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ON THE NUMBER OF LIMIT CYCLES IN DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH TWO PIECES SEPARATED BY A STRAIGHT LINE

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Abstract. In this paper we study the maximum number N of limit cycles that can exhibit a planar piecewise linear differential system formed by two pieces separated by a straight line. More precisely, we prove that this maximum number satisfies $2 \leq N \leq 4$ if one of the two linear differential systems has its equilibrium point on the straight line of discontinuity.

1. Introduction and statement of the main result

The study of piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1] and still continues to receive attention by researchers. These last years a renewed interest has appeared in the mathematical community working in differential equations for understanding the dynamical richness of the piecewise linear differential systems, because these systems are widely used to model processes appearing in electronics, mechanics, economy, ..., see for instance the books of di Bernardo, Budd, Champneys and Kowalczyk [3], and Simpson [25], and the survey of Makarenkov and Lamb [23], and the hundreds of references quoted in these last three works.

We recall that a *limit cycle* is a periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system.

The simplest possible continuous but nonsmooth piecewise linear differential systems are the ones having only two pieces separated by a straight line. In 1990 Lum and Chua [22] conjectured that a continuous piecewise linear vector field in the plane with two pieces has at most one limit cycle. We note that even in this apparent simple case, only after a difficult analysis it was possible to prove the existence of at most one limit cycle, thus in 1998 this conjecture was proved by Freire,

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Ponce, Rodrigo and Torres [7]. There are two reasons that difficult the analysis of these differential systems. First, even one can easily integrate the solutions of every linear differential system, the time that an orbit expends in each half–plane governed by each linear differential system is in general unknown, consequently the matching of the corresponding solutions is a difficult problem. Second, the number of parameters to consider in order to be sure that we take into account all possible cases is in general not small. Of course, these difficulties increase when we work with discontinuous piecewise linear differential systems. Recently, a new an easier proof that at most one limit cycle exists for the continuous piecewise linear differential systems with two pieces separated by a straight line has been done by Llibre, Ordoñez and Ponce in [19].

The objective of this paper is to study the problem of Lum and Chua but now for the class of discontinuous piecewise linear differential systems in the plane with two pieces separated by a straight line. In some sense this problem can be seen as an extension of the 16th Hilbert's problem to the discontinuous piecewise linear differential systems in the plane with two pieces separated by a straight line. We recall that the 16th Hilbert's problem essentially ask for the maximum number of limit cycles that a polynomial differential system in the plane can have in function of the degree of the system. For the moment this problem remains open, for more details on the 16th Hilbert's problem see for instance [12, 16, 18].

Several authors tried to determine the maximum number of nested limit cycles surrounding a unique equilibrium point for the class of all discontinuous piecewise linear differential systems with two pieces separated by a straight line. Thus in the paper of Han and Zhang [11] some results about the existence of two limit cycles appeared, so that the authors conjectured that the maximum number of limit cycles for this class of piecewise linear differential systems is exactly two. But Huan and Yang in [13] provided numerical evidence about the existence of three nested limit cycles surrounding a unique equilibrium. Llibre and Ponce in [20] inspired in the numerical example of [13] proved that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three limit cycles. Later on other authors obtained also three limit cycles for those differential systems following different ways, see the papers of Braga and Mello [4], of Buzzi, Pessoa and Torregrosa [5], and of Freire, Ponce and Torres [9].

The linear differential systems that we consider in every half–plane extended to the full plane is either a focus (F) (we include in this class of foci the centers), or a node (N), or a saddle (S). We recall that there are three classes of linear nodes: nodes with different eigenvalues, nodes with equal eigenvalues whose linear part does not diagonalize, and nodes with equal eigenvalues whose linear part diagonalize, called *star nodes*. Clearly if a piecewise linear differential system with two pieces separated by a straight line has a star node, this prevents the existence of periodic orbits.

An equilibrium point p of a linear differential system defined in a half–plane having in the full plane a node, a focus or a saddle is *real* when p belongs to the closure of the half–plane where the system is defined the mentioned linear differential system, and p is called *virtual* otherwise.

We distinguish six classes or types of planar discontinuous piecewise linear differential systems: FF, FN, FS, NN, NS and SS. Inside these classes and *in this paper we only consider limit cycles surrounding a unique equilibrium point or a unique sliding segment*. So we do not consider sliding limit cycles. Now we recall the definitions of sliding segment and non–sliding limit cycle, for more details on these definitions see for instance [10] and [26].

Let $Z = (X, Y)$ be a discontinuous piecewise linear differential vector field with two pieces separated by a straight line Σ, in one piece we have the linear vector field X and in the other the linear vector field Y. Following Filippov [6] we distinguish three open regions in the discontinuity straight line Σ.

1) The *sliding region* Σ^{sl} where the vectors $X(p)$ and $Y(p)$ with $p \in \Sigma$ point inward Σ.
2) The *escaping region* Σ^{e} where the vectors $X(p)$ and $Y(p)$ with $p \in \Sigma$ point outward Σ.
3) The *sewing region* Σ^{s} where the vectors $X(p)$ and $Y(p)$ with $p \in \Sigma$ point to the same direction and are transverse to Σ.

Any segment contained in $\Sigma^e \cup \Sigma^{sl}$ is called a *sliding segment*. Any limit cycle γ of Z such that $\gamma \cap (\Sigma^e \cup \Sigma^{sl}) = \emptyset$ is called a *non–sliding limit cycle*.

Limit cycles of discontinuous piecewise linear differential systems with two pieces separated by a straight line have been studied by many authors, see for instance the articles [2, 8, 9, 11, 13, 14, 15, 20, 21, 24]. Summarizing the results of these articles we have that the maximum number of known limit cycles that one of these systems can exhibit is given in Table 1. In that table the symbol "–" indicates that those cases appear repeated in the table, because for instance the case NF is the same as the case FN, which appears with a 3 in the table. For more details on Table 1 see mainly the references [21] of Llibre, Teixeira and Torregrosa and [9] of Freire, Ponce and Torres.

	F	N	S
F	3	3	3
N	–	2	2
S	–	–	2

TABLE 1. Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with two pieces separated by a straight line.

Our main result is the following.

Theorem 1. *If one of the linear differential systems has its equilibrium point on the straight line of separation, then the maximum number N of limit cycles of discontinuous piecewise linear differential systems with two pieces separated by a straight line satisfies $2 \leq N \leq 4$.*

Theorem 1 is proved in section 4. As far as we know it is the first time that an upper bound for the maximum number of limit cycles is given for a class of discontinuous piecewise linear differential systems with two pieces separated by a straight line.

Under the assumptions of Theorem 1 we must remark that if the linear differential system whose equilibrium point is on the straight line of discontinuity is a saddle or a node, then due to the existence of the invariant straight lines of the saddle and of the node such discontinuous piecewise linear differential systems cannot have periodic solutions, and consequently limit cycles. So we only must prove Theorem 1 when the linear differential system whose equilibrium point is on the straight line of discontinuity is a focus or a center.

There are two main tools in the proof of Theorem 1. The first are the canonical forms of all the possible configurations of the discontinuous piecewise linear differential systems in the plane with two pieces separated by a straight line, see section 2. These canonical forms only depend of five parameters and are due to Freire, Ponce and Torres, see [9]. The second tool are the extended complete Chebyshev systems, see for more details section 3.

2. Canonical forms

We assume without loss of generality that the two pieces in the plane where are defined the discontinuous piecewise linear differential systems are the left and the right half–planes

$$S^- = \{(x,y) \in \mathbb{R}^2 : x < 0\}, \qquad S^+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$

Consequently $x = 0$ is the straight line of separation between the two linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix}, \tag{1}$$

defined for the $(x,y) \in S^-$, and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix}, \tag{2}$$

defined for the $(x,y) \in S^+$. Note that both systems together depend on twelve parameters.

Now we consider the discontinuous piecewise linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2\ell & -1 \\ \ell^2 - \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{3}$$

defined for the $(x,y) \in S^-$, and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2r & -1 \\ r^2 - \beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}, \tag{4}$$

defined for the $(x,y) \in S^+$, where $\alpha, \beta \in \{i, 0, 1\}$. Of course $i = \sqrt{-1}$. Note that both systems together depend on five parameters. We remark that if $\alpha = i$ then the equilibrium point of system (3) has eigenvalues $\ell \pm i$, so it is a focus if $\ell \neq 0$, and a center if $\ell = 0$. If $\alpha = 0$ then system (3) is a node with eigenvalue $\ell \neq 0$ of multiplicity 2 whose linear part does not diagonalize. If $\alpha = 1$ then system (3) is a saddle with eigenvalues $\ell - 1$ and $\ell + 1$ when $|\ell| < 1$, and a node with eigenvalues $\ell - 1$ and $\ell + 1$ whose linear part diagonalize when $|\ell| > 1$.

Let U be an open subset of $\mathbb{R}^2$. We say that the homeomorphism h between U and its image by h is a *topological equivalence* between the discontinuous piecewise linear differential system (1)+(2) and the discontinuous piecewise linear differential system (3)+(4) if h applies orbits of system (1)+(2) contained in U into orbits of system (3)+(4) contained in $h(U)$.

From Propositions 1 and 2 of [9] it follows that there exists a topological equivalence between the phase portrait of the discontinuous piecewise linear differential system (1)+(2) and the phase portrait of the discontinuous piecewise linear differential system (3)+(4) restricted to the orbits that do not have points in common with the sliding set of these systems. Therefore, since we are interested in studying the limit cycles of the system (1)+(2) which do not intersect its sliding set, it will be sufficient to study the limit cycles of the system (3)+(4).

3. Extended Complete Chebyshev systems

The set of functions $\{f_0, f_1, ..., f_n\}$ defined on the interval I forms an *Extended Chebyshev* system on I, if and only if any nontrivial linear combination of these functions has at most n zeros counting their multiplicities and this number is reached.

The set of functions $\{f_0, f_1, ..., f_n\}$ is an *Extended Complete Chebyshev* system or simply an ECT–*system* on I if and only if for $k = 0, 1, \ldots, n$, the subset of functions $\{f_0, f_1, ..., f_k\}$ form an Extended Chebyshev system.

For proving that the set of functions $\{f_0, f_1, ..., f_n\}$ is an ECT–system on I it is sufficient and necessary to show that the Wronskians

$$W(f_0, ..., f_k)(s) = \begin{vmatrix} f_0(s) & f_1(s) & \cdots & f_k(s) \\ f_0'(s) & f_1'(s) & \cdots & f_k'(s) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(s) & f_1^{(k)}(s) & \cdots & f_k^{(k)}(s) \end{vmatrix} \neq 0,$$

on I for $k = 0, 1, \ldots, n$. For more details on ECT–system see the book [17].

4. Proof of Theorem 1

We separate the proof of Theorem 1 in three cases.

Case 1: The discontinuous piecewise linear differential system (3)+(4) has one real focus on the discontinuity straight line and another focus, real or virtual. Then in system (3)+(4) we must take $\alpha = \beta = i$. We note that in this case the sliding segment is $\{(0, y) : 0 < y < b\}$.

The solution of system (3) starting at the point $(x, y) = (0, y_0)$ is

$$
\begin{aligned}
x(t) &= \frac{e^{\ell t}(a\cos t - (y_0 + \ell(a + \ell y_0))\sin t) - a}{\ell^2+1},\\
y(t) &= \frac{e^{\ell t}\left((y_0\ell^2 + 2a\ell + y_0)\cos t - (a(\ell^2-1) + \ell(\ell^2+1)y_0)\sin t\right) - 2a\ell}{\ell^2+1}.
\end{aligned}
\tag{5}
$$

The solution of system (4) starting at the point $(x, y) = (0, y_0)$ is

$$
\begin{aligned}
x(t) &= \frac{e^{rt}(c\cos t - (b(r^2+1) + y_0 + r(c + ry_0))\sin t) - c}{r^2+1},\\
y(t) &= -\frac{1}{r^2+1}\Big(br^2 + 2cr + b + e^{rt}\big((c(r^2-1) + r(r^2+1)(b+y_0))\sin t\\
&\quad - (br^2 + y_0r^2 + 2cr + b + y_0)\cos t\big)\Big).
\end{aligned}
\tag{6}
$$

Let t_- be the finite positive time that an orbit of system (3) expends inside S^- starting at the point $(0, y_0)$ and entering in S^- in forward time, and let $-t_-$ be the finite positive time that an orbit of system (3) expends inside S^- starting at the point $(0, y_0)$ and entering in S^- in backward time. If such orbits do not exist for system (3), then system (3)+(4) cannot have periodic solutions and we are done. So we assume that there are orbits for which the times t_- or $-t_-$ are well defined.

In a similar way let t_+ be the finite positive time that an orbit of system (4) expends inside S^+ starting at the point $(0, y_0)$ and entering in S^+ in forward time, and let $-t_+$ be the finite positive time that an orbit of system (4) expends inside S^+ starting at the point $(0, y_0)$ and entering in S^+ in backward time. Again we assume that there are orbits for which the times t_+ or $-t_+$ are well defined, otherwise the system (3)+(4) cannot have periodic solutions.

Note that if we have a periodic solution of system (3)+(4) for such an orbit the times t_- and t_+ satisfy that $t_-t_+ < 0$.

Assume that system (3)+(4) has a periodic solution and let t_- and t_+ be the times associated to the two pieces of this periodic solution. Then one of the following sets of three equations must be satisfied for a such periodic solution, either

$$x(t_+) = 0, \quad x(-t_-) = 0, \quad y(-t_-) - y(t_+) = 0, \tag{7}$$

or

$$x(-t_+) = 0, \quad x(t_-) = 0, \quad y(t_-) - y(-t_+) = 0. \tag{8}$$

Now we shall assume that equations (7) hold. The proof of Theorem 1, in Case 1 when $c = 0$ and if equations (8) hold, is completely analogous to the proof when equations (7) hold when $c = 0$.

Using (5) and (6) the three equations (7) become

$$\begin{aligned} e_1 = \quad & e^{rt_+}\left(c\cos t_+ - \left(b\left(r^2+1\right) + y_0 + r(c+ry_0)\right)\sin t_+\right) - c = 0, \\ e_2 = \quad & e^{\ell t_-}(a\cos t_- + (y_0 + \ell(a+\ell y_0))\sin t_-) - a = 0, \\ e_3 = \quad & -(1+r^2)\Big(e^{-\ell t_-}\big((2a\ell + y_0 + \ell^2 y_0)\cos t_- \\ & + (a(\ell^2-1) + \ell(\ell^2+1)y_0)\sin t_- - 2a\ell\big)\Big) \\ & +(\ell^2+1)\Big(e^{rt_+}\big(\,(b + br^2 + 2cr + y_0 + r^2 y_0)\cos t_+ - br^2 \\ & - \left(c\left(r^2-1\right) + r\left(r^2+1\right)(b+y_0)\right)\sin t_+ - 2cr - b\big)\Big) = 0. \end{aligned}$$

By assumption one of the two foci must be on the discontinuity line $x = 0$. We assume that the focus of system (4) is on $x = 0$, i.e. we take $c = 0$. We must also study the case when the focus of system (3) is on $x = 0$ (i.e. $a = 0$), because in the expression of system (3)+(4) both foci do not play exactly the same role due to the parameter b. Now we will do the study when the focus of system (4) is on $x = 0$, i.e. $c = 0$. Later on we shall study the case $a = 0$.

Taking $c = 0$ equation $e_1 = 0$ becomes

$$-e^{rt_+}(1+r^2)(y_0+b)\sin t_+ = 0.$$

If $y_0 + b = 0$ and $\sin t_+ \neq 0$, then at most there is one periodic solution, because from this equation $y_0 = -b$, and we are done.

We suppose that $y_0 + b \neq 0$ and $\sin t_+ = 0$, i.e. $t_+ = \pi$. Now solving equation $e_2 = 0$ with respect to the variable y_0 we get

$$y_0 = a\,\frac{e^{-\ell t_-} - \cos t_- - \ell\sin t_-}{(\ell^2+1)\sin t_-}.$$

Therefore the equation $e_3 = 0$ writes

$$\begin{aligned} & -\left(e^{\pi r}+1\right)\left(b\ell^2 - a\ell + b\right)e^{\ell t_-} + a\left(e^{\pi r}-1\right)e^{\ell t_-}\cot t_- \\ & \qquad\qquad + a\csc t_- - ae^{\pi r}e^{2\ell t_-}\csc t_- = 0, \end{aligned} \tag{9}$$

or equivalently

$$\begin{aligned} & a\big(\ell(e^{\pi r}+1)e^{\ell t_-} + (e^{\pi r}-1)e^{\ell t_-}\cot t_- + \csc t_-(1 - e^{\pi r}e^{2\ell t_-})\big) \\ & -b(1+\ell^2)(e^{\pi r}+1)e^{\ell t_-} = 0, \end{aligned} \tag{10}$$

where

$$
\begin{aligned}
f_0(t_-) &= e^{\ell t_-},\\
f_1(t_-) &= e^{\ell t_-}\cot t_-,\\
f_2(t_-) &= \csc t_-,\\
f_3(t_-) &= e^{2\ell t_-}\csc t_-.
\end{aligned}
\tag{11}
$$

Now we claim that the set of functions $\{f_0, f_1, f_2, f_3\}$ is an extended complete Chebyshev system, and consequently the system (7) can have at most 3 solutions. From section 3 this number of solutions is reached if the coefficients of the functions f_0, f_1, f_2, f_3 in (10), as functions of the parameters of the discontinuous piecewise differential system (3)+(4), are functionally independent, which is not the case because the coefficients $a(e^{\pi r} - 1)$, a and $ae^{\pi r}$ are not functionally independent. But since the three coefficients of f_0, f_1, f_2 are functionally independent the maximum number of limit cycles of system (3)+(4) $y_0 + b \neq 0$ and $\sin t_+ = 0$ in Case 1 will be at least 2.

Note that if $y_0 + b = \sin t_+ = 0$, then the system $e_1 = e_2 = e_3 = 0$ eventually can have at most 4 solutions. Hence, once the claim be proved, the proof of Theorem 1 in Case 1 with $c = 0$ will be completed.

Now we prove the claim. For proving that the set of functions $\{f_0, f_1, f_2, f_3\}$ is an ECT–system it is sufficient and necessary to show that the Wronskians $W(f_0, ..., f_k)(t_-)$ are not zero for $k = 0, 1, 2, 3$. Indeed, we have

$$
\begin{aligned}
W(f_0)(t_-) &= \ e^{\ell t_-} \neq 0,\\
W(f_0, f_1)(t_-) &= \ -e^{2\ell t_-}\csc^2 t_- \neq 0,\\
W(f_0, f_1, f_2)(t_-) &= \ -e^{2\ell t_-}(\ell^2 + 1)\csc^3 t_- \neq 0,\\
W(f_0, f_1, f_2, f_3)(t_-) &= \ -2e^{4\ell t_-}\ell(1 + \ell^2)^2\csc^4 t_-.
\end{aligned}
$$

Therefore, if $\ell \neq 0$ (i.e. if the equilibrium point of system (3) is a focus), then the set of functions $\{f_0, f_1, f_2, f_3\}$ is an extended complete Chebyshev system.

Assume now that $\ell = 0$, i.e. the equilibrium point of system (3) is a center. Then equation (9) becomes

$$-b\,(1 + e^{\pi r}) + a\,(e^{\pi r} - 1)\,(\cot t_- - \csc t_-) = 0,$$

or equivalently

$$-b\,(1 + e^{\pi r})\,f_0(t_-) + a\,(e^{\pi r} - 1)\,f_1(t_-) = 0,$$

where

$$f_0(t_-) = 1,$$
$$f_1(t_-) = \cot t_- - \csc t_-.$$

The set of functions $\{f_0, f_1\}$ is an ECT–system because the Wronskians $W(f_0)(t_-)$ and $W(f_0, f_1)(t_-)$ are not zero, because

$$\begin{aligned} W(f_0)(t_-) = &\ 1 \neq 0, \\ W(f_0, f_1)(t_-) = &\ (\cot t_- - \csc t_-) \csc t_- \leq -\frac{1}{2}. \end{aligned}$$

In short this completes the proof of Theorem 1 in the Case 1 when $c = 0$.

Assume $a = 0$. Then the equation $e_2 = 0$ reduces to

$$e^{-\ell t_-} \left(\ell^2 + 1\right) y_0 \sin(t_-) = 0.$$

If $y_0 = 0$, then at most there is one periodic solution, and we are done, but we note that this periodic solution would be a non–sliding limit cycle and consequently we must not take it into account. So we suppose that $y_0 \neq 0$ and $\sin t_- = 0$, i.e. $t_- = \pi$. Now solving equation $e_1 = 0$ with respect to the variable y_0 we get

$$y_0 = -\frac{br^2 + cr + b - c\cot t_+ + ce^{-rt_+}\csc t_+}{r^2 + 1}.$$

Therefore the equation $e_3 = 0$ writes

$$\begin{aligned} -\left(1 + e^{\ell\pi}\right)(br^2 + cr + b)e^{rt_+} + c(1 - e^{\ell\pi})e^{rt_+}\cot t_+ & \\ -c\csc t_+ + ce^{\pi\ell}e^{2rt_+}\csc t_+ &= 0, \end{aligned}$$

or equivalently

$$\begin{aligned} -\left(1 + e^{\ell\pi}\right)(br^2 + cr + b)f_0(t_+) + c(1 - e^{\ell\pi})f_1(t_+) & \\ -cf_2(t_+) + ce^{\pi\ell}f_3(t_+) &= 0, \end{aligned}$$

where

$$\begin{aligned} f_0(t_-) &= e^{rt_+}, \\ f_1(t_-) &= e^{rt_+}\cot t_+, \\ f_2(t_-) &= \csc t_+, \\ f_3(t_-) &= e^{2rt_+}\csc t_+. \end{aligned}$$

These functions f_0, f_1, f_2, f_3 coincide with the functions (11) if we change r by ℓ, and t_+ by t_-. So the rest of the proof of Theorem 1 in Case 1 with $a = 0$ follows as in the Case 1 with $c = 0$. In summary we have proved Theorem 1 in Case 1. Note that here when $y_0 = \sin t_- = 0$ also we have the upper bound of $N \leq 4$.

Case 2: The discontinuous piecewise linear differential system (3)+(4) has one real focus on the discontinuity straight line and one real or virtual node outside the discontinuity straight line with the eigenvalue $\ell \neq 0$ of multiplicity two and whose linear part does not diagonalize. Then in system (3)+(4) we must take $\alpha = 0$ and $\beta = i$. The case $\alpha = i$ and $\beta = 0$ follows in a similar way.

The solution of system (3) starting at the point $(x, y) = (0, y_0)$ is

$$
\begin{aligned}
x(t) &= \frac{e^{\ell t}(a - \ell^2 y_0 t - a\ell t) - a}{\ell^2}, \\
y(t) &= -\frac{2a + e^{\ell t}(a(\ell t - 2) + \ell(\ell t - 1)y_0)}{\ell}.
\end{aligned} \tag{12}
$$

The solution of system (4) starting at the point $(x, y) = (0, y_0)$ is given in (6).

Let t_- and t_+ be the finite positive times defined in a similar way to the Case 1. Again if we have a periodic solution of system (3)+(4) for such an orbit the times t_- and t_+ satisfy that $t_- t_+ < 0$. Suppose that system (3)+(4) has a periodic solution and let t_- and t_+ the times associated to the two pieces of this periodic solution. Then one of the following sets of three equations (7), or (8) must be satisfied.

Now we shall assume that equations (7) hold. Again the proof of Theorem 1 in Case 2 if equations (8) hold is completely analogous to the proof when equations (7) hold.

Using (6), (12), taking into account that $\ell \neq 0$ and that the focus must be on the discontinuity straight line (i.e. $c = 0$) the three equations (7) become

$$
\begin{aligned}
e_1 = &\ -e^{rt_+}(b + y_0)\sin t_+ = 0, \\
e_2 = &\ \ell^2 y_0 t_- + a\ell t_- - ae^{\ell t_-} + a = 0, \\
e_3 = &\ \ell y_0(1 + \ell t_-) + b\ell e^{\ell t_-} + a\ell t_- + 2a(1 - e^{\ell t_-}) \\
&\ -(b + y_0)\ell e^{\ell t_- + rt_+}(\cos t_+ - r\sin t_+) = 0.
\end{aligned}
$$

From equation $e_1 = 0$ if $y_0 + b = 0$ and $\sin t_+ \neq 0$, then at most there is one periodic solution, and we are done. So we suppose that $y_0 + b \neq 0$ and $\sin t_+ = 0$, i.e. $t_+ = \pi$. Now solving equation $e_2 = 0$ with respect to the variable y_0 we get

$$
y_0 = \frac{a\left(e^{\ell t_-} - \ell t_- - 1\right)}{\ell^2 t_-}.
$$

Therefore the equation $e_3 = 0$ writes

$$
a\left(1 - e^{\pi r}\right) + ae^{\pi r}e^{\ell t_-} - ae^{-\ell t_-} - \left(1 + e^{\pi r}\right)\ell(a - b\ell)t_- = 0,
$$

or equivalently

$$a\left(1-e^{\pi r}\right)f_0(t_-)+ae^{\pi r}f_1(t_-)-af_2(t_-)-\left(1+e^{\pi r}\right)\ell(a-b\ell)f_3(t_-)=0,$$

where

$$\begin{aligned}
f_0(t_-)&=1,\\
f_1(t_-)&=e^{\ell t_-},\\
f_2(t_-)&=e^{-\ell t_-},\\
f_3(t_-)&=t_-.
\end{aligned}$$

The set of functions $\{f_0, f_1, f_2, f_3\}$ is an ECT–system because the Wronskians $W(f_0, ..., f_k)(t_-)$ are not zero for $k = 0, 1, 2, 3$. Indeed, we have

$$\begin{aligned}
W(f_0)(t_-)=&\ \ 1\neq 0,\\
W(f_0,f_1)(t_-)=&\ \ \ell e^{\ell t_-}\neq 0,\\
W(f_0,f_1,f_2)(t_-)=&\ \ 2\ell^3\neq 0,\\
W(f_0,f_1,f_2,f_3)(t_-)=&\ \ -2\ell^5.
\end{aligned}$$

Therefore system (7) with $y_0+b\neq 0$ and $\sin t_+=0$ can have at most 3 solutions, and as in the previous cases system (7) with $y_0+b=\sin t_+=0$ can have at most 4 solutions. Since the three coefficients of f_0, f_1, f_2 are functionally dependent the system (3)+(4) perhaps do not reach the three solutions. But since the coefficients of f_1, f_2, f_3 are functionally independent the system (3)+(4) with $y_0+b\neq 0$ and $\sin t_+=0$ can have two solutions. This completes the proof of Theorem 1 in Case 2.

Case 3: The discontinuous piecewise linear differential system (3)+(4) has one real focus on the discontinuity straight line and one real or virtual node or saddle outside the discontinuity straight line. This node has two different eigenvalues. Then in system (3)+(4) we must take $\alpha = 1$ and $\beta = i$. The case $\alpha = i$ and $\beta = 1$ follows in a similar way.

We recall that if $|\ell| > 1$ then system (3) has a real or virtual node with two different eigenvalues, while if $|\ell| < 1$ the system has a real or virtual saddle. Both cases are studied simultaneously.

The solution of system (3) starting at the point $(x, y) = (0, y_0)$ is

$$
\begin{aligned}
x(t) = & -\frac{e^{-t}}{2(\ell^2-1)}\Big(a\left(e^{(\ell+2)t}(\ell-1)+2e^t-e^{\ell t}(\ell+1)\right)\\
&\qquad +e^{\ell t}\left(e^{2t}-1\right)(\ell^2-1)y_0\Big),\\
y(t) = & \frac{e^{-t}}{2\left(\ell^2-1\right)}\Big(a\left(-e^{(\ell+2)t}(\ell-1)^2+e^{\ell t}(\ell+1)^2-4e^t\ell\right)\\
&\qquad +2e^{\ell t+t}\left(\ell^2-1\right)y_0(\cosh t-\ell\sinh t)\Big).
\end{aligned}
\tag{13}
$$

The solution of system (4) starting at the point $(x, y) = (0, y_0)$ is given in (6).

Let t_- and t_+ be again the finite positive times defined in a similar way to the Case 1, and if we have a periodic solution of system (3)+(4) one of the sets of three equations (7), or (8) must be satisfied. Now we shall assume that equations (7) hold. Again the proof of Theorem 1 in Case 2 if equations (8) hold is completely analogous to the proof when equations (7) hold.

Using (6), (13), taking into account that $\ell \neq \pm 1$ and that the focus must be on the discontinuity straight line (i.e. $c = 0$) the three equations (7) become

$$
\begin{aligned}
e_1 = & -e^{rt_+}(b+y_0)\sin t_+ = 0,\\
e_2 = & \ a\left(1-\ell-2e^{\ell t_-+t_-}+e^{2t_-}(\ell+1)\right)+\left(e^{2t_-}-1\right)(\ell^2-1)y_0=0,\\
e_3 = & \ 2(\ell^2-1)\left(-b+e^{rt_+}(b+y_0)(\cos t_+-r\sin t_+)\right)\\
& -e^{t_-}\Big(ae^{(-\ell-2)t_-}\left(-(\ell-1)^2+e^{2t_-}(\ell+1)^2-4e^{\ell t_-+t_-}\ell\right)\\
&\qquad +2e^{(-\ell-1)t_-}(\ell^2-1)y_0(\cosh t_-+\ell\sinh t_-)\Big)=0.
\end{aligned}
$$

From equation $e_1 = 0$, if $y_0 + b = 0$ and $\sin t_+ \neq 0$, then at most there is one periodic solution, and we are done. So we suppose that $y_0 + b \neq 0$ and $\sin t_+ = 0$, i.e. $t_+ = \pi$. Now solving equation $e_2 = 0$ with respect to the variable y_0 we get

$$
y_0 = -\frac{a(1-\ell+e^{2t_-}+e^{2t_-}\ell-2e^{t_-+\ell t_-})}{(e^{2t_-}-1)(\ell^2-1)}.
$$

Therefore the equation $e_3 = 0$ writes

$$
\begin{aligned}
&2ae^{t_-}+\left(b\left(1+e^{\pi r}\right)(\ell^2-1)-a\left(e^{\pi r}(\ell-1)+\ell+1\right)\right)e^{\ell t_-}+\\
&a\left(\ell+e^{\pi r}(\ell+1)-1\right)e^{(\ell+2)t_-}-\left(2e^{\pi r}a+b\left(1+e^{\pi r}\right)(\ell^2-1)\right)e^{(2\ell+1)t_-}=0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&2af_0(t_-)+\left(b\left(1+e^{\pi r}\right)(\ell^2-1)-a\left(e^{\pi r}(\ell-1)+\ell+1\right)\right)f_1(t_-)+\\
&a\left(\ell+e^{\pi r}(\ell+1)-1\right)f_2(t_-)-\left(2e^{\pi r}a+b\left(1+e^{\pi r}\right)(\ell^2-1)\right)f_3(t_-)=0,
\end{aligned}
$$

where

$$
\begin{aligned}
f_0(t_-) &= e^{t_-},\\
f_1(t_-) &= e^{\ell t_-},\\
f_2(t_-) &= e^{(\ell+2)t_-},\\
f_3(t_-) &= e^{(2\ell+1)t_-}.
\end{aligned}
$$

The set of functions $\{f_0, f_1, f_2, f_3\}$ is an ECT–system because the Wronskians $W(f_0, ..., f_k)(t_-)$ are not zero for $k = 0, 1, 2, 3$. Indeed, we have

$$
\begin{aligned}
W(f_0)(t_-) = &\quad e^{t_-} \neq 0,\\
W(f_0, f_1)(t_-) = &\quad (\ell-1)e^{(\ell+1)t_-} \neq 0,\\
W(f_0, f_1, f_2)(t_-) = &\quad 2(\ell^2-1)e^{(2\ell+3)t_-} \neq 0,\\
W(f_0, f_1, f_2, f_3)(t_-) = &\quad 4\ell(\ell^2-1)^2 e^{4(\ell+1)t_-} \neq 0 \quad \text{if } \ell \neq 0.
\end{aligned}
$$

Therefore, if $\ell \neq 0$ the system (7) with $y_0 + b \neq 0$ and $\sin t_+ = 0$ can have at most 3 solutions. Since the four coefficients of f_0, f_1, f_2, f_3 are functionally dependent the system (3)+(4) perhaps do not reach the three solutions. But since the coefficients of f_1, f_2, f_3 are functionally independent the system (3)+(4) with $y_0+b \neq 0$ and $\sin t_+ = 0$ can have two solutions. Similarly to the previous cases when $y_0 + b = \sin t_+ = 0$ we can have at most 4 solutions. This completes the proof of Theorem 1 in Case 3 when $\ell \neq 0$.

Assume $\ell = 0$. Then the equation $e_3 = 0$ becomes

$$
\begin{aligned}
a\left(e^{\pi r}-1\right) - b\left(1+e^{\pi r}\right) - \left(2a(e^{\pi r}-1) - b\left(1+e^{\pi r}\right)\right)e^{t_-}&\\
+a\left(e^{\pi r}-1\right)e^{2t_-} &= 0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\left(a\left(e^{\pi r}-1\right) - b\left(1+e^{\pi r}\right)\right)f_0(t_-) - \left(2a(e^{\pi r}-1) - b\left(1+e^{\pi r}\right)\right)f_1(t_-)&\\
+a\left(e^{\pi r}-1\right)f_2(t_-) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
f_0(t_-) &= 1,\\
f_1(t_-) &= e^{t_-},\\
f_2(t_-) &= e^{2t_-}.
\end{aligned}
$$

The set of functions $\{f_0, f_1, f_2\}$ is an ECT–system because the Wronskians $W(f_0, ..., f_k)(t_-)$ are not zero for $k = 0, 1, 2$. Indeed, we have

$$
\begin{aligned}
W(f_0)(t_-) = &\quad 1 \neq 0,\\
W(f_0, f_1)(t_-) = &\quad e^{t_-} \neq 0,\\
W(f_0, f_1, f_2)(t_-) = &\quad 2e^{3t_-} \neq 0.
\end{aligned}
$$

Since the three coefficients of f_0, f_1, f_2 are functionally dependent the system (3)+(4) perhaps do not reach the two solutions. But since the coefficients of f_1, f_2 are functionally independent the system (3)+(4) can have one solution. This completes the proof of Theorem 1 in Case 3 when $\ell = 0$.

In short, Theorem 1 is proved.

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