This is a preprint of: "Dynamic systems behaviour analysis and design based on the qualitative theory of differential equations: the Boost power converter case", Jaume Llibre, Mario Spinetti-Rivera, Eliezer Colina-Morles, *Int. J. Control*, vol. 88(6), 1103–1111, 2015.
DOI: [10.1080/00207179.2014.994037]

Characterization of the Trajectories Space of a Class of Second Order Nonlinear Systems: The Linear plus Integral Controller-Based Buck-Boost Power Converter Case of Study

Jaume Llibre, Mario Spinetti-Rivera and Eliezer Colina-Morles.

Abstract The behaviour of the Buck-Boost power converter with a linear plus integral controller, operating to regulate the output voltage, is analyzed. The study is performed by means of the qualitative theory of differential equations. The restrictions imposed by the parameters of the controller help to depict a map of all trajectories of the controlled power converter in terms of a restriction parameter ε, which allows to obtain conclusions for the 3 dimensional differential system in terms of a 2 dimensional differential system.

Keywords Power converters · nonlinear systems · control theory · Poincaré compactification.

1 Introduction

The Buck-Boost power converter corresponds to a circuital configuration whose output gain is higher or lower than the power supply and includes a polarity change, hence its name of inverter converter. Usually, a proper selection of a control law forces the system to become semi globally stable in closed loop, see [5]. The variation and perturbations on the parameter included in the control law have a significant disadvantage upon the system dynamics, because it may

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displace the equilibrium point. In order to eliminate the offset an integral term is added to the control law. As a consequence, analyzing the stability of the system using Lyapunov functions is a very tough problem. This paper is about the space of trajectories (or solutions) of the closed loop system. The tools used are the Routh–Hurwitz criterion, the Poincaré compactification method and the theorem of continuous dependence on initial conditions and parameters in [3]. The first tool allows describing the behaviour of the finite equilibrium points. The second tool is to analyze the infinite equilibrium points, and the third tool helps describing part of the space of trajectories between finite and infinite equilibrium points. It is worth mentioning that, because of its complexity, it is not possible to perform a global phase portrait analysis in $\mathbb{R}^3$. Therefore, the control law includes a restriction that allows to describe the trajectories where the power converter operates and, at the same time, facilitates the use of the Poincaré compactification method as if the closed loop system were defined in $\mathbb{R}^2$. The results are not conclusive when the restriction is not enforced and the present analysis is done when the power converter is operating in continuous mode.

For a practical approach, this analysis may be redone taking into consideration the particular characteristics of a real system, using for instance the Joule model presented in [7]; however, the use of a pulse width modulator (PWM) to drive the Buck-Boost power converter makes the analysis harder, because the proposed technique only allows to address systems of polynomial nonlinear differential equations.

2 Buck-Boost power converter dynamics

Figure 1 depicts a circuital scheme of the *Buck-Boost power converter*. Here, the output voltage v increases or decreases with respect to the power supply E, and has an inverted polarity. This is, if $E > 0$, then $v < 0$.

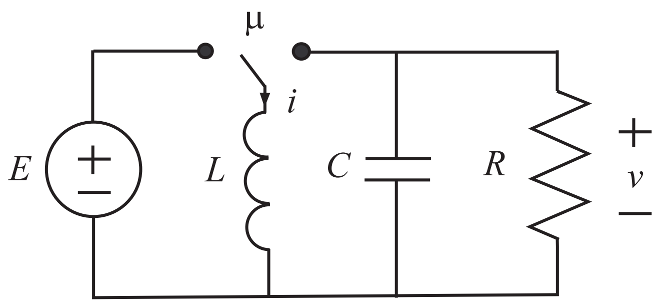

Fig. 1 The Buck–Boost power converter.

An average model of the Buck-Boost power converter may be readily obtained using Kirchhoff and Ohm laws. This model is as follows:

$$L\frac{di}{dt} = (1-\mu)\,E + \mu\,v, \quad C\frac{dv}{dt} = -\mu\,i - \frac{v}{R}, \tag{1}$$

where the variable i represents the inductor current, v is the capacitor voltage or output voltage, and μ is the control action. The parameters are the inductance L, the capacitance C, the power supply E and the load resistor R.

Let T be a nonsingular transformation defined as

$$T = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{E}\sqrt{\frac{L}{C}} & 0 \\ 0 & \frac{1}{E} \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix}, \quad \tau = \frac{t}{\sqrt{LC}}. \tag{2}$$

It is possible to write a normalized system by using the linear transformation (2) in the differential system (1). This is

$$\frac{dx}{d\tau} = (1-\mu) + \mu\,y, \quad \frac{dy}{d\tau} = -\mu\,x - \frac{y}{Q}, \tag{3}$$

where $Q = R\sqrt{C/L}$, see [5].

Based on the exact error dynamics passive output feedback, the control action may be represented as

$$\mu = \bar{\mu} - \gamma\left(-(1-V_d)(x-\bar{x}) + V_d(1-V_d)\frac{(y-\bar{y})}{Q}\right), \tag{4}$$

where $\bar{\mu} = \dfrac{1}{1-V_d}$, $\bar{x} = -V_d\dfrac{(1-V_d)}{Q}$, $\bar{y} = V_d$ (for more details see [5] and [6]).

The construction characteristic V_d is negative ($V_d < 0$). Note that the parameter Q appears both in (3) and (4). When the controller for the Buck-Boost power converter is implemented, the parameter Q in equation (3) may change because it corresponds to an external resistor. In this case the finite equilibrium point is moved and generates an offset level. To eliminate the offset level and return the finite equilibrium point to its original coordinates, an integral output error term is added to the control law, which becomes

$$\mu = \frac{1}{1-\bar{y}} - \gamma\left((1-\bar{y})(y-\bar{y})\frac{\bar{y}}{Q} - (1-\bar{y})\left(x + (1-\bar{y})\frac{\bar{y}}{Q}\right)\right) + \beta\int (y-\bar{y})\,d\tau. \tag{5}$$

Note that the notation might be abused because the normalized system is function on τ instead of t.

With the control action (5), and writing $\beta = \varepsilon b$, $\gamma = \varepsilon d$, the normalized Buck-Boost power converter (3) becomes the differential system

$$\begin{aligned}
\dot{x} &= -\frac{\left(Q + (1-\bar{y})^2 \varepsilon d\right)\bar{y}}{Q\,(1-\bar{y})} - (1-\bar{y})\,\varepsilon d x + \frac{Q + 2(1-\bar{y})^2 \bar{y}\varepsilon d}{Q\,(1-\bar{y})} y \\
&\quad -\varepsilon b z + (1-\bar{y})\,\varepsilon d x y + \varepsilon b y z - \frac{(1-\bar{y})\,\bar{y}\varepsilon d}{Q} y^2, \\
\dot{y} &= -\frac{Q + (1-\bar{y})^2 \bar{y}\varepsilon d}{Q\,(1-\bar{y})} x - \frac{1}{Q} y + \frac{(1-\bar{y})\,\bar{y}\varepsilon d}{Q} x y - \varepsilon b x z \\
&\quad - (1-\bar{y})\,\varepsilon d x^2, \\
\dot{z} &= y - \bar{y}.
\end{aligned} \tag{6}$$

Furthermore, when $\varepsilon = 0$, the differential system (6) becomes

$$\begin{aligned}
\dot{x} &= -\frac{\bar{y}}{1-\bar{y}} + \frac{1}{1-\bar{y}} y, \\
\dot{y} &= -\frac{1}{1-\bar{y}} x - \frac{1}{Q} y, \\
\dot{z} &= y - \bar{y}.
\end{aligned} \tag{7}$$

2.1 Finite equilibrium points of systems (6) and (7)

Proposition 1 *The differential system* (7) *has a straight line of finite equilibrium points. The system has a family of invariant parallel planes under its flow. Every plane contains one of these equilibrium points. On every invariant plane the equilibrium point is a global attractor. (The second column of table 1 contains the finite equilibrium points of the system, projected over the Poincaré disc, depicted in figure 6 of the appendix).*

Region	Finite equilibrium point p	Infinite Equilibria in $\mathbb{S}^1$
$0 < Q < \frac{1-\bar{y}}{2}$	Stable Node	2 Saddles 2 Repeller Nodes
$Q = \frac{1-\bar{y}}{2}$	Stable Node	2 Saddle-Nodes
$Q > \frac{1-\bar{y}}{2}$	Stable Focus	$\emptyset$ (Periodic Orbit in ∞)

Table 1 Equilibrium points in the Poincaré disc of system (7) restricted to the invariant planes $Z = Z_0$.

Proof The finite equilibrium points of system (7) filled the straight line $p_z = p_z(x,y,z) = \left(-\dfrac{(1-\bar{y})\bar{y}}{Q}, \bar{y}, z\right)$, for all $z \in \mathbb{R}$. The Jacobian matrix at the point p_z is given by

$$\begin{pmatrix} 0 & \dfrac{1}{1-\bar{y}} & 0 \\ -\dfrac{1}{1-\bar{y}} & -\dfrac{1}{Q} & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$p(\lambda) = -\lambda^3 - \frac{\lambda^2}{Q} - \frac{\lambda}{(-1+\bar{y})^2}.$$

The roots of the polynomial $p(\lambda)$ are

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= -\frac{1}{2Q} - \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q(1-\bar{y})}, \\ \lambda_3 &= -\frac{1}{2Q} + \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q(1-\bar{y})}. \end{aligned}$$

In order to study the roots λ_2 and λ_3 we define the quantity $\Delta = (1-\bar{y})^2 - 4Q^2$ and use Theorem 2.15 of [2]. Then there are three possibilities:

1. If $0 < Q < \dfrac{1-\bar{y}}{2}$ in each invariant plane there is a stable node.
2. If $Q = \dfrac{1-\bar{y}}{2}$ in each invariant plane there is a stable node with equal values, which is not diagonalizable as it will be show later on.
3. If $Q > \dfrac{1-\bar{y}}{2}$ in each invariant plane there is a stable focus.

Since system (7) restricted to every invariant plane is a linear differential system in $\mathbb{R}^2$, then the basin of attraction of the equilibrium p_z contained in this plane is the full plane. Next, the dynamics on each one of these invariant planes, plus the behaviour at infinity will be described. For this purpose, the equilibrium point p_z of the differential system (7) will be moved to the origin of coordinates using the change of variables $(x,y,z) \rightarrow \left(x_1 - \dfrac{(1-\bar{y})\bar{y}}{Q}, y_1 + \bar{y}, z_1\right)$. Thus we get the following differential system:

$$\begin{aligned} \dot{x}_1 &= \frac{y_1}{1-\bar{y}}, \\ \dot{y}_1 &= -\frac{1}{1-\bar{y}} x_1 - \frac{1}{Q} y_1, \\ \dot{z}_1 &= y_1, \end{aligned} \tag{8}$$

whose equilibrium point is located at $(0, 0, 0)$.

The differential system (8) will be written in its real Jordan normal form for each of the range of values that Q may attain. This is:

1. When $0 < Q < \dfrac{1-\bar{y}}{2}$ doing the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \dfrac{1-\bar{y}-\sqrt{(1-\bar{y})^2-4Q^2}}{2Q} & 1 & 0 \\ \dfrac{1-\bar{y}-\sqrt{(1-\bar{y})^2-4Q^2}}{2Q} & 1 & 0 \\ -(1-\bar{y}) & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

system (8) may be written as

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2Q} - \dfrac{\sqrt{(1-\bar{y})^2-4Q^2}}{2Q\,(1-\bar{y})} & 0 & 0 \\ 0 & -\dfrac{1}{2Q} + \dfrac{\sqrt{(1-\bar{y})^2-4Q^2}}{2Q\,(1-\bar{y})} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{9}$$

In other words, $\dot{Z} = 0$ and therefore, all planes $Z = Z_0$ are invariant with respect to the flow. Thus the differential (9) has a global stable node in each invariant plane $Z = Z_0$.

2. When $Q = \dfrac{1-\bar{y}}{2}$ doing the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1-\bar{y} & 0 & 0 \\ 1 & 1 & 0 \\ -(1-\bar{y}) & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

system (8) may be written as

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2Q} & 1 & 0 \\ 0 & -\dfrac{1}{2Q} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{10}$$

Again $\dot{Z} = 0$, and therefore all planes $Z = Z_0$ are invariant with respect to the flow. Thus the differential system (10) has a global stable node with equal eigenvalues, but is not diagonalizable in each invariant plane $Z = Z_0$.

3. When $Q > \dfrac{1-\bar{y}}{2}$ using the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \dfrac{1-\bar{y}}{\sqrt{4Q^2-(1-\bar{y})^2}} & \dfrac{2Q}{\sqrt{4Q^2-(1-\bar{y})^2}} & 0 \\ 1 & 0 & 0 \\ -(1-\bar{y}) & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

system (8) may be written as

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2Q} & \dfrac{\sqrt{4Q^2-(1-\bar{y})^2}}{2Q\,(1-\bar{y})} & 0 \\ -\dfrac{\sqrt{4Q^2-(1-\bar{y})^2}}{2Q\,(1-\bar{y})} & -\dfrac{1}{2Q} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{11}$$

and $\dot{Z} = 0$, therefore all planes $Z = Z_0$ are invariant with respect to the flow. Thus, the differential system (11) has a global stable focus in each invariant plane $Z = Z_0$.

Proposition 2 *For $\varepsilon = 0$ there is a straight line filled of equilibrium points for system* (6)*. For $\varepsilon \neq 0$ this straight line disappears and there appears a unique finite equilibrium point p. If ε is sufficiently small and $b\varepsilon < 0$, then the equilibrium point p is locally stable.*

Proof The finite equilibrium points for system (6) are given by $p = \left(-\dfrac{(1-\bar{y})\,\bar{y}}{Q},\right.$ $\bar{y}, 0)$. A way to study its local phase portrait is by computing its Jacobian matrix at p which is

$$\begin{pmatrix} -d(1-\bar{y})^2\varepsilon & -\dfrac{d(1-\bar{y})^2\bar{y}\varepsilon - Q}{Q\,(1-\bar{y})} & -b(1-\bar{y})\varepsilon \\ \dfrac{d(1-\bar{y})^3\bar{y}\varepsilon + Q}{Q\,(1-\bar{y})} & -\dfrac{d(1-\bar{y})^2\bar{y}^2\varepsilon + Q}{Q^2} & \dfrac{b\,(1-\bar{y})\,\bar{y}\varepsilon}{Q} \\ 0 & 1 & 0 \end{pmatrix}.$$

The associated characteristic polynomial $p(\lambda)$ is

$$\lambda^3 + \frac{Q + d(1-\bar{y})^2\left(Q^2+\bar{y}^2\right)\varepsilon}{Q^2}\lambda^2 + \frac{Q-(1-\bar{y})^3\left(b\bar{y} - d\,(1-\bar{y})\right)\varepsilon}{Q(1-\bar{y})^2}\lambda - b\varepsilon.$$

By virtue of the Routh-Hurwitz stability criterion, the following conditions must be satisfied:

$$b\varepsilon < 0,$$
$$s_1 = Q + d(1-\bar{y})^2 \left(Q^2 + \bar{y}^2\right)\varepsilon > 0,$$
$$s_2 = Q - (1-\bar{y})^3 \left(b\bar{y} - d\,(1-\bar{y})\right)\varepsilon > 0,$$
$$s_3 = b\varepsilon + \frac{\left(Q + d(1-\bar{y})^2 \left(Q^2+\bar{y}^2\right)\varepsilon\right)\left(Q + \left(Q^2+\bar{y}^2\right)^3 \left(\bar{y}\,(b+d)\right)\varepsilon - d\right)}{Q^3(1-\bar{y})^2} > 0.$$

It is easy to check that

$$\lim_{\varepsilon\to 0} s_1 = Q, \quad \lim_{\varepsilon\to 0} s_2 = Q, \quad \lim_{\varepsilon\to 0} s_3 = \frac{1}{Q(1-\bar{y})^2},$$

when $\varepsilon \to 0$. Therefore if ε is sufficiently small and $b\varepsilon < 0$, then the finite equilibrium point p is locally stable.

2.2 The neighborhood of infinity

The next step is to describe the complete behaviour of the polynomial differential system (3) at infinity, when the control action (5) is enforced, with $\beta = b\varepsilon$ and $\gamma = d\varepsilon$. Then two of the three eigenvalues are equal to cero at the infinity equilibrium point, this fact makes very complex to describe the dynamics near the infinity of $\mathbb{R}^3$. An alternative and easier approach is resorting to use the Poincaré compactification method on $\mathbb{R}^2$ for describing the global phase portrait of systems (9), (10) and (11) at infinity and near infinity. Using the notation given in appendix, the degree of the polynomial components for systems (9), (10) and (11) is one ($m = 1$).

Proposition 3 *The local phase portraits at the infinite equilibrium points on the Poincaré disc, corresponding to an invariant plane of systems* (9)*,* (10) *and* (11) *are shown in the third column of table 1.*

Proof There are three cases

1. Assume $0 < Q < \dfrac{1-\bar{y}}{2}$. Then in the local chart U_1 the coordinates (u, v) are defined by $x = 1/v$ and $y = u/v$. Using equation (15) from the appendix, the differential system (9) may be rewritten as

$$\dot{u} = \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{Q\,(1-\bar{y})}u, \quad \dot{v} = \left(\frac{1}{2Q} - \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q\,(1-\bar{y})}\right)v.$$

In the chart U_1 the infinite equilibrium point corresponds to $(0, 0)$. Now, since the eigenvalues are

$$\lambda_1 = \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q\,(1-\bar{y})} > 0, \quad \lambda_2 = \frac{1}{2Q} + \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{Q\,(1-\bar{y})} > 0,$$

then the origin $(0,0)$ of U_1 is a repeller node.

In the local chart U_2 the coordinates (u,v) are defined by $x = u/v$ and $y = 1/v$. Using equation (16) from the appendix, the differential system (9) may be written as

$$\dot{u} = -\frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{Q\,(1-\bar{y})}u, \quad \dot{v} = \left(\frac{1}{2Q} - \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q\,(1-\bar{y})}\right)v.$$

In the local chart U_2 the infinite equilibrium point is $(0,0)$. Since the eigenvalues are

$$\lambda_1 = \frac{1}{2Q} - \frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{2Q\,(1-\bar{y})} > 0, \quad \lambda_2 = -\frac{\sqrt{(1-\bar{y})^2 - 4Q^2}}{Q\,(1-\bar{y})} < 0,$$

then the origin $(0,0)$ of U_2 is a saddle point.

For the local charts V_1 and V_2 it is found that the origin of V_1 is a repeller node, and the origin of V_2 is a saddle point.

2. Assume $Q = \dfrac{1-\bar{y}}{2}$. In the local chart U_1 the differential system (9) becomes

$$\dot{u} = -u^2, \quad \dot{v} = \frac{1}{2Q}v - uv.$$

In the chart U_1 the infinite equilibrium point corresponds to $(0,0)$. The eigenvalues are $\lambda_1 = \dfrac{1}{2Q}$ and $\lambda_2 = 0$. So the origin is a semi–hyperbolic equilibrium point, which is analyzed using the Theorem 2.19 of [2], to conclude that the origin of U_1 is a saddle–node.

In the local chart U_2 the differential system (9) may be expressed as

$$\dot{u} = 1, \quad \dot{v} = \frac{1}{2Q}v.$$

Hence in the chart U_2 does not exist infinite equilibrium point.

3. Finally suppose $Q > \dfrac{1-\bar{y}}{2}$. In the local chart U_1 the differential system (9) writes

$$\begin{aligned}\dot{u} &= -\frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})} - \frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})}u^2,\\ \dot{v} &= \frac{1}{2Q}v - \frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})}uv.\end{aligned}$$

Hence in the local chart U_1 there are no infinite equilibrium points.

In the local chart U_2 the differential system (9) may be expressed as

$$\dot{u} = -\frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})} + \frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})}u^2,$$
$$\dot{v} = \frac{1}{2Q}v + \frac{\sqrt{4Q^2 - (1-\bar{y})^2}}{2Q\,(1-\bar{y})}uv.$$

Again in the local chart U_2 the are no infinite equilibrium points.

Theorem 1 *The global phase portraits in the Poincaré disc, corresponding to the invariant planes $Z = Z_0$ of system (7), with $\varepsilon = 0$ are described in Figure 2.*

Proof It follows immediately from Propositions 1 and 3, and the fact that system (7) with $\varepsilon = 0$ is a linear differential system.

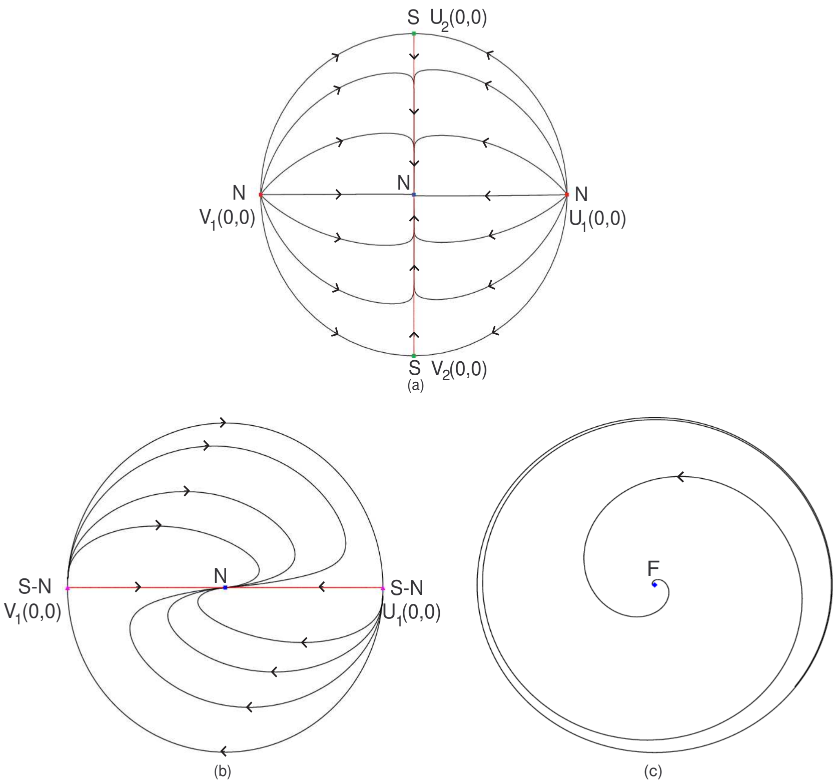

Fig. 2 The three global phase portraits in the Poincaré disc of system (7), corresponding to the invariant plane $Z = Z_0$ of systems (9), (11) and (10), respectively.

Three comments on the results of Figure 2.

(i) If $0 < Q < \dfrac{1-\bar{y}}{2}$, then there exists a global stable node p in $\mathbb{R}^2$ (the interior of the Poincarè disc). The infinite equilibrium points are composed by a pair of diametrically opposed repeller nodes and a pair of diametrically opposed saddles, corresponding to the origins of the charts U_1, U_2, V_1 and V_2, respectively. See Figure 2(a).

(ii) If $Q = \dfrac{1-\bar{y}}{2}$, then there exists a global stable node in $\mathbb{R}^2$, and at infinity there is a pair of diametrically opposed saddle–nodes. These two saddle–nodes correspond to the origins of U_1 and V_1, respectively. See Figure 2(b).

(iii) If $Q > \dfrac{1-\bar{y}}{2}$, then there exists a global stable focus in $\mathbb{R}^2$. Since there is no infinite equilibrium point and the infinity $\mathbb{S}^1$ of the Poincaré disc is invariant, it follows that $\mathbb{S}^1$ is a periodic orbit. See Figure 2(c).

In summary, the results of Theorem 1 show that any solution of system (7) with $\varepsilon = 0$, defined in a space $\mathbb{R}^3$, tends in forward time to a unique finite equilibrium point belonging to the straight line filled of equilibrium points.

The Theorem of Continuous Dependence on Initial Conditions and Parameters, in [3], states that any solution γ of system (6), with ε sufficiently small and with initial conditions (x_0, y_0, z_0), follows as close as it is desired the solution γ_0 with the same initial conditions of (7), for $\varepsilon = 0$. Consequently, the solution γ turns out close to the curve of equilibrium points which exist for $\varepsilon = 0$; but for ε sufficiently small, such curve does not exist and only exists the stable equilibrium p if $b\varepsilon < 0$. If γ enters in the basin of attraction of the equilibrium point p, then γ tends to this equilibrium point. Otherwise, it is not known where the orbit γ tends. It might be the infinity, or some other invariant object created by a bifurcation, which appears when the parameter ε goes away from zero.

3 Example

As an example, it is taken $\bar{y} = V_d = -1$ and $Q = \{1/2, 1, 2\}$, which are used to represent the three regions $0 < Q < \dfrac{1-\bar{y}}{2}$, $Q = \dfrac{1-\bar{y}}{2}$ and $Q > \dfrac{1-\bar{y}}{2}$. For each value of Q there exists a straight line of finite equilibrium points p_z of system (7) with $\varepsilon = 0$. Some of these straight lines of equilibrium points are shown in Figure 3.

An analysis of the three regions for different values of Q gives the following results:

(i) With $Q = 1/2$ the differential equations system (9) is transformed into

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -1.866.. & 0 & 0 \\ 0 & -0.1339.. & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{12}$$

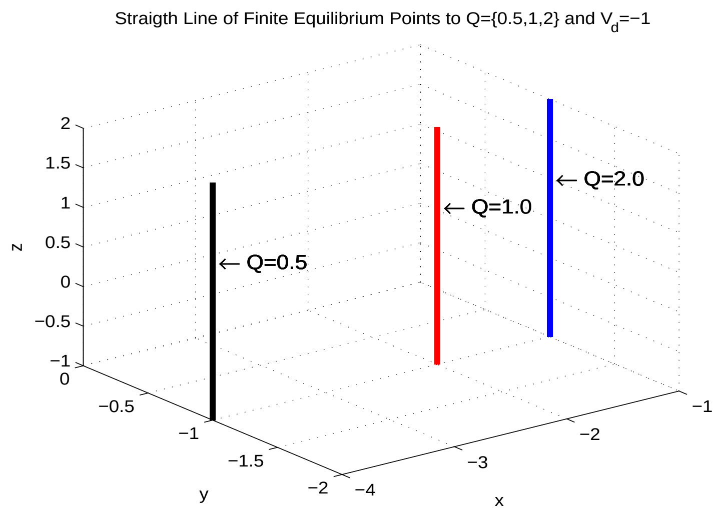

Fig. 3 Finite equilibrium points of system (6) with $\varepsilon = 0$.

Figure 4(a) shows the behaviour near the finite node in the plane X, Y of the differential system (12).

(ii) With $Q = 1$ the differential equations system (10) is transformed into

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -1/2 & 1 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{13}$$

Figure 4(b) shows the behaviour near the finite node, not diagonalizable, in the plane X, Y of the differential system (13).

(iii) With $Q = 2$ the differential equations system (11) is transformed into

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -1/4 & \sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & -1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{14}$$

Figure 4(c) shows the behaviour near the finite focus in the plane X, Y of the differential system (14).

All these phase portrais are in accordance with Proposition 1.

If it is taken $\bar{y} = V_d = -1$, $Q = \{1/2, 1, 2\}$ and the differential system (6) the following equilibrium points are obtained $p_{0.5} = (4, -1, 0)$, $p_1 = (2, -1, 0)$ and $p_2 = (1, -1, 0)$, respectively, where now the subindex indicates the value of the parameter Q. With $\varepsilon = 0.01$ and $b = -1$, it results in $\varepsilon b = -0.01 < 0$. Thus, for any value of d, for example $d = 1$, the eigenvalues of the differential system (6) for each of the mentioned values of Q, are calculated, and the results

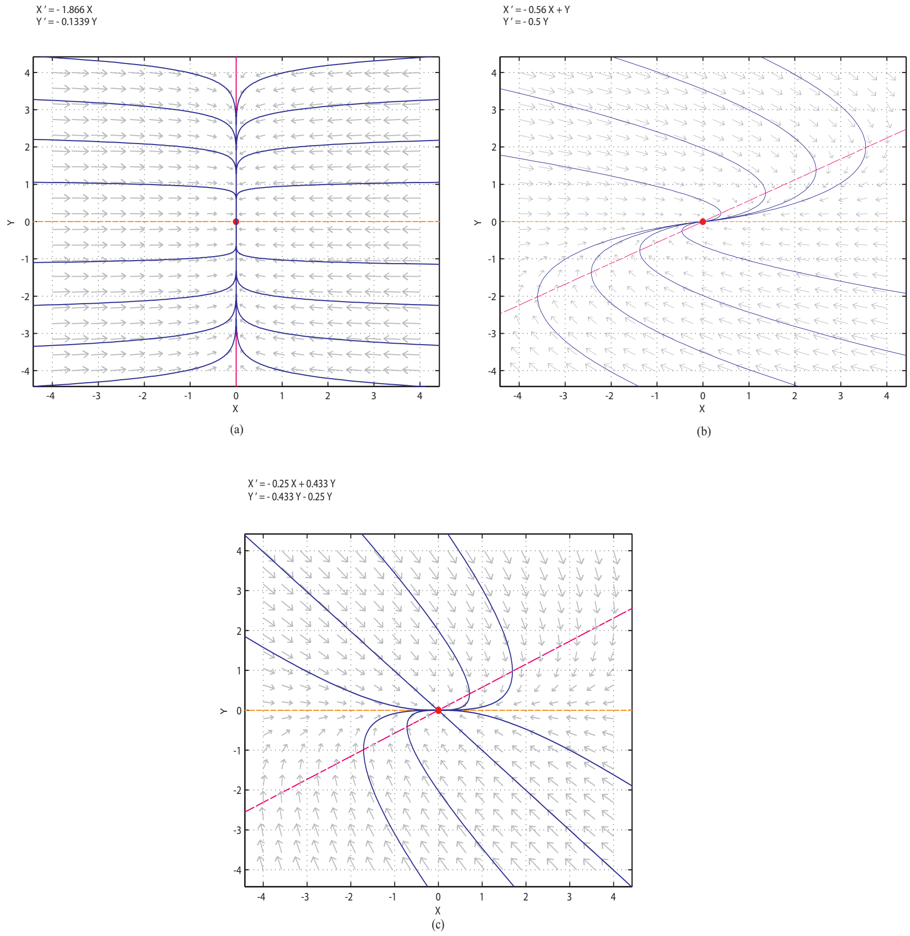

Fig. 4 Phase portraits near the finite equilibrium point of systems (12), (13) and (14), respectively.

are summarized in Table 2. These results are in concordance with Proposition 2.

Note that when the restriction $\varepsilon = 0$ is imposed; the differential system (6) is transformed into the linear differential system (7). When $\varepsilon \neq 0$; the differential system (6) becomes nonlinear and the equilibrium point depends on the parameters of the system. By choosing these parameters using the Routh–Hurwitz criterion, it is possible to assure the local stability of the finite equilibrium point in accordance with Proposition 2.

Now it has been set that for the infinite equilibrium points of the differential systems (12), (13) and (14) correspond the three phase portraits that system (7) may exhibit.

Analyzing the infinity of differential systems (12), (13) and (14) yields:

Q	Eigenvalues	Equilibrium point	Coordinates p_Q
1/2	$\lambda_1 = -2.06169..,$ $\lambda_{2,3} = -0.0691543.. \pm 0.00825049..\,\mathrm{i}$	Stable Focus-Node	$(4, -1, 0)$
1	$\lambda_1 = -0.72798.., \lambda_2 = -0.307322..$ $\lambda_3 = -0.0446979..$	Stable Node	$(2, -1, 0)$
2	$\lambda_{1,2} = -0.254055.. \pm 0.417338..\,\mathrm{i},$ $\lambda_3 = -0.041891..$	Stable Focus-Node	$(1, -1, 0)$

Table 2 Finite equilibrium point of the differential system (5) with $\bar{y} = -1$, $\varepsilon = 0.01$, $b = -1$ and $d = 1$ for the values of Q in $\{1/2, 1, 2\}$.

(i) In the local chart U_1 system (12) becomes

$$\dot{u} = 1.73205..u, \quad \dot{v} = 1.86603..v.$$

Then the infinite equilibrium point is the origin of U_1 and its eigenvalues are $\lambda_1 = 1.73205, \lambda_2 = 1.86603$. So it is an unstable node.

On the other hand, in the local chart U_2 the system (12) corresponds to

$$\dot{u} = -1.73205..u, \quad \dot{v} = 0.133975..v.$$

The infinite equilibrium point is the origin of U_2 and its eigenvalues are $\lambda_1 = 0.133975.., \lambda_2 = -1.73205..$, which is a saddle.

In the local charts V_1 and V_2 it is found the same behaviour, (i.e. the origin of V_1 is an unstable node and the origin of V_2 is a saddle.

(ii) In the local chart U_1 system (13) becomes

$$\dot{u} = -u^2, \quad \dot{v} = 0.5v - uv.$$

Therefore, the infinite equilibrium point is the origin of U_1 and its eigenvalues are $\lambda_1 = 0.5, \lambda_2 = 0$. This is a semi–hyperbolic infinite equilibrium point, which corresponds to a saddle–node, (see Theorem 2.19 of [2]). Therefore the origin of the local chart V_1 is also a saddle–node.

In the local chart U_2 the system has no infinite equilibrium points.

(iii) System (14) has no infinite equilibrium points in the local charts U_1 and U_2; so there is a periodic orbit at infinity.

All the above results are in concordance with Proposition 3.

Finally Theorem 1 describes the global phase portraits of system (7) in its invariant planes using the Poincaré disc. This is shown in Figure 2.

With $V_d = -1, Q = 0.5, b = -1, d = 1$, four solutions δ_1 and δ_2 of the differential system (6), and ζ_1 and ζ_2 of the differential system (7) are simulated. The solutions δ_1 and ζ_1 have the same initial conditions $ic_{\delta_1,\zeta_1} = (0, 0, 0)$. The solutions δ_2 and ζ_2 also have the same initial conditions $ic_{\delta_2,\zeta_2} = (1, -1, 0)$. The differential system (6) has a unique finite equilibrium point $p_{\delta_1,\delta_2} = (4, -1, 0)$, as it is stated in Proposition 2. The differential system (7) has one straight line of equilibrium points, (i.e. the two equilibrium points $p_{\zeta_1} = (4, -1, 8)$ and $p_{\zeta_2} = (4, -1, 6)$, see Proposition 1. As stated by the Theorem of Continuous Dependence on Initial Conditions and Parameters, both solutions δ_1 and ζ_1, with same initial condition, are closed to one another until they are near to the position of the stable node p_{ζ_1}. Similar behaviour can be observed for the solutions δ_2 and ζ_2. All information contained in this paragraph is summarized in Figure 5.

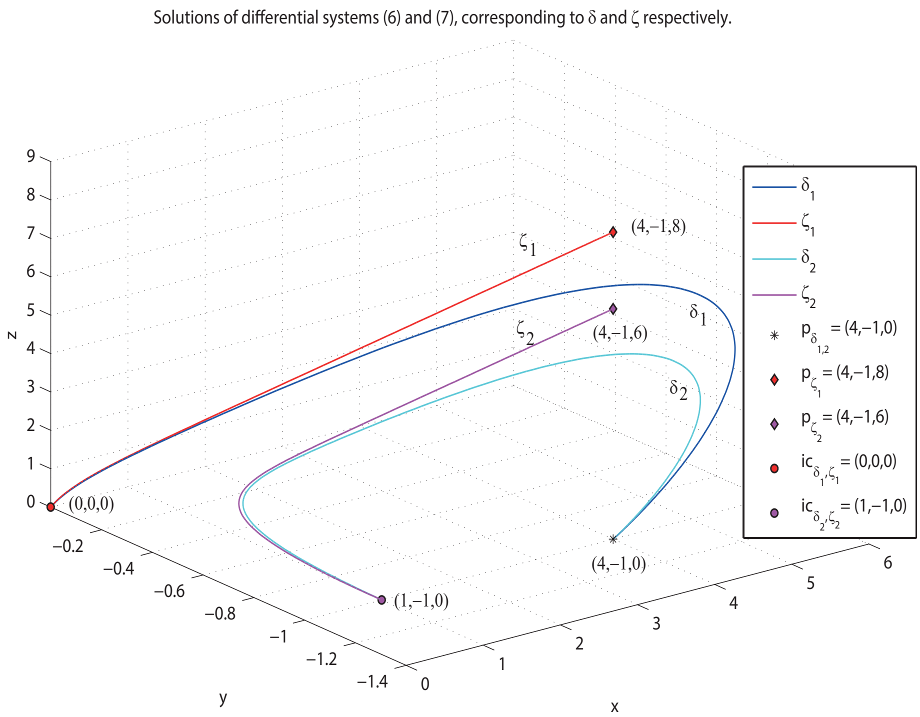

Fig. 5 Some solutions of the differential systems (6) and (7).

4 Comments and Perspectives concerning the Operation of Buck-Boost Converters

When it is intended to design a controller, the basic strategy may involve obtaining a globally stable closed loop system in the Lyapunov sense. In some cases this strategy is too much complicated, if not impossible to obtain. In the example considered in this paper, a control law which leads to a semi-globally stable system is used, (see (1)). This control law depends on the parameters R,

L and C of the Buck-Boost converter; which is a disadvantage as variations of these parameters produce a displacement of the equilibrium point. Normally, this technique includes an integral term to eliminate the offset and return the finite equilibrium point to the corresponding coordinates (see (2)). When the control law with the integral term is included, the resulting closed loop system is a nonlinear one (see (4)), and the stability in the Lyapunov sense is not guaranteed.

A question to deal with is related to the possibility to operate the controlled system in a globally or semi globally stable behaviour. This question has no definite answer, yet a proper answer is connected to a study of the local stability, which requires knowing the basin of attraction of the finite equilibrium point. A new approach proposed here, is in terms of describing the global phase portrait of the controlled differential system for some values of its parameters. This global phase portrait includes the infinity equilibrium points, using the Poincaré compactification theorem for describing the behaviour of the system solutions near the infinity. However, it is not possible to describe the dynamics of the Poincaré compactification in $\mathbb{R}^3$ because there may appear equilibrium points at infinity with two eigenvalues equal to zero.

To obtain a semi–global solution of this problem, the following three tools are applied:

(1) Use the Routh–Hurwitz criterion for local stability analysis of the finite equilibrium point (see Proposition 2).

(2) For a set of values of the parameters, some of them very small, the controlled differential system becomes a set of linear differential systems. For these linear differential systems, it is possible to describe the global phase portrait, including the infinity, because they have a family of invariant planes filling the whole space $\mathbb{R}^3$. Use the Poincaré compactification in $\mathbb{R}^2$ for each invariant plane in order to studying the behavior near the infinity, including the infinite equilibrium points (see Proposition 3 and the appendix).

(3) The Theorem of Continuous Dependence on Initial Conditions and Parameters [3] allows to explain partially the behavior of the solutions lying between the infinity and the finite equilibrium point.

With these tools it is possible to have a broader understanding of the basin of attraction of the finite equilibrium point, where it is easier to proceed with the controller design. The term semi–globally stable is used because it has been shown that the basin of attraction extends near infinity, which is a new piece of information. The practical experience gained in the implementation of the Buck-Boost controller suggests that for small values of parameters, the converter operates properly.

5 Conclusions

The approach proposed in this paper has allowed a qualitative analysis of the trajectories associated with a polynomial nonlinear dynamic system. The analysis has included both a local characterization of the equilibrium points; by the method of Routh–Hurwitz, as the specification of the infinite equilibrium points; using the method of compactification of Poincaré.

In order to proceed with the proposed approach, when the infinite equilibrium points are associated with more than two zero eigenvalues; as the case of the Buck-Boost power converter considered in this work, it has been necessary to multiply the controller parameters by an auxiliary value ε, and make a qualitative analysis when $\varepsilon = 0$ as well as when $\varepsilon \to 0$. This has allowed a characterization of a portion of the solution space of the dynamical system, where all the possible behaviors of the trajectories are shown; from a neighborhood of the infinity equilibrium point to the finite equilibrium point.

The work has demonstrated that the Buck-Boost power converter presents a globally stable behaviour, provided that the controller parameters are small enough; i.e. when $\varepsilon = 0$ or when $\varepsilon \to 0$.

6 Appendix

Here there are some basic results on the qualitative theory for two-dimensional systems used in this paper.

6.1 Poincaré Compactification

Consider the polynomial vector field (P, Q) associated to the differential polynomial system

$$\begin{aligned} \dot{x}_1 &= P\left(x_1, x_2\right), \\ \dot{x}_2 &= Q\left(x_1, x_2\right), \end{aligned}$$

of degree $m = \max\{\deg P, \deg Q\}$. To study the Poincaré compactification of (P, Q) on the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ we use six local charts: $U_k = \left\{y \in \mathbb{S}^2 : y_k > 0\right\}$, $V_k = \left\{y \in \mathbb{S}^2 : y_k < 0\right\}$ for $k = 1, 2, 3$.

In the local chart U_1 with coordinates (u, v) given by $x_1 = 1/v$, $x_2 = u/v$ the expression of the Poincaré compactification of (P, Q) is

$$\begin{aligned} \dot{u} &= v^m \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \\ \dot{v} &= -v^{m+1} P\left(\frac{1}{v}, \frac{u}{v}\right). \end{aligned} \tag{15}$$

In the local chart U_2 with coordinates (u, v) given by $x_1 = u/v$, $x_2 = 1/v$ the expression of the Poincaré compactification of (P, Q) is

$$\begin{aligned} \dot{u} &= v^m \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \\ \dot{v} &= -v^{m+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned} \tag{16}$$

The expression of chart U_3 is given by

$$\begin{aligned} \dot{u} &= P(u, v), \\ \dot{v} &= Q(u, v). \end{aligned}$$

Finally, the expression in the local chart V_k is equal to the expression U_k multiplied by $(-1)^{m-1}$ for $k = 1, 2, 3$.

Note that in all the local charts the infinity is given by $v = 0$. Moreover, the construction of the Poincaré compactification forces that the equilibrium points at infinity appear in pairs diametrally opposite, and the local phase portraits in these diametrally opposite equilibria is symmetric with respect to the center of the Poincaré sphere $\mathbb{S}^2$, but the flow in a neighborhood of these equilibria is the same if the maximum degree of the polynomials P and Q is odd, or one of these equilibria has the sense of all the orbits reversed with respect to the other if the degree is even. For more details on the Poincaré compactification see the chapter 5 of [2].

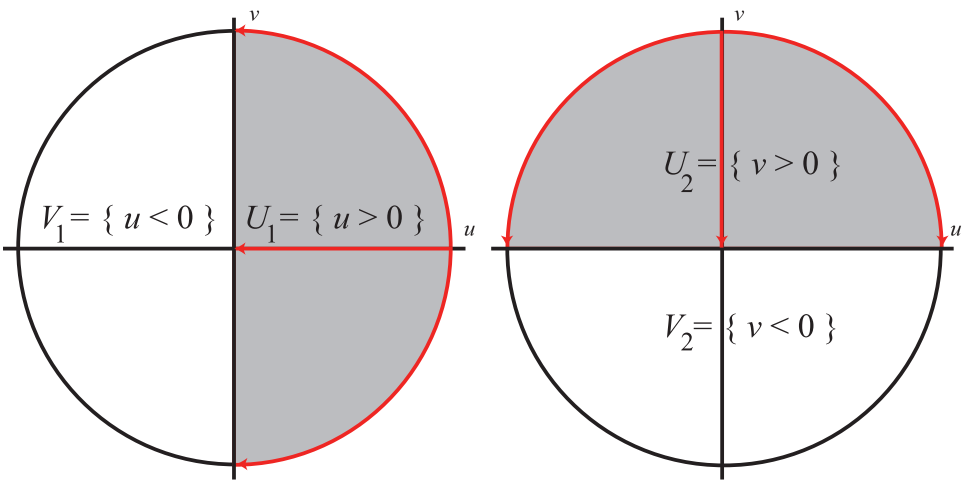

Fig. 6 The local charts U_1, U_2, V_1 and V_2 projected on the Poincaré disc.

Acknowledgements The first author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2013SGR-568, an ICREA

Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, FEDER-UNAB-10-4E-378.

The second author wants to acknowledge his gratitude to D. Biel, E. Fossas and J. Olm for their unconditional support.

The third author is supported by the Prometeo project, sponsored by Secretariat of Higher Education, Science, Technology and Innovation of Ecuador.

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