 + \delta_2^1).$$

Clearly $f_1(r)$ has no solution in I. Thus there is no small limit cycles bifurcating from the uniform isochronous center at the origin by the averaging theory of first order. Now setting $\gamma_1^1 = -(\alpha_1^1 + \beta_2^1 + \delta_2^1)$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of second order, obtaining

$$f_2(r) = r(A_2 r + A_1),$$

where

$$
\begin{aligned}
A_2 =&\frac{2}{3}(\alpha_4^1 - \gamma_4^1 + \beta_3^1 + 2\beta_5^1 - \delta_3^1 - 2\delta_5^1),\\
A_1 =&\frac{\pi}{4}(\alpha_1^1\alpha_2^1 + 2\alpha_1^2 + 2\pi(\alpha_1^1)^2 - \alpha_1^1\gamma_2^1 + 2\gamma_1^2 - \alpha_1^1\beta_1^1 + \alpha_2^1\beta_2^1 + 4\pi\alpha_1^1\beta_2^1\\
&- \gamma_2^1\beta_2^1 - \beta_1^1\beta_2^1 + 2\pi(\beta_2^1)^2 + 2\beta_2^2 + \alpha_1^1\delta_1^1 + \beta_2^1\delta_1^1 + 2\delta_2^2).
\end{aligned}
$$

Thus $f_2(r)$ has one solution in I if $0 < -A_1/A_2 < 1$. Therefore applying the averaging theory of order 2 it is proved that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of third order we need that $f_2(r) = 0$. Thus we solve $A_1 = 0$ for γ_4^1 and $A_2 = 0$ for γ_1^2 from these coefficients. Calculating the next averaging function we have

$$f_3(r) = r(B_3r^2 + B_2r + B_1),$$

where

$$
\begin{aligned}
B_3 =&\frac{1}{8}\pi(-4\beta_2^1 + 3\alpha_6^1 + \beta_7^1 + \alpha_8^1 + 3\beta_9^1 - 4\delta_2^1 + \delta_7^1 + 3\delta_9^1 + 3\gamma_6^1 + \gamma_8^1),\\
B_2 =&\frac{2}{9}(\alpha_1^1\alpha_3^1 - 3\beta_1^1\beta_3^1 + 6\pi\alpha_1^1\beta_3^1 - \alpha_1^1\beta_4^1 + 6\pi\alpha_1^1\alpha_4^1 + 2\alpha_1^1\alpha_5^1 + 12\pi\alpha_1^1\beta_5^1\\
&- \alpha_1^1\delta_4^1 + \alpha_1^1\gamma_3^1 + 2\alpha_1^1\gamma_5^1 + 6\pi\beta_2^1\beta_3^1 + 3\alpha_2^1\alpha_4^1 - 4\beta_2^1\beta_4^1 + 6\alpha_2^1\beta_5^1\\
&+ 12\pi\beta_2^1\beta_5^1 - \beta_2^1\delta_4^1 - 5\beta_2^1\alpha_3^1 + 6\pi\beta_2^1\alpha_4^1 - 3\alpha_4^1\gamma_2^1 + 2\beta_2^1\alpha_5^1 + 3\beta_3^2\\
&+ 3\alpha_4^2 + 6\beta_5^2 + 3\delta_1^1\delta_3^1 + 3\delta_2^1\delta_4^1 - 3\beta_3^1\gamma_2^1 - 6\beta_5^1\gamma_2^1 + 3\delta_3^1\gamma_2^1 + \beta_2^1\gamma_3^1\\
&+ 6\delta_2^1\gamma_3^1 + 2\beta_2^1\gamma_5^1 - 3\delta_3^2 - 3\gamma_4^2 - 6\delta_5^2),\\
B_1 =&\frac{1}{16}\pi(10\pi^2(\alpha_1^1)^3 - 8\pi\beta_1^1(\alpha_1^1)^2 + 30\pi^2(\alpha_1^1)^2\beta_2^1 - 4(\alpha_1^1)^2\beta_2^1\\
&+ 8\pi(\alpha_1^1)^2\alpha_2^1 + 3(\beta_1^1)^2\beta_2^1 + 4\pi(\alpha_1^1)^2\delta_1^1 - 4(\alpha_1^1)^2\delta_2^1 - 4\pi(\alpha_1^1)^2\gamma_2^1\\
&+ 3(\beta_1^1)^2\alpha_1^1 - 16\pi\beta_1^1\alpha_1^1\beta_2^1 - 2\beta_1^1\alpha_1^1\delta_1^1 + 3\alpha_1^1(\alpha_2^1)^2 + 30\pi^2\alpha_1^1(\beta_2^1)^2\\
&- 4\alpha_1^1(\beta_2^1)^2 - 8\pi\beta_1^1(\beta_2^1)^2 - 2\beta_1^1\alpha_1^1\alpha_2^1 + 16\pi\alpha_1^1\alpha_2^1\beta_2^1 + 2\alpha_1^1\alpha_2^1\delta_1^1\\
&+ 8\pi\alpha_1^1\beta_2^1\delta_1^1 - 2\beta_1^1\beta_2^1\delta_1^1 - 8\alpha_1^1\beta_2^1\delta_2^1 - 2\alpha_1^1\alpha_2^1\gamma_2^1 - 4\alpha_1^1\beta_1^2\\
&+ 16\pi\alpha_1^1\alpha_1^2 + 4\alpha_1^1\alpha_2^2 + 16\pi\alpha_1^1\beta_2^2 - 4\beta_1^1\beta_2^2 - \alpha_1^1(\delta_1^1)^2\\
&- 4\alpha_1^1(\delta_2^1)^2 - \alpha_1^1(\gamma_2^1)^2 + 2\beta_1^1\alpha_1^1\gamma_2^1 - 8\pi\alpha_1^1\beta_2^1\gamma_2^1 - 2\alpha_1^1\delta_1^1\gamma_2^1\\
&+ 4\alpha_1^1\delta_1^2 - 4\alpha_1^1\gamma_2^2 + 10\pi^2(\beta_2^1)^3 + 3(\alpha_2^1)^2\beta_2^1 + 4\pi(\beta_2^1)^2\delta_1^1\\
&- 4(\beta_2^1)^2\delta_2^1 - 2\beta_1^1\alpha_2^1\beta_2^1 + 8\pi\alpha_2^1(\beta_2^1)^2 + 2\alpha_2^1\beta_2^1\delta_1^1 + 4\alpha_2^1\alpha_1^2\\
&- 4\beta_2^1\beta_1^2 + 4\alpha_2^1\beta_2^2 + 16\pi\beta_2^1\beta_2^2 - \beta_2^1(\delta_1^1)^2 - 4\beta_2^1(\delta_2^1)^2\\
&- 2\alpha_2^1\beta_2^1\gamma_2^1 + 4\beta_2^1\delta_1^2 - 4\beta_1^1\alpha_1^2 + 16\pi\beta_2^1\alpha_1^2 + 4\alpha_1^2\delta_1^1
\end{aligned}
$$

$$- 4\alpha_1^2\gamma_2^1 + 4\beta_2^1\alpha_2^2 + 4\beta_2^2\delta_1^1 + 8\alpha_1^3 + 8\beta_2^3 - \beta_2^1(\gamma_2^1)^2 + 2\beta_1^1\beta_2^1\gamma_2^1$$
$$- 4\pi(\beta_2^1)^2\gamma_2^1 - 2\beta_2^1\delta_1^1\gamma_2^1 - 4\beta_2^2\gamma_2^1 - 4\beta_2^1\gamma_2^2 + 8\gamma_1^3 + 8\delta_2^3).$$

Since $f_3(r)$ can have at most 2 solutions in I, we conclude that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. In order to apply the averaging theory of order 4 we need that $f_3(r) = 0$, so we vanish its coefficients B_1, B_2 and B_3 by conveniently isolating δ_2^3, δ_5^2 and δ_9^1 from these coefficients. The resulting averaging function of fourth order is

$$f_4(r) = r(C_4r^3 + C_3r^2 + C_2r + C_1),$$

where the expressions of C_j for $j = 1, \ldots, 4$ are too long and we do not provide them here.

Of course $f_4(r)$ can have at most 3 solutions in I, so at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. In order to apply the averaging method of order 5 we must have that $f_4(r) = 0$. Thus we solve $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ and C_4=0 isolating β_2^4, β_5^3, β_9^2 and β_5^1 respectively. Now we can apply the averaging theory of order 5, and its averaging function is

$$f_5(r) = r(D_5r^4 + D_4r^3 + D_3r^2 + D_2r + D_1),$$

where again we do not provide the explicit expressions of D_j for $j = 1, \ldots, 5$. Hence $f_5(r)$ has at most 4 solutions in I. Doing analogous arguments than in the proof of Theorem 4 it is proved that at most 4 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 5, and this number can be reached.

To apply the averaging theory of order 6 we solve $D_1 = 0$ for δ_2^5, $D_2 = 0$ for δ_5^4, $D_3 = 0$ for δ_9^3, D_4 for δ_3^2, and $D_5 = 0$ for γ_6^1, so we get $f_5(r) = 0$. Calculating the averaging function of order 6 we obtain

$$f_6(r) = r(E_6r^5 + E_5r^4 + E_4r^3 + E_3r^2 + E_2r + E_1).$$

We do not provide the expressions of E_i for $i = 1, \ldots, 6$ because they are too long. Thus $f_6(r)$ has at most 5 solutions in I. Doing analogous arguments than in the proof of Theorem 4 we can show that at most 5 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 6, and this number can be reached.

Collins second form

Similarly to the previous arguments used in the Collins first form case, to study the Hopf bifurcation for system (6) with $f(x, y) =$

$x + Axy$ we introduce a small parameter ε by doing the change of coordinates $x = \varepsilon X$, $y = \varepsilon Y$ and then the standard polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$. Doing a Taylor expansion truncated at the 5^{th} order in ε we obtain an expression for $dr/d\theta$ similar to (8) with $F_0 = 0$. The explicit expression is very large so we omit it. We shall have I of Theorem 10 as $I = \{r : 0 < r < r_0 < 1\}$, where the unperturbed system has periodic solutions passing through the point $(r < r_0, \theta = 0)$. Moreover we also apply the regularization theory to fulfill the other conditions of Theorem 10 as previously done for the Collins first form. Hence, applying Theorem 10 we obtain the averaging function of first order

$$f_1(r) = \frac{1}{2}\pi r(\alpha_1^1 + \beta_2^1 + \delta_2^1 + \gamma_1^1).$$

Therefore $f_1(r)$ has no solution in I. Setting $\gamma_1^1 = -(\alpha_1^1 + \beta_2^1 + \delta_2^1)$ we have $f_1(r) = 0$. So we can apply the averaging theory of order 2 obtaining

$$f_2(r) = r(A_2 r + A_1),$$

where

$$\begin{aligned} A_2 =& \frac{2}{3}(-3\beta_2^1 + \beta_3^1 + \alpha_4^1 + 2\beta_5^1 + 3\delta_2^1 - \delta_3^1 - 2\delta_5^1 - \gamma_4^1),\\ A_1 =& \frac{\pi}{4}(2\pi(\alpha_1^1)^2 + \alpha_1^1(-\beta_1^1 + \alpha_2^1 + 4\pi\beta_2^1 + \delta_1^1 - \gamma_2^1) - \beta_1^1\beta_2^1 + 2\pi(\beta_2^1)^2\\ &+ \alpha_2^1\beta_2^1 + \beta_2^1\delta_1^1 + 2\alpha_1^2 + 2\beta_2^2 - \beta_2^1\gamma_2^1 + 2\gamma_1^2 + 2\delta_2^2). \end{aligned}$$

Thus $f_2(r)$ can have one solution in I if $0 < -A_1/A_2 < r_0$, i.e. applying the averaging theory of order 2 we can show that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging theory of order 3 we solve $A_1 = 0$ and $A_2 = 0$ isolating γ_4^1 and γ_1^2 respectively. Calculating the next averaging function we have

$$f_3(r) = r(B_3 r^2 + B_2 r + B_1),$$

where

$$\begin{aligned} B_3 =& \frac{\pi}{8}(-4A\beta_2^1 - 4A\delta_2^1 - 3\beta_2^1 - 2\beta_3^1 + 2\alpha_4^1 + \beta_5^1 + 3\alpha_6^1 + \beta_7^1 + \alpha_8^1 + 3\beta_9^1\\ &+ 3\delta_2^1 - 4\delta_3^1 - 3\delta_5^1 + \delta_7^1 + 3\delta_9^1 + 3\gamma_6^1 + \gamma_8^1),\\ B_2 =& + \frac{2}{9}(9\beta_1^1\beta_2^1 - 18\pi\alpha_1^1\beta_2^1 + \alpha_1^1\alpha_3^1 - 3\beta_1^1\beta_3^1 + 6\pi\alpha_1^1\beta_3^1 - \alpha_1^1\beta_4^1\\ &+ 6\pi\alpha_1^1\alpha_4^1 + 2\alpha_1^1\alpha_5^1 + 12\pi\alpha_1^1\beta_5^1 - \alpha_1^1\delta_4^1 + \alpha_1^1\gamma_3^1 + 2\alpha_1^1\gamma_5^1 - 18\pi(\beta_2^1)^2\\ &+ 6\pi\beta_2^1\beta_3^1 - 4\beta_2^1\beta_4^1 + 3\alpha_2^1\alpha_4^1 + 12\pi\beta_2^1\beta_5^1 + 6\alpha_2^1\beta_5^1 - \beta_2^1\delta_4^1 - 5\beta_2^1\alpha_3^1\\ &+ 6\pi\beta_2^1\alpha_4^1 - 3\alpha_4^1\gamma_2^1 + 2\beta_2^1\alpha_5^1 - 9\beta_2^2 + 3\beta_3^2 + 3\alpha_4^2 + 6\beta_5^2 - 9\delta_1^1\delta_2^1 \end{aligned}$$

$$
\begin{aligned}
&+3\delta_1^1\delta_3^1+3\delta_2^1\delta_4^1+9\beta_2^1\gamma_2^1-3\beta_3^1\gamma_2^1-6\beta_5^1\gamma_2^1+3\delta_3^1\gamma_2^1-9\alpha_2^1\beta_2^1\\
&+\beta_2^1\gamma_3^1+6\delta_2^1\gamma_3^1+2\beta_2^1\gamma_5^1+9\delta_2^2-3\gamma_3^2-3\gamma_4^2-6\delta_5^2),\\
B_1=&+\frac{\pi}{16}(10\pi^2(\alpha_1^1)^3-8\pi\beta_1^1(\alpha_1^1)^2+30\pi^2(\alpha_1^1)^2\beta_2^1-4(\alpha_1^1)^2\beta_2^1\\
&+8\pi(\alpha_1^1)^2\alpha_2^1+3(\beta_1^1)^2\beta_2^1+4\pi(\alpha_1^1)^2\delta_1^1-4(\alpha_1^1)^2\delta_2^1-4\pi(\alpha_1^1)^2\gamma_2^1\\
&-16\pi\beta_1^1\alpha_1^1\beta_2^1-2\beta_1^1\alpha_1^1\delta_1^1+3\alpha_1^1(\alpha_2^1)^2+30\pi^2\alpha_1^1(\beta_2^1)^2-4\alpha_1^1(\beta_2^1)^2\\
&-8\pi\beta_1^1(\beta_2^1)^2-2\beta_1^1\alpha_1^1\alpha_2^1+16\pi\alpha_1^1\alpha_2^1\beta_2^1+2\alpha_1^1\alpha_2^1\delta_1^1+8\pi\alpha_1^1\beta_2^1\delta_1^1\\
&-2\beta_1^1\beta_2^1\delta_1^1-8\alpha_1^1\beta_2^1\delta_2^1-2\alpha_1^1\alpha_2^1\gamma_2^1-4\alpha_1^1\beta_1^2+16\pi\alpha_1^1\alpha_1^2+4\alpha_1^1\alpha_2^2\\
&+16\pi\alpha_1^1\beta_2^2-4\beta_1^1\beta_2^2-\alpha_1^1(\delta_1^1)^2-4\alpha_1^1(\delta_2^1)^2-\alpha_1^1(\gamma_2^1)^2\\
&+2\beta_1^1\alpha_1^1\gamma_2^1-8\pi\alpha_1^1\beta_2^1\gamma_2^1-2\alpha_1^1\delta_1^1\gamma_2^1+4\alpha_1^1\beta_1^3-4\alpha_1^1\gamma_2^2+10\pi^2(\beta_2^1)^3\\
&+3(\alpha_2^1)^2\beta_2^1+4\pi(\beta_2^1)^2\delta_1^1-4(\beta_2^1)^2\delta_2^1-2\beta_1^1\alpha_2^1\beta_2^1+8\pi\alpha_2^1(\beta_2^1)^2\\
&+2\alpha_2^1\beta_2^1\delta_1^1+4\alpha_2^1\alpha_1^2-4\beta_2^1\beta_1^2+4\alpha_2^1\beta_2^2+16\pi\beta_2^1\beta_2^2-\beta_2^1(\delta_1^1)^2\\
&-4\beta_2^1(\delta_2^1)^2-2\alpha_2^1\beta_2^1\gamma_2^1+4\beta_2^1\delta_1^2-4\beta_1^1\alpha_1^2+16\pi\beta_2^1\alpha_1^2\\
&+4\alpha_1^2\delta_1^1-4\alpha_1^2\gamma_2^1+4\beta_2^1\alpha_2^2+4\beta_2^2\delta_1^1+8\alpha_1^3+8\beta_2^3+3(\beta_1^1)^2\alpha_1^1\\
&-\beta_2^1(\gamma_2^1)^2+2\beta_1^1\beta_2^1\gamma_2^1-4\pi(\beta_2^1)^2\gamma_2^1-2\beta_2^1\delta_1^1\gamma_2^1-4\beta_2^2\gamma_2^1\\
&-4\beta_2^1\gamma_2^2+8\gamma_1^3+8\delta_2^3).
\end{aligned}
$$

Then $f_3(r)$ has at most 2 solutions in I, i.e. applying the averaging theory of order 3 it is proved that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of order 4 we solve $B_1=0$, $B_2=0$ and $B_3=0$ isolating δ_2^3, δ_5^2, δ_9^1 respectively. The next averaging function is

$$f_4(r)=r(C_4r^3+C_3r^2+C_2r+C_1).$$

We do not provide the expressions of C_j for $j=1,\ldots,4$ because they are too long.

Of course $f_4(r)$ has at most 3 solutions in I, that is, applying the averaging theory of order 4 we can show that at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of order 5 we solve $C_1=0$, $C_2=0$, $C_3=0$ and $C_4=0$ isolating β_2^4, β_5^3, β_9^2 and β_9^1 respectively. The next averaging function is

$$f_5(r)=r(D_5r^4+D_4r^3+D_3r^2+D_2r+D_1),$$

where again we do not give the expressions of D_j for $j=1,\ldots,5$. Hence $f_5(r)$ has at most 4 solutions in I. Using analogous arguments than in the proof of Theorem 4 we can show that at most 4 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

In order to apply the averaging theory of order 6 we solve $D_1 = 0$ for δ_2^5, $D_2 = 0$ for δ_5^4, $D_3 = 0$ for δ_9^3, D_4 for δ_9^2, and $D_5 = 0$ for γ_6^1, so we get $f_5(r) = 0$. Calculating the averaging function of order 6 we obtain

$$f_6(r) = r(E_6r^5 + E_5r^4 + E_4r^3 + E_3r^2 + E_2r + E_1).$$

We do not provide the expressions of E_i for $i = 1, \dots, 6$ because they are too long. Thus $f_6(r)$ has at most 5 solutions in I. Doing analogous arguments than in the proof of Theorem 4 it follows that at most 5 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 6, and this number can be reached.

This ends the proof of Theorem 6.

6. Proof of Theorem 7

We proceed as in the proof of Theorem 5 in section 4 since the unperturbed system (2) is the same. Hence a first integral H, its corresponding integrating factor μ, and a function ρ satisfying the hypotheses of Theorem 11 are $H(x,y) = (x^2+y^2)/(1-x^2)$, $\mu = -2/(x^2-1)^2$, and $\rho(R,\theta) = R/(R^2\cos^2\theta + 1)$ for all $0 < R < 1$ and $\theta \in [0, 2\pi)$.

Applying Theorem 11 we transform the perturbed differential system (7) into the form

$$\frac{dR}{d\theta} = \begin{cases} \varepsilon \dfrac{\sum_{i=0}^{5} M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta} + O(\varepsilon^2) & \text{if} \quad y > 0, \\ \varepsilon \dfrac{\sum_{i=0}^{5} N_i(\theta,\gamma,\delta)R^i}{1+R^2\cos^2\theta} + O(\varepsilon^2) & \text{if} \quad y < 0, \end{cases} \tag{17}$$

where the functions $M_i(\theta,\alpha,\beta)$ coincide with the ones given in system (15), and $N_i(\theta,\gamma,\delta) = M_i(\theta,\gamma,\delta)$ for $i = 0, \dots, 5$, with $\gamma = (\gamma_0, \dots, \gamma_9)$, $\delta = (\delta_0, \dots, \delta_9)$.

The discontinuous differential system (17) is under the assumptions of Theorem 11. Hence we must study the zeros of the averaged function $f : (0,1) \to \mathbb{R}$

$$f(R) = \int_0^{\pi} \frac{\sum_{i=0}^{5} M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta} d\theta + \int_{\pi}^{2\pi} \frac{\sum_{i=0}^{5} N_i(\theta,\gamma,\delta)R^i}{1+R^2\cos^2\theta} d\theta$$

We compute these integrals obtaining

$$\begin{aligned} f(R) =& \pi(\alpha_6 - \alpha_8 - \beta_7 + \beta_9 + \gamma_6 - \gamma_8 - \delta_7 + \delta_9)g_0 + \pi/2(\alpha_6 \\ & - \alpha_1 - 3\alpha_8 - \beta_2 - \beta_7 + 3\beta_9 - \gamma_1 + \gamma_6 - 3\gamma_8 - \delta_2 - \delta_7 \end{aligned}$$

$$
\begin{aligned}
& + 3\delta_9)g_1 - \pi/2(\alpha_1 + \alpha_6 + \alpha_8 + \gamma_1 + \gamma_6 + \gamma_8)g_2 + (\beta_5 \\
& - \alpha_4 - \beta_0 - \beta_3 + \gamma_4 + \delta_0 + \delta_3 - \delta_5)g_3 + \pi(\alpha_8 - \beta_9 + \gamma_8 \\
& - \delta_9)g_4 + (\gamma_4 - \alpha_4)g_5 + (\alpha_4 - \beta_0 + \beta_3 - \beta_5 - \gamma_4 + \delta_0 \\
& - \delta_3 + \delta_5)g_6 + (\alpha_4 - 2\beta_5 - \gamma_4 + 2\delta_5)g_7,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
g_0 =& (1 - \sqrt{1 + R^2})/R, \qquad g_1 = R, \qquad g_2 = R^3, \\
g_3 =& \sqrt{1 + R^2}, \qquad g_4 = R\sqrt{1 + R^2}, \qquad g_5 = R^2\sqrt{1 + R^2}, \\
g_6 =& (\operatorname{arcsinh} R)/R, \qquad g_7 = R \operatorname{arcsinh} R.
\end{aligned}
$$

In order to find the maximum number of simple zeros of function f we need to prove that the eight functions $g_i : (0, 1) \to \mathbb{R}$, $i \in \{0, \dots, 7\}$ given in (18) form an ECT-system and according to Theorem 13 this is the case if each Wronskian $W_j(g_0, \dots, g_j) \neq 0$, $j \in \{0, \dots, 7\}$. More precisely

$$
\begin{aligned}
W_0 =& (1 - K)/R, \quad W_1 = (2K - 2 - R^2)/(RK), \\
W_2 =& 2K^{-3}(1 - 6K^2 + 8K^3 - 3K^4), \\
W_3 =& 6R^{-3}K^{-7}(8 - 8K + 4R^6K + R^4(16 - 7K) + 4R^2(6 - 5K)), \\
W_4 =& -36R^{-2}K^{-10}(4R^6K + R^2(76 - 56K) + R^4(40 - 17K) \\
& - 40(K - 1)), \\
W_5 =& 1080R^{-5}K^{-15}(24(K - 1) + R^2(R^2K(3R^2 - 5) + 4(4K - 7))), \\
W_6 =& 25920R^{-7}K^{-20}(64(1 - K) + R^2(R^2K(6R^2 - 17) + 32(7 - 6K)) \\
& + 105R^3 \operatorname{arcsinh} R), \\
W_7 =& 1244160R^{-8}K^{-26}(4R^8 - 515R^4 - 12R^6 - 256(K - 1) + R^2(896K \\
& - 243) + 105RK(2R^2 - 5) \operatorname{arcsinh} R),
\end{aligned}
$$

where $K = \sqrt{1 + R^2}$. For $0 < R < 1$ we have that all the Wronskians above are nonzero. Moreover the rank of the Jacobian matrix of the coefficients of g_i for $i \in \{0, \dots, 7\}$ in (18) in the variables $\alpha_1, \alpha_4, \alpha_6, \alpha_8$, $\beta_0, \beta_2, \beta_3, \beta_5, \beta_7, \beta_9$, $\gamma_1, \gamma_4, \gamma_6, \gamma_8$, $\delta_0, \delta_2, \delta_3, \delta_5, \delta_7, \delta_9$ is 8. Hence applying the averaging theory of first order and Theorem 13 it is proved that at most 7 medium limit cycles can bifurcate from the periodic solutions of the cubic uniform isochronous center of the Collins first form and this number can be reached. This completes the proof of Theorem 7.

7. Proof of Theorem 8

We analyze each distinct case in order to compute the first integrals, considering the condition

$$a_1^2 a_3 - a_2^2 a_3 + a_1 a_2 a_4 = 0. \tag{19}$$

presented in Theorem 2.

Case 1: $\mathbf{a_1^2 - a_2^2 \neq 0}$. The condition (19) can be expressed as

$$a_3 = -\frac{a_1 a_2 a_4}{a_1^2 - a_2^2}, \tag{20}$$

and in polar coordinates the system can be written as

$$\begin{aligned} \frac{dr}{d\theta} =& r^2(a_1 \cos\theta + a_2 \sin\theta) \\ &+ \frac{a_4 r^3(-a_2 \cos\theta + a_1 \sin\theta)(a_1 \cos\theta + a_2 \sin\theta)}{a_1^2 - a_2^2}. \end{aligned} \tag{21}$$

Subcase 1.1: $\mathbf{a_4 \neq 0}$.

Subcase 1.1.1: $\mathbf{a_4 \neq a_1^2 - a_2^2}$. It is easy to verify that the H presented in this subcase is a first integral of system (21).

Subcase 1.1.2: $\mathbf{a_4 = a_1^2 - a_2^2}$. In polar coordinates system (1) is written as

$$\frac{dr}{d\theta} = Ar^3 + Br^2.$$

where $A = 1/4(a_1 a_2 \sin^2\theta + (a_1^2 - a_2^2)\sin\theta\cos\theta - a_1 a_2 \cos^2\theta$, $B = a_1 \cos\theta + a_2 \sin\theta$. This is an *Abel differential equation* satisfying

$$\frac{dA(\theta)}{d\theta} B(\theta) - A(\theta)\frac{dB(\theta)}{d\theta} = aB(\theta)^3,$$

with $a = 1/4$. Therefore the H given in this subcase is a first integral for this system, for more details see [14].

Subcase 1.2: $\mathbf{a_4 = 0}$. System (21) is reduced to

$$\frac{dr}{d\theta} = r^2(a_1 \cos\theta + a_2 \sin\theta),$$

and the H given in this subcase is a first integral for this system.

Case 2: $\mathbf{a_1^2 - a_2^2 = 0}$,

Subcase 2.1: $\mathbf{a_2 = a_1}$. The expression (19) is reduced to $a_1^2 a_4 = 0$. Therefore we have the following possibilities.

Subcase 2.1.1: $\mathbf{a_1 = 0}$. Applying the condition $a_1 = a_2 = 0$ in system (1), we obtain in polar coordinates

$$\frac{dr}{d\theta} = r^3(a_3 \cos^2\theta + a_4 \sin\theta\cos\theta - a_3 \sin^2\theta).$$

The expression of H in this subcase is a first integral of this system.

Subcase 2.1.2: $a_1 \neq 0$, $\mathbf{a_4 = 0}$. Under this condition, system (1) is expressed in polar coordinates as follows

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta + \sin\theta) + r^3[a_3(\cos^2\theta - \sin^2\theta)].$$

Subcase 2.1.2.1: $\mathbf{a_3(a_1^2 + 4a_3) \neq 0}$. It is easy to check that the H given in this subcase is a first integral of the system.

Subcase 2.1.2.2: $\mathbf{a_3 = 0}$. In this case system (1) becomes in polar coordinates

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta + \sin\theta),$$

and the H given in this subcase is a first integral for this system.

Subcase 2.1.2.3: $\mathbf{a_3 = -a_1^2/4}$. In polar coordinates system (1) is written as

$$\frac{dr}{d\theta} = -\frac{1}{4}a_1 r^2[a_1\cos(2\theta)r - 4(\cos\theta + \sin\theta)].$$

Applying the same arguments as in subcase 1.1.2 we have that this is an *Abel differential equation* with $A(\theta) = (-1/4)a_1^2\cos(2\theta)$, $B(\theta) = a_1(\cos\theta + \sin\theta)$ and $a = 1/4$. Therefore the H given in this subcase is a first integral for this system, , see [14].

Subcase 2.2: $\mathbf{a_2 = -a_1}$.

Subcase 2.2.2: $a_1 \neq 0$, $\mathbf{a_4 = 0}$. In polar coordinates system (1) becomes

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta - \sin\theta) + r^3[a_3(\cos^2\theta - \sin^2\theta)].$$

Subcase 2.2.2.1: $\mathbf{a_3(4a_3 - a_1^2) \neq 0}$. The expression of H presented in this subcase is a first integral of the system.

Subcase 2.2.2.2: $\mathbf{a_3 = 0}$. System (1) becomes in in polar coordinates

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta - \sin\theta).$$

and the expression of H in this subcase is a first integral of this system.

Subcase 2.2.2.3: $\mathbf{a_3 = a_1^2/4}$. In polar coordinates system (1) can be written as

$$\frac{dr}{d\theta} = a_1 r^2(\cos\theta - \sin\theta) + \frac{1}{4}[a_1^2 r^3 \cos(2\theta)].$$

Applying the same arguments as in subcase 1.1.2 we have that this is an *Abel differential equation* with $A(\theta) = 1/4(a_1^2\cos(2\theta))$, $B(\theta) = a_1(\cos\theta - \sin\theta)$ and $a = 1/4$. Using the results presented in [14], we conclude that the H given in this subcase is a first integral for the system, see [14].

8. Proof of Theorem 9

We provide all the possible phase portraits for the planar cubic differential systems with a uniform isochronous center at the origin, in the Poincaré disc, by studying the finite and infinite singular points of such systems.

Finite singular points

In polar coordinates a planar cubic differential system with a uniform isochronous center at the origin can always be written as $\dot{r} = rf(r\cos\theta, r\sin\theta)$, $\dot{\theta} = 1$. Hence, since $\dot{\theta} = 1$ there are no finite singular points except at the origin.

Infinite singular points

For studying the infinite singular points in the Poincaré disc, we use the definitions and notations given in Chapter 5 of [10].

We perform the analysis of the vector field at infinity. In the chart U_1 the differential system (1) becomes

$$\dot{u} = (1+u^2)v^2, \quad \dot{v} = (-a_3 - a_4 u + a_3 u^2 - a_1 v - a_2 uv + uv^2)v, \tag{22}$$

and therefore $(u, 0)$, for all $u \in \mathbb{R}$ is an infinite singular point of the differential system (1) in U_1, which means that the equator of $\mathbb{S}^2$ is formed by singularities. In order to obtain the phase portraits, we perform a change of coordinates of the form $dt = vds$, and system (22) becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_4 u + a_3 u^2 - a_1 v - a_2 uv + uv^2, \tag{23}$$

where the prime denotes derivative with respect to s.

In the chart U_2 system (1) becomes

$$\dot{u} = -(1+u^2)v^2, \quad \dot{v} = (a_3 - a_4 u - a_3 u^2 - a_2 v - a_1 uv - uv^2)v.$$

We only need to study the point $(0,0)$ of U_2. By performing a change of coordinates of the form $dt = vds$ we obtain the system

$$u' = -(1+u^2)v, \quad v' = a_3 - a_4u - a_3u^2 - a_2v - a_1uv - uv^2. \tag{24}$$

In order to study the singular points at infinity of systems (23) and (24), we have to consider several cases. We apply Theorems 2.15, 2.19 and 3.15 of [10] for the characterization of the local phase portraits at each singular point.

Case I: $\mathbf{a_1^2 - a_1^2 \neq 0}$. The condition (19) is written as (20). If $a_4 = 0$, then $a_3 = 0$, and hence system (1) degenerates to a quadratic differential system, which has already been exhaustively studied, as previously mentioned in this article. Therefore, we are going to omit the cases in which $a_4 = 0$.

Subcase I.1: $\mathbf{a_1 a_2 \neq 0}$. The expression (23) for our system in U_1 becomes

$$\begin{aligned} u' &= (1+u^2)v, \\ v' &= \frac{a_1a_2a_4}{a_1^2 - a_2^2} - a_4u - \frac{a_1a_2a_4}{a_1^2 - a_2^2}u^2 - a_1v - a_2uv + uv^2. \end{aligned} \tag{25}$$

The singular points at the infinity are $p_1 = (-a_1/a_2, 0)$ and $p_2 = (a_2/a_1, 0)$. The linear parts of system (25) at p_1 and p_2 are, respectively

$$\begin{pmatrix} 0 & \left(\dfrac{a_1}{a_2}\right)^2 \\ \dfrac{(a_1^2+a_2^2)a_4}{a_1^2-a_2^2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \left(\dfrac{a_2}{a_1}\right)^2 \\ -\dfrac{(a_1^2+a_2^2)a_4}{a_1^2-a_2^2} & -\dfrac{a_1^2+a_2^2}{a_1} \end{pmatrix}.$$

These singularities are studied later on. For U_2 the expression (24) becomes

$$\begin{aligned} u' &= -(1+u^2)v, \\ v' &= -\frac{a_1a_2a_4}{a_1^2 - a_2^2} - a_4u + \frac{a_1a_2a_4}{a_1^2 - a_2^2}u^2 - a_2v - a_1uv - uv^2. \end{aligned}$$

Since we are assuming $a_1a_2 \neq 0$, the origin of U_2 is not a singular point.

Subcase I.1.1: $\mathbf{a_4(a_1^2 - a_2^2) > 0}$.

Subcase I.1.1.1: $\mathbf{a_4 \leq \dfrac{a_1^2 - A^2}{4}}$.

Subcase I.1.1.1.1: $\mathbf{a_1 > 0}$. p_1 is a saddle and p_2 is a stable node.

Subcase I.1.1.1.2: $\mathbf{a_1 < 0}$. p_1 is a saddle and p_2 is an unstable node.

Subcase I.1.1.2: $\mathbf{a_4 > \dfrac{a_1^2 - A^2}{4}}$. p_1 is a saddle and p_2 is a focus.

Subcase I.1.2: $\mathbf{a_4(a_1^2 - A^2) < 0}$. p_1 is a focus/center and p_2 is saddle.

Subcase I.2: $\mathbf{a_1 = 0}$. In chart U_1, we have

$$u' = (1+u^2)v, \quad v' = -a_4u - a_2uv + uv^2, \tag{26}$$

and therefore the only infinite singular point is the origin, which we will designate by O_{U_1}. Similarly, in chart U_2 we have the origin O_{U_2} as the unique infinite singular point, since the expression of the vector field becomes

$$u' = -(1+u^2)v, \quad v' = -a_4u - a_2v - uv^2. \tag{27}$$

The linear parts of systems (26) and (27) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & -a_2 \end{pmatrix}.$$

Hence we have the following cases.

Subcase I.2.1: $\mathbf{a_4 > 0}$. O_{U_1} is a focus/center and O_{U_2} is a saddle.

Subcase I.2.2: $\mathbf{-\frac{a_2^2}{4} \leq a_4 < 0}$.

Subcase I.2.2.1: $\mathbf{a_2 > 0}$. O_{U_1} is a saddle and O_{U_2} is a stable node.

Subcase I.2.2.2: $\mathbf{a_2 < 0}$. O_{U_1} is a saddle and O_{U_2} is an unstable node.

Subcase I.2.3: $\mathbf{a_4 < -\frac{a_2^2}{4}}$. O_{U_1} is a saddle and O_{U_2} is a focus.

Subcase I.3: $\mathbf{a_2 = 0}$. In chart U_1, we have

$$u' = (1+u^2)v, \quad v' = -a_4u - a_1v + uv^2, \tag{28}$$

and therefore the only infinite singular point is the origin, which we will designate by O_{U_1}. Similarly, in chart U_2 we have the origin O_{U_2} as the unique infinite singular point, since the expression of the vector field becomes

$$u' = -(1+u^2)v, \quad v' = -a_4u - a_1uv - uv^2. \tag{29}$$

The linear parts of systems (28) and (29) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & -a_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & 0 \end{pmatrix}.$$

Hence we have the following cases.

Subcase I.3.1: $\mathbf{a_4 < 0}$. O_{U_1} is a saddle and O_{U_2} is a focus/center.

Subcase I.3.2: $\mathbf{0 < a_4 \leq \frac{a_1^2}{4}}$.

Subcase I.3.2.1: $\mathbf{a_1 > 0}$. O_{U_1} is a stable node and O_{U_2} is a saddle.

Subcase I.3.2.2: $\mathbf{a_1 < 0}$. O_{U_1} is an unstable node and O_{U_2} is a saddle.

Subcase I.3.3: $\mathbf{a_4 > \frac{a_1^2}{4}}$. O_{U_1} is a focus and O_{U_2} is a saddle.

Case II: $\mathbf{a_1^2 - a_2^2 = 0}$. The condition (19) is simplified to $a_1 a_2 a_4 = 0$ and therefore the following cases might occur.

Subcase II.1: $\mathbf{a_1 = a_2 = 0}$ **and** $\mathbf{a_4 \neq 0}$.
Subcase II.1.1: $\mathbf{a_3 \neq 0}$. p_1 is a focus/center and p_2 is a saddle. In fact the expression (23) for our system in U_1 becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_4 u + a_3 u^2 + uv^2. \tag{30}$$

The singular points at the infinity are $p_{1,2} = ((a_4 \mp \sqrt{4a_3^2 + a_4^2})/2a_3, 0)$. The linear parts of system (30) at p_1 and p_2 are, respectively

$$\begin{pmatrix} 0 & 2 + \dfrac{a_4(a_4 - \sqrt{4a_3^2 + a_4^2})}{2a_3^2} \\ -\sqrt{4a_3^2 + a_4^2} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 + \dfrac{a_4(a_4 + \sqrt{4a_3^2 + a_4^2})}{2a_3^2} \\ \sqrt{4a_3^2 + a_4^2} & 0 \end{pmatrix}.$$

It is easy to see that p_1 is a focus/center and p_2 is a saddle.

For U_2 the expression (24) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 - a_4 u - a_3 u^2 - uv^2.$$

The singular points at the infinity are $p_{3,4} = ((-a_4 \mp \sqrt{4a_3^2 + a_4^2})/2a_3, 0)$. Since $-a_4 \mp \sqrt{4a_3^2 + a_4^2} \neq 0$ for all $a_3, a_4 \in \mathbb{R}\backslash\{0\}$, the origin of U_2 is not a singular point and hence, the only infinite singular points are p_1 and p_2.

Subcase II.1.2: $\mathbf{a_3 = 0}$. The expression (23) for our system in U_1 becomes

$$u' = (1+u^2)v, \quad v' = -a_4 u + +uv^2, \tag{31}$$

and therefore the origin O_{U_1} is the unique infinite singular point in U_1. Similarly, in the chart U_2 the origin O_{U_2} is an infinite singular point because system (24) becomes

$$u' = -(1+u^2)v, \quad v' = -a_4 u - uv^2. \tag{32}$$

The linear parts of systems (31) and (32) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & 0 \end{pmatrix}.$$

Hence we have the following cases.

Subcase II.1.2.1: $\mathbf{a_4 < 0}$. O_{U_1} is a saddle and O_{U_2} is a focus/center.

Subcase II.1.2.2: $\mathbf{a_4 > 0}$. O_{U_1} is a focus/center and O_{U_2} is a saddle.

Subcase II.2: $\mathbf{a_2 = -a_1 \neq 0}$ **and** $\mathbf{a_4 = 0}$. We are only interested in the cases that $a_3 \neq 0$, because as previously mentioned, when $a_3 = a_4 = 0$ system (1) becomes a quadratic differential system, which has already been exhaustively studied.

The expression (23) for our system in U_1 becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_1 v + a_3 u^2 + a_1 uv + uv^2. \tag{33}$$

The singular points at the infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts of system (33) at p_1 and p_2 are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & -2a_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & 0 \end{pmatrix}.$$

For U_2 the expression (24) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 + a_1 v - a_3 u^2 - a_1 uv - uv^2.$$

The singular points at infinity are $p_{3,4} = (\mp 1, 0)$. The origin of U_2 is not a singular point and hence, the only infinite singular points are p_1 and p_2. These singularities are studied in what follows.

Subcase II.2.1: $\mathbf{a_3 < 0}$. p_1 is a saddle and p_2 is a focus/center.

Subcase II.2.2: $\mathbf{0 < a_3 \leq a_1^2/4}$.

Subcase II.2.2.1: $\mathbf{a_1 > 0}$. p_1 is a stable node and p_2 is a saddle.

Subcase II.2.2.2: $\mathbf{a_1 < 0}$. p_1 is an unstable node and p_2 is a saddle.

Subcase II.2.3: $\mathbf{a_3 > a_1^2/4}$. p_1 is a focus and p_2 is a saddle.

Subcase II.3: $\mathbf{a_2 = a_1 \neq 0}$ **and** $\mathbf{a_4 = 0}$. Again we are only interested in the cases that $a_3 \neq 0$.

The expression (23) for our system in U_1 becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_1 v + a_3 u^2 - a_1 uv + uv^2. \tag{34}$$

The singular points at infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts of system (34) at p_1 and p_2 are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & -2a_1 \end{pmatrix}.$$

These singularities are studied later on.

For U_2 the expression (24) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 + a_1 v - a_3 u^2 - a_1 uv - uv^2. \tag{35}$$

The singular points at infinity for (35) are $p_{3,4} = (\mp 1, 0)$. The origin of U_2 is not a singular point.

Subcase II.3.1: $\mathbf{a_3 > 0}$. p_1 is a focus/center and p_2 is a saddle.

Subcase II.3.2: $\mathbf{-a_1^2/4 \leq a_3 < 0}$.

Subcase II.3.2.1: $\mathbf{a_1 > 0}$. p_1 is a saddle and p_2 is a stable node.

Subcase II.3.2.2: $\mathbf{a_1 < 0}$. p_1 is a saddle and p_2 is an unstable node.

Subcase II.3.3: $\mathbf{a_3 < -a_1^2/4}$. p_1 is a saddle and p_2 is a focus.

Subcase II.4: $\mathbf{a_1 = a_2 = a_4 = 0}$. Again we are only interested in the cases that $a_3 \neq 0$. In this case system (1) has the particular form

$$\dot{x} = -y + a_3x^3 - a_3xy^2, \quad \dot{y} = x + a_3x^3 - a_3xy^2.$$

The expression (23) for our system in U_1 becomes

$$u' = (1 + u^2)v, \quad v' = -a_3 + a_3u^2 + uv^2. \tag{36}$$

The singular points at the infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts of system (34) at p_1 and p_2 are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & 0 \end{pmatrix}.$$

These singularities are studied in the next subcases.

For U_2 the expression (24) becomes

$$u' = -(1 + u^2)v, \quad v' = a_3 - a_3u^2 - -uv^2.$$

The singular points at infinity are $p_{3,4} = (\mp 1, 0)$. The origin of U_2 is not a singular point.

Subcase II.4.1: $\mathbf{a_3 > 0}$. p_1 is a focus/center and p_2 is a saddle.

Subcase II.4.2: $\mathbf{a_3 < 0}$. p_1 is a saddle and p_2 is a focus/center.

Finally, the global phase portraits for the uniform isochronous cubic polynomial systems are obtained using the study of the finite and infinite singular points in the local phase portraits and the first integrals calculated in Theorem 8. Hence Theorem 9 is proved.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant number MTM2009-03437, by an AGAUR grant number 2014SGR-568, by an ICREA Academia, two FP7+PEOPLE+2012+IRS8S numbers 316338 and 318999, and FEDER-UNAB10-4E-378. The second author is supported by a Ciência sem Fronteiras-CNPq grant number 201002/2012-4.

References

[1] A. Algaba and M. Reyes, *Computing center conditions for vector fields with constant angular speed*, J. Computational and Applied Mathematics **154** (2003) 143–159.

[2] A. Buică and J. Llibre, Averaging methods for finding periodic orbits via Brower degree, Bull. Sci. Math. **128** (2004), 7–22.

[3] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qualitative Theory of Dynamical Systems **1** (1999), 1–70.

[4] C. Christopher and C. Li, *Limit cycles in differential equations*, Birkhäuser, Boston, 2007.

[5] A.G. Choudhury and P. Guha, *On commuting vector fields and Darboux functions for planar differential equations*, Lobachevskii Journal of Mathematics **34** (2013), 212–226.

[6] C.B. Collins, *Conditions for a center in a simples class of cubic systems*, Differential and Integral Equations **10** (1997), 333-356.

[7] R. Conti, *Uniformly isochronous centers of polynomial systems in* $\mathbb{R}^2$, Lecture Notes in Pure and Appl. Math. **152** (1994), 21–31.

[8] J. Devlin, N.G. Lloyd and J.M. Pearson, *Cubic Systems and Abel Equations*, J. Differential Equations **147** (1998), 435–454.

[9] F.S. Dias and L.F. Mello, *The center-focus problem and small amplitude limit cycles in rigid systems*, Discrete and Continuous Dynamical Systems **32** (2012), 1627–1637.

[10] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Spring-Verlag, 2006.

[11] A. Gasull, R. Prohens and J. Torregrosa, *Limit cycles for rigid cubic systems*, J. Math. Anal. Appl. **303** (2005), 391–404.

[12] J. Karlin and W.J. Studden, *T-systems: with applications in analysis and statistics*, Pure Appl. Math., Interscience Publishers, New York, London, Sidney, 1966.

[13] J.M. Lee, *Introduction to Smooth Manifolds*, University of Washington, USA, 2000.

[14] C. Li, W. Li, J. Llibre and Z. Zhang, *New families of centers and limit cycles for polynomial differential systems with homogeneous nonlinearities*, Ann. Differential Equations **19** (2003), 302–317.

[15] J. Llibre, D. D. Novaes and M. A. Teixeira, *Higher order averaging theory for finding periodic solutions via Brouwer degree*, Nonlinearity **27** (2014), 563–583.

[16] J. Llibre and A.C. Mereu, *Limit cycles for discontinuous quadratic differential systems with two zones*, J. Math. Anal. Appl. **413** (2014), 763–775.

[17] J. Llibre and G. Świrszcz, *On the limit cycles of polynomial vector fields*, Dynamics of Continuous, Discrete and Impulsive Systems, Serie A **18** (2011), 203–214.

[18] L.S. Pontrjagin, *Über Autoschwingungssysteme, die den hamiltonshen nahe liegen*, Physikalische Zeitschrift der Sowjetunion **6** (1934), 25–28.

[19] J.A. Sanders, F. Verhulst and J. Murdock, *Averaging methods in nonlinear dynamical systems*, 2nd, edition, Applied Mathematical Sciences 59, Springer, New York, 2007.

[1]Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain. Fax +34 935812790. Phone +34 93 5811303

E-mail address: jllibre@mat.uab.cat, itikawa@mat.uab.cat