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PERIOD SETS OF LINEAR TORAL ENDOMORPHISMS ON $\mathbb{T}^2$

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Abstract. The period set of a dynamical system is defined as the subset of all integers n such that the system has a periodic orbit of length n. Based on known results on the intersection of period sets of torus maps within a homotopy class, we give a complete classification of the period sets of (not necessarily invertible) toral endomorphisms on the 2–dimensional torus $\mathbb{T}^2$.

1. Introduction and statement of the main results

The sets of periods that are present in a dynamical system is one of the key quantities which characterises the system. For continuous maps on the interval, the celebrated theorem by Sharkovsky provides a complete characterisation of period sets in terms of the Sharkovsky ordering. A generalisation of Sharkovsky's Theorem to the circle was obtained by the authors Block, Coppel, Guckenheimer, Misiurewicz and Young [4, 5, 6, 19], for a unified proof see [1]. For other classes of maps where the period sets have been studied see for instance [2, 11, 14, 18].

Toral or torus endomorphisms are continuous mappings of the torus that preserve its group structure, hence in the additive notation $\mathbb{T}^m \cong \mathbb{R}^m/\mathbb{Z}^m$, they can be represented as $m \times m$ integer matrices, see for instance [21, Chapter 0]. In the present article, the term 'toral endomorphism' is always used in this sense, that is, for a map on the torus which is induced by the action of an integer matrix modulo 1. They serve as a standard example in the theory of discrete dynamical systems and ergodic theory, and particularly the case of hyperbolic toral automorphisms, corresponding to integer matrices with determinant ± 1 and no eigenvalues on the unit circle, has been studied extensively because of its interesting dynamical properties, compare [21, 16]. In [15], the period sets of toral automorphisms on $\mathbb{T}^2$ were investigated.

Minimal period sets are the intersection of all period sets arising from maps in the same homotopy class; the origins of their study go back to Alsedá, Baldwin, Llibre, Swanson and Szlenk [3], see also [12, 13]. Associated with each homotopy class, one has a unique integer matrix A, which defines the action on the first homology group, and this integer matrix, in turn, defines an endomorphism of the torus, f_A, hence it is itself a member of the homotopy class. An important result in arbitrary dimension is that the minimal period set of a homology class "essentially" coincides with the period set of the associated toral endomorphism, apart from possibly those periods that may arise from roots of unity

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among the eigenvalues of this matrix, see [13]. However, in many of the cases in which ± 1 is among the eigenvalues, one finds that the minimal period set and the period set of the endomorphism totally differ and it can be concluded that, in these cases, the endomorphism is not a good model for the dynamics of its homotopy class with respect to periodic orbits.

For circle maps, the situation is comparatively simple and is displayed in Table 1. The period sets only depend on the degree d of the map, listed in the first column; the second and the third column refer to the minimal period set and the period set of the corresponding endomorphism, respectively; the last column answers the question whether the endomorphism defining the class under consideration is invertible, hence an automorphism. Here, the linear endomorphism f_A is given by $f_{(d)}(x) = d \cdot x$.

d	$\mathrm{MPer}(f_A)$	$\mathrm{Per}(f_A)$	$\in \mathrm{Aut}(\mathbb{T})$?
1	$\emptyset$	$\{1\}$	yes
0	$\{1\}$	$\{1\}$	no
-1	$\{1\}$	$\{1,2\}$	yes
-2	$\mathbb{N} \setminus \{2\}$	$\mathbb{N} \setminus \{2\}$	no
$d \in \mathbb{Z} \setminus \{-2,-1,0,1\}$	$\mathbb{N}$	$\mathbb{N}$	no

Table 1. Period sets for circle maps.

In this article, we compare the minimal period sets of maps on $\mathbb{T}^2$ with the period set of the associated toral endomorphism, aiming for an analogue of the above table for dimension 2.

To obtain a complete classification, a variety of partly very different techniques is employed. The minimal period sets were derived in [3] by means of estimating Nielsen numbers, see [11] for background reading on this approach. The study of period sets in the special case of toral automorphisms in [15] is based on an extensive distinction of cases; we note that much of the reasoning in [15] makes use of the assumption of a determinant ± 1 and hence does not directly generalise to arbitrary 2×2 matrices. We complete the classification of period sets by making use of results on *local conjugacy*, that is conjugacy modulo n, $n \in \mathbb{N}$, which corresponds to the action of the matrix on the invariant subsets of points with rational coordinates.

The outline of this article is as follows. In Section 2 we compile the theory of minimal period sets and periods of toral endomorphisms as far as needed for our purpose, derive the period sets $\mathrm{Per}(f_A)$ from $\mathrm{MPer}(f_A)$ wherever possible and identify the cases that require individual treatment. In Section 3, we complete the classification of period sets on the two-dimensional torus by considering normal forms for matrices with an eigenvalue ± 1. At the end of this introductory section, we summarise the results of our later analysis in form of a comprehensive table.

The following Table 2 constitutes the 2–dimensional analogue of Table 1 for circle maps. Starting from some $A \in \mathrm{Mat}(2, \mathbb{Z})$, the columns (from left to right order) refer to the eigenvalues of A, the pair of the trace and the determinant of A, i.e. $(t, d) = (\mathrm{tr}(A), \det(A))$, the minimal polynomial μ_A, the minimal period set $\mathrm{MPer}(f_A)$ of the homotopy class of f_A, the period set $\mathrm{Per}(f_A)$, and finally an answer to the question whether $f_A \in \mathrm{End}(\mathbb{T}^2)$ is invertible, hence an element of $\mathrm{Aut}(\mathbb{T}^2) \subset \mathrm{End}(\mathbb{T}^2)$. The symbol χ_A denotes the characteristic polynomial of the matrix A; the symbol E refers to the set of exceptional integer values $E = \{-2, -1, 0, 1\}$.

	eigenvalues	(t, d)	μ_A	$\mathrm{MPer}(f_A)$	$\mathrm{Per}(f_A)$	$\in \mathrm{Aut}(\mathbb{T}^2)$?
1.	1	$(2, 1)$	$x - 1$	$\emptyset$	$\{1\}$	y
2.	1	$(2, 1)$	χ_A	$\emptyset$	$\mathbb{N}$	y
3.	-1	$(-2, 1)$	$x + 1$	$\{1\}$	$\{1, 2\}$	y
4.	-1	$(-2, 1)$	χ_A	$\{1\}$	$2\mathbb{N} \cup \{1\}$	y
5.	± 1	$(0, -1)$	χ_A	$\emptyset$	$\{1, 2\}$	y
6.	$e^{2\pi i/3}, e^{-2\pi i/3}$	$(-1, 1)$	χ_A	$\{1\}$	$\{1, 3\}$	y
7.	$\pm i$	$(0, 1)$	χ_A	$\{1, 2\}$	$\{1, 2, 4\}$	y
8.	$e^{\pi i/3}, e^{-\pi i/3}$	$(1, 1)$	χ_A	$\{1, 2, 3\}$	$\{1, 2, 3, 6\}$	y
9.	0	$(0, 0)$	x^2 or x	$\{1\}$		n
10.	$0, 1$	$(1, 0)$	$x^2 - x$	$\emptyset$	$\{1\}$	n
11.	$0, -1$	$(-1, 0)$	χ_A	$\{1\}$	$\{1, 2\}$	n
12.	$\notin \mathbb{R}$	$(-2, 2)$	χ_A	$\mathbb{N} \setminus \{2, 3\}$		n
13.	$\notin \mathbb{R}$	$(-1, 2)$	χ_A	$\mathbb{N} \setminus \{3\}$		n
14.	$\notin \mathbb{R}$	$(0, 2)$	χ_A	$\mathbb{N} \setminus \{4\}$		n
15.	$\notin \mathbb{R}$	none of the above	χ_A	$\mathbb{N}$		y/n
16.	real, both $\neq \pm 1$	$t + d \notin \{0, -2\}$	χ_A or $x - a$, $a \in \mathbb{Z} \setminus E$	$\mathbb{N}$		y/n
17.	real	$t + d \in \{0, -2\}$, $(t, d) \neq (0, 0)$	χ_A or $x + 2$	$\mathbb{N} \setminus \{2\}$		y/n
18.	$-1, -d$	$t + d = -1$, $d \in \mathbb{Z} \setminus \{-1, 0, 1\}$	χ_A	$2\mathbb{N} - 1$	$\mathbb{N}$	n
19.	$1, -2$	$(-1, -2)$	χ_A	$\emptyset$	$\mathbb{N} \setminus \{2\}$	n
20.	$1, d$	$t - d = 1$, $(t, d) \neq (-1, -2)$	χ_A	$\emptyset$	$\mathbb{N}$	n

Table 2. Period sets for torus maps.

2. Definitions and overview of results

The set of periods of a map $f : \mathbb{T}^m \to \mathbb{T}^m$ will be denoted by

$$\mathrm{Per}(f) \coloneqq \{n \in \mathbb{N} \mid f \text{ has an orbit of length } n\}.$$

The *minimal* set of periods is defined as the intersection of all period sets in a given homotopy class,

$$\mathrm{MPer}(f_A) \coloneqq \bigcap_{g \simeq f} \mathrm{Per}(g), \tag{1}$$

where $g \simeq f$ means that g is homotopic to f.

Associated with a torus map $f : \mathbb{T}^m \to \mathbb{T}^m$, one has the first induced homology map $f_{*1} : H_1(\mathbb{T}^m, \mathbb{Z}) \to H_1(\mathbb{T}^m, \mathbb{Z})$, which corresponds to some $m \times m$ integer matrix A. In the following, we will denote by $\mathrm{Mat}(m, R)$ the ring of $m \times m$ matrices with entries from the ring R. By $\mathrm{Mat}(m, R)^\times$ we will refer to the invertible matrices over R or, equivalently, to those elements of $\mathrm{Mat}(m, R)$ whose determinant is a unit in R. The linear map defined by the matrix $A : \mathbb{R}^m \to \mathbb{R}^m$, covers a unique torus endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$ whose action is given by matrix multiplication modulo 1, that is, two points $x, y \in \mathbb{R}^m$ cover the same element of $\mathbb{T}^m$ if and only if $x - y \in \mathbb{Z}^m$. By $\mathrm{End}(\mathbb{T}^m)$ we denote the set of toral endomorphisms, and by $f_A \in \mathrm{End}(\mathbb{T}^m)$ we refer to the map induced by $A \in \mathrm{Mat}(m, \mathbb{Z})$. As $f_A \simeq f$, the intersection on the right-hand side of Equation (1) comprises the period set of f_A, so one always has $\mathrm{MPer}(f_A) \subset \mathrm{Per}(f_A)$. In fact, these two sets are typically more closely related; in [3, Proposition 3.4] the following general result is proved

$$\mathrm{MPer}(f) = \mathrm{Per}(f_A) \setminus \{k \in \mathbb{N} : 1 \text{ is an eigenvalue of } A^n\}. \tag{2}$$

In other words, the period set of the endomorphism f_A and the minimal period set $\mathrm{MPer}(f)$ coincide if and only if A does not have any eigenvalues that are roots of unity. In this case, the number of fixed points of A^n can be calculated in terms of the *Nielsen numbers*: f_A^n has $N(f_A^n)$ isolated fixed points, where

$$N(f_A^n) = |\det(\mathbb{1} - A^n)|,$$

and $\mathbb{1}$ denotes the $m \times m$ identity matrix. For $m = 2$, this simplifies to

$$N(f_A^n) = |1 + \lambda_1^n \lambda_2^n - (\lambda_1^n + \lambda_2^n)|,$$

where λ_1, λ_2 are the eigenvalues of A.

When the matrix A has an eigenvalue that is an n-th root of unity, the Nielsen numbers vanish for all multiples of n. In this case, n is not in the minimal period set; however, the endomorphism f_A admits periodic points of period n, which form subtori of $\mathbb{T}^m$. A treatment of this case can be found in the appendix of [8].

The rational points of $\mathbb{T}^m$ form an invariant subset of $\mathbb{T}^m$ and can be written as the countable union of the finite sets of n-division points, $n \in \mathbb{N}$,

$$[0,1)^m \cap \mathbb{Q}^m = \bigcup_{n\in\mathbb{N}} \left\{ \left(\frac{k_1}{n}, \dots, \frac{k_m}{n} \right) \mid 0 \le k_1, \dots, k_m < n \right\} = \bigcup_{n\in\mathbb{N}} L_n.$$

Each rational lattice L_n in the union on the right-hand side is invariant under the action of the integer matrix A, hence L_n is partitioned into finite orbits by f_A. Denoting by $\mathrm{PerP}(f)$ the periodic *points* of a map f, we can state that, if $\gcd(n, \det(A)) = 1$, $L_n \subset \mathrm{PerP}(f_A)$, hence

$$\bigcup_{n\in\mathbb{N}\,:\,\gcd(n,\det(A))=1} L_n \subset \mathrm{PerP}(f_A), \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} L_n \subset \mathrm{PerP}\,(f_A)$$

if $\det(A) = \pm 1$, that is, if A is an automorphism. In the absence of roots of unity among the eigenvalues, all periodic points are isolated and live on the rational lattices, whence the converse $\mathrm{PerP}(f_A) \subset \bigcup_{n\in\mathbb{N}} L_n$ follows, and one has strict equality if A is invertible, as all lattice points are then consumed in periodic orbits. There is an abundance of literature dealing with the periods of toral endomorphisms (mostly restricted to the invertible case, i.e. automorphisms) on the rational lattices L_n depending on n; for an overview, see, for instance, the articles [20, 10, 7, 9] and references therein. The rational lattice L_n can be identified with the free module over the finite ring $\mathbb{Z}/n\mathbb{Z}$, so studying the action of an integer matrix on L_n amounts to studying the matrix action modulo n, and the question whether two integer matrices share the same orbit structure on L_n is closely related to the question of conjugacy over the residue class ring $\mathbb{Z}/n\mathbb{Z}$. Two matrices that are conjugate over $\mathbb{Z}/n\mathbb{Z}$ necessarily exhibit the same orbit structure on L_n.

On the two-dimensional torus, one can formulate a necessary and sufficient criterion for matrices to be locally conjugate on a lattice L_n for some or all $n \in \mathbb{N}$ in terms of trace, determinant and a third invariant, which we now define.

Definition 1. The matrix gcd (short mgcd) of a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{Z})$ is defined as $\mathrm{mgcd}(M) \coloneqq \gcd(\beta, \gamma, \delta - \alpha)$.

We note that the square of $\mathrm{mgcd}(M)$ always divides the discriminant $\Delta(M) = \mathrm{tr}(M)^2 - 4\det(M)$,

$$\Delta(M) = (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) = (\alpha+\delta)^2 - 4\beta\gamma. \tag{3}$$

The following theorem asserts trace, determinant and mgcd to be a complete set of invariants with respect to local conjugacy.

Theorem 2 ([7, Theorem 2])**.** *For two integer matrices $M, M' \in \mathrm{Mat}(2,\mathbb{Z})$, the following statements are equivalent.*

(a) *The reductions mod n of M and M' are $\mathrm{Mat}(2,\mathbb{Z}/n\mathbb{Z})^\times$ conjugate.*
(b) *M and M' share the same trace, determinant and mgcd.*

We will make use of this theorem in Section 3 in order to determine the period sets of certain toral endomorphisms with integer eigenvalues. Now we first treat the cases of complex and irrational real eigenvalues, for which the period sets of toral endomorphisms can be gained from the minimal period sets of their homotopy classes.

2.1. **Complex eigenvalues and finite orders.** If $\xi \in \mathbb{C} \setminus \mathbb{R}$ is the eigenvalue of a matrix $A \in \mathrm{Mat}(2,\mathbb{Z})$, its complex conjugate $\bar{\xi}$ is the second eigenvalue of A and the determinant $d = \xi\bar{\xi} = |\xi|^2$ is consequently a positive integer.

If $d = 1$, both roots lie on the unit circle, that is, they have modulus 1 and they are actually roots of unity. (This is a special case of a more general fact: if all algebraic conjugates of an algebraic integer α have modulus 1, α is a root of unity.) But if all eigenvalues of the matrix A are distinct roots of unity, A has finite order, that is, there is some integer n such that $A^n = \mathbb{1}$.

It is well-known that for each n the finite orders of subgroups of $\mathrm{GL}(n,\mathbb{Z})$ can only assume certain values, determined by all factors of the cyclotomic polynomials that are elements of $\mathbb{Z}[x]$ and have degree n, hence are the characteristic polynomial of a matrix in $\mathrm{GL}(n,\mathbb{Z})$. Furthermore, for $2n$ and $2n+1$, the lists of possible orders coincide, more specifically, $\mathrm{GL}(n,\mathbb{Z})$ for $n=2$ and 3 can have subgroups of the finite orders $1,2,3,4,6$, see [17], for example.

For 2×2 integer matrices, these cases correspond to the characteristic polynomials x^2+x+1, x^2+1, x^2-x+1 whose roots are $e^{2i\pi/3}$, $e^{-2i\pi/3}$, $\pm i$ and $e^{i\pi/3}$, $e^{-i\pi/3}$, respectively. In [15] the periods of the corresponding automorphisms are derived from the characteristic equations; using the knowledge about the minimal period sets, one can treat these three cases at once. *They are represented in rows 6–8 of Table 2.*

Lemma 3. *Assume* $A\in\mathrm{Mat}(2,\mathbb{Z})$ *is of finite order* $k=\mathrm{ord}(A)$. *Then* $k\in\mathrm{Per}(f_A)$. *Consequently, for* $A\in\mathrm{Mat}(2,\mathbb{Z})$ *whose eigenvalues are complex roots of unity, one finds* $\mathrm{Per}(f_A)=\mathrm{MPer}(f_A)\cup\{\mathrm{ord}(A)\}$.

Proof. The fact that A has order k means $A^k-\mathbb{1}$ is the zero matrix but for every $d<k$, $A^d-\mathbb{1}$ is not zero. But that means, there is some $n\in\mathbb{N}$ such that $A^d-\mathbb{1}\not\equiv 0 \mod n$ for every $d<k$. Consequently, there is some $x=(\alpha,\beta)\in(\mathbb{Z}/n\mathbb{Z})^2$ such that $(A^d-\mathbb{1})x\neq 0$ over $\mathbb{Z}/n\mathbb{Z}$ for every $d<k$. Hence, the point x has period k. The last statement follows because the period set of a matrix of order k can contain divisors of k only, and in the cases under consideration, all divisors d of k with $d<k$ are already contained in the set $\mathrm{MPer}(f_A)\subset\mathrm{Per}(f_A)\subset\{d : d|k\}$. □

Note that the matrix order is absent from the set of minimal periods of the homotopy class, as for $A^n=\mathbb{1}$, one has $N(f_A^{j\cdot n})=0$ for all $j\in\mathbb{N}$. The cases of integer roots of unity will be dealt with in Section 3.

If the eigenvalues in $\mathbb{C}\setminus\mathbb{R}$ are not roots of unity, the determinant has to be greater than 1. The period sets of the remaining cases with non-real eigenvalues are given by the following theorem [3, Thm. 4.24].

Theorem 4 ([3])**.** *If* $d\geq 2$ *and* $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$ *then* $\mathrm{MPer}(f_A)=\mathbb{N}$, *unless the pair* (t,d) *is one of the following cases:*

(a) *If* $(t,d)=(-2,2)$, *then* $\mathrm{MPer}(f_A)=\mathbb{N}\setminus\{2,3\}$.
(b) *If* $(t,d)=(-1,2)$, *then* $\mathrm{MPer}(f_A)=\mathbb{N}\setminus\{3\}$.
(c) *If* $(t,d)=(0,2)$, *then* $\mathrm{MPer}(f_A)=\mathbb{N}\setminus\{4\}$.

In none of the cases addressed in Theorem 4 the eigenvalues are roots of unity, so Equation (2) implies $\mathrm{MPer}(f_A)=\mathrm{Per}(f_A)$. *These cases are listed in rows 12-15 of Table 2.*

2.2. **Real eigenvalues.** Real eigenvalues either come in pairs of irrational quadratic integers (that is, zeros of polynomials of the shape x^2-tx+d, $t,d\in\mathbb{Z}$) or elements of $\mathbb{Z}$. The fact that rational eigenvalues have to be integers is shown in the following lemma.

Lemma 5. *If* A *has one (and thus two) rational eigenvalues, they are in fact integers, hence the polynomial splits into two linear factors over* $\mathbb{Z}$.

Proof. Let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be the zeros of the characteristic polynomial $\chi_A(x)$. Without loss of generality, we can assume that $\gcd(p_1,q_1)=\gcd(p_2,q_2)=1$. We know that $\frac{p_1}{q_1}+\frac{p_2}{q_2}=\frac{p_1q_2+p_2q_1}{q_1q_2}=\mathrm{tr}(A)\in\mathbb{Z}$.

Consequently, the numerator r has to be divisible by q_1q_2, hence it is divisible by both q_1 and q_2, or, equivalently, $r = p_1q_2 + p_2q_1 \equiv 0 \mod q_i$, $i \in \{1,2\}$. But $r \equiv 0 \equiv p_1q_2 \mod q_1$, whence p_1q_2 is divisible by q_1. As $\gcd(p_1, q_1) = 1$ by assumption, one has $q_1|q_2$. Considering the equation modulo q_2 analogously gives $q_2|q_1$, hence $q := q_1 = \pm q_2$. Furthermore, $\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \det(A) \in \mathbb{Z}$, thus $\pm q^2|p_1p_2$ such that the assumed coprimality implies $q \in \{-1, 1\}$. $\square$

The following theorem is the summary of Propositions 4.18, 4.19 and 4.23 from [3].

Theorem 6 ([3])**.** *Let A be in $\mathrm{Mat}(2,\mathbb{Z})$ with real eigenvalues.*

(a) *Suppose A has eigenvalues, neither of which is ± 1. If $d + t \notin \{0, -2\}$, then $\mathrm{MPer}(f_A) = \mathbb{N}$.*
(b) *If $d = t = 0$ then $\mathrm{MPer}(f_A) = \{1\}$.*
(c) *Suppose $t + d \in \{0, -2\}$ and $(t, d) \neq (0, 0)$. Then $\mathrm{MPer}(f_A) = \mathbb{N} \setminus \{2\}$.*

In the situation of (a) and (b), no root of unity is eigenvalue of the matrix A and hence $\mathrm{MPer}(f_A) = \mathrm{Per}(f_A)$. *The rows in Table 2 corresponding to Theorem 6 are 9, 16 and 17.*

So according to Equation (2) and in view of Theorem 6, the only cases of real eigenvalues in which $\mathrm{MPer}(f_A) \neq \mathrm{Per}(f_A)$ correspond to ± 1 being among the eigenvalues. As Lemma 5 tells us, the other eigenvalue then has to be an integer as well. These cases will be dealt with in the following section.

3. Filling the gaps: the remaining cases

The cases where the general theory of minimal period sets does not immediately give the period set of the corresponding toral endomorphism is the case when the eigenvalues are 1 or -1 and some second integer, which is in fact decisive for the determination of the period set. The simple example of $\mathbb{1}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shows that the same eigenvalues can a priori imply different period sets, the former being $\{1\}$, the latter $\mathbb{N}$, although both matrices share the (double) eigenvalue 1.

Let $E = \{-2, -1, 0, 1\}$ and $\mathrm{ord}_n\, a$ the order of a modulo n, that is, the least integer $k > 0$ such that $a^k \equiv 1 \mod n$. The order $\mathrm{ord}_n(a)$ is well-defined for all $n \in \mathbb{N}$, $n > 1$, such that $\gcd(a, n) = 1$.

Lemma 7. *Let $a \in \mathbb{Z} \setminus \{0\}$. Then*

$$P_a := \{\mathrm{ord}_n\, a \mid n \in \mathbb{N} \setminus \{1\}, \gcd(a, n) = 1\}$$

equals one of the following sets:

(a) $\{1\}$ *if $a = 1$,*
(b) $\{1, 2\}$ *if $a = -1$,*
(c) $\mathbb{N} \setminus \{2\}$ *if $a = -2$,*
(d) $\mathbb{N} \setminus \{1\}$ *if $a = 2$,*
(e) $\mathbb{N}$ *if $a \in \mathbb{Z} \setminus (E \cup \{2\})$.*

Proof. The statements for $a = \pm 1$ are clear, so we exclude these cases from our further considerations. We first check that $2 \notin P_{-2}$: if $(-2)^2 \equiv 1 \mod n$, clearly, $n|[(-2)^2-1] = 3$, so the only possible choice for n is 3 (as 1 is excluded). But $-2 \equiv 1 \mod 3$, and thus $\mathrm{ord}_3(-2) = 1$, whence $1 \in P_{-2}$ and $2 \notin P_{-2}$ follows.

Next, we note that $1 \notin P_2$, for $2 \equiv 1 \mod n$ would imply $n|(2-1) = 1$; however, $1 \in P_a$ for all other a because $a \equiv 1 \mod (a-1)$ if $a > 2$ and $a \equiv 1 \mod (-a+1)$ if $a < -1$.

We first consider the case where a is positive and assume $a > 2$ or $k > 1$, such that $a^k - 1 > 1$. Clearly, $a^k \equiv 1 \mod (a^k - 1)$, so assume there is a $d|k$, $d \neq k$, with $a^d \equiv 1 \mod (a^k-1)$. But then $(a^k-1)|(a^d-1)$, which is not possible for $d < k$, hence $\mathrm{ord}_{a^k-1}(a) = k$ and $k \in P_a$.

Next, consider the case where a is negative and k even, $a \neq -2$ or $k \neq 2$. One has $a^k \equiv 1 \mod (a^k - 1)$, and $(a^k - 1)|(a^d - 1)$ for the divisor $d = \mathrm{ord}_{a^k-1}(a)$ of k. If d is even, this gives the same contradiction as above; if d is odd, one has $(a^k - 1)|(-a^d + 1)$ and thus $|a|^k - |a|^d \leq 2$. But the only positive integers d, k with $d < k$ satisfying this inequality are $k = 2$, $d = 1$ in the case $|a| = 2$. For $a < -2$ and k odd, one readily checks that $a^k \equiv 1 \mod (-a^k + 1)$ and similar reasoning as above shows that k is indeed the order modulo $(-a^k + 1)$. Hence it follows that if $|a| > 2$ or $k > 2$ then $k \in P_a$, and in summary, the lemma follows. □

We stipulate that $P_0 = \emptyset$. Then we can formulate the following

Proposition 8. *Let A be in $\mathrm{Mat}(2, \mathbb{Z})$ and assume A has the eigenvalues $a, b \in \mathbb{Z}$. Then $\mathrm{Per}(f_A) \supset \{1\} \cup P_a \cup P_b$.*

Proof. According to Theorem 2, two matrices in $\mathrm{Mat}(2, \mathbb{Z})$ induce the same orbit structure on the rational lattices L_n for all n, if trace, determinant and mgcd coincide. The square of the mgcd always divides the discriminant, see Equation (3), hence the set of possible triplets $(\mathrm{tr}, \det, \mathrm{mgcd})$ can be itemised explicitly. Fixing the trace and the determinant as $\mathrm{tr}(A) = a + b, \det(A) = a \cdot b$, the discriminant is $\Delta = \mathrm{tr}(A)^2 - 4\det(A) = (b - a)^2$, whence the mgcd has to divide $b - a$. Consequently, one has an upper diagonal matrix in the following set corresponding to each possible value of the mgcd

$$\left\{ \begin{pmatrix} a & r \\ 0 & b \end{pmatrix} \mid r^2|\Delta \right\}. \tag{4}$$

In other words, given an integer matrix A, there is some upper diagonal matrix in the above set that has exactly the same orbit structure on each of the finite sets L_n, $n \in \mathbb{N}$. Furthermore, the matrices $\begin{pmatrix} a & r \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} b & r \\ 0 & a \end{pmatrix}$ have the same orbit structure on each L_n. But an upper diagonal matrix $\begin{pmatrix} d_1 & r \\ 0 & d_2 \end{pmatrix}$ clearly has points of all orders contained in P_{d_1} and P_{d_2}, as, for instance, for $k \in P_{d_1}$, the point $(\frac{1}{n}, 0)$ has period $k = \mathrm{ord}_n(d_1)$ and by the previous sentence, the roles of d_1 and d_2 are interchangeable. Thus, for eigenvalues a, b, the set $\mathrm{Per}(f_A)$ comprises the union $P_a \cup P_b$. Since the point $(0, 0)$ has period 1 for every integer matrix, the claim follows. □

Remark 9. Note that the eigenvalue 0 does not contribute any periods. If both eigenvalues are 0, the matrix is nilpotent, and the only periodic point is $(0,0)$. If the second eigenvalue is non-zero, it has to be an integer, and it determines the period set according to Lemma 7 and Proposition 8. If the second eigenvalue is 1 or -1, one has $A^2 = A$ or $A^3 = A$, respectively, which immediately gives the periodsets. *The cases with* 0 *among the eigenvalues are treated in rows 9–11 of Table 2 if not covered by row 15 or 16.*

The symbol '$\supset$' in Proposition 8 cannot be replaced by an equality sign, because a non-zero upper-diagonal entry r may give rise to additional periods, as in the aforementioned example of $\mathbb{1}$ versus $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, as well as for the matrices $-\mathbb{1}$ and $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. These cases are covered by Proposition 3.6 in [15], where the powers of suitably parametrised 2×2 matrices are considered. Using our local approach modulo n, one could restrict the considerations to upper diagonal matrices as in Equation 4, which, together with the Nielsen numbers whenever they do not vanish, yield the period sets as listed in Table 2. We note that the cases of a double eigenvalue 1 or -1, respectively, are the only ones in which the knowledge of the eigenvalues is not enough to determine the period set of the endomorphism.
The cases corresponding to the eigenvalues ± 1 *are listed in rows 1–5 in Table 2.*

With help of Proposition 8, we can write down the period set for all remaining cases where 1 or -1 is among the eigenvalues of the matrix.

Corollary 10. *Let* (t, d) *denote the pair of trace and determinant of* $A \in \mathrm{Mat}(2, \mathbb{Z})$*.*

(a) *If the eigenvalues are* $-1, -d$*, hence* $t + d = -1$*,* $d \notin \{-1, 0, 1\}$ *the period set* $\mathrm{Per}(f_A)$ *is* $\mathbb{N}$*.*
(b) *If the eigenvalues are* $-2, 1$*, hence* $t - d = 1$*, then* $\mathrm{Per}(f_A) = \mathbb{N} \setminus \{2\}$*.*
(c) *If the eigenvalues are* $1, d$*,* $t - d = 1$*,* $(t, d) \neq (-1, -2)$ *then* $\mathrm{Per}(f_A) = \mathbb{N}$*.*

Proof. Case (a) and (c) immediately follow from Proposition 8; for (b), we note that for each x with $A^2 x = x \mod 1$, one finds $A^2 x - \mathbb{1} = -Ax + x = -(A - \mathbb{1})x = 0 \mod 1$ by the Cayley-Hamilton Theorem, so x is a fixed point and therefore, A does not admit any points of least period 2. □

We state again that the cases treated in Corollary 10 are the ones in which $\mathrm{Per}(f_A)$ and $\mathrm{MPer}(f_A)$ significantly differ; the latter can be found on page 30 in [3].
Corollary 10 yields the entries of rows 18–20 in Table 2.

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