

On the Automorphism Groups of the $\mathbb{Z}_2\mathbb{Z}_4$ -Linear Hadamard Codes and Their Classification

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Abstract It is known that there are exactly $\lfloor \frac{t-1}{2} \rfloor$ and $\lfloor \frac{t}{2} \rfloor$ nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes of length 2^t , with $\alpha = 0$ and $\alpha \neq 0$, respectively, for all $t \geq 3$. In this paper, it is shown that each $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code with $\alpha = 0$ is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code with $\alpha \neq 0$, so there are only $\lfloor \frac{t}{2} \rfloor$ nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes of length 2^t . Moreover, the orders of the permutation automorphism groups of the $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes are given.

Key words: $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, Additive codes, Hadamard codes, Automorphism group

1 Introduction

Let \mathbb{Z}_2 and \mathbb{Z}_4 be the rings of integers modulo 2 and modulo 4, respectively. Let \mathbb{Z}_2^n be the set of all binary vectors of length n and let \mathbb{Z}_4^n be the set of all quaternary vectors of length n . For a vector $x = (x_1, \dots, x_n) \in \mathbb{Z}_2^n$ and a set $I \subseteq \{1, \dots, n\}$, we denote by $x|_I$ the vector x restricted to the coordinates in I .

Any nonempty subset C of \mathbb{Z}_2^n is a binary code and a subgroup of \mathbb{Z}_2^n is called a *binary linear code*. Similarly, any nonempty subset \mathcal{C} of \mathbb{Z}_4^n is a quaternary code and a subgroup of \mathbb{Z}_4^n is called a *quaternary linear code*. Let \mathcal{C} be a quaternary linear code. Since \mathcal{C} is a subgroup of \mathbb{Z}_4^n , it is isomorphic to an Abelian group $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, and we say that \mathcal{C} is of type $2^\gamma 4^\delta$ as a group. Quaternary codes can be seen as binary codes under the usual Gray map defined as $\varphi(0) = (0, 0)$, $\varphi(1) = (0, 1)$, $\varphi(2) =$

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$(1, 1)$, $\varphi(3) = (1, 0)$ in each coordinate. If \mathcal{C} is a quaternary linear code, then the binary code $C = \varphi(\mathcal{C})$ is called a \mathbb{Z}_4 -linear code.

Additive codes were first defined by Delsarte in 1973 as subgroups of the underlying Abelian group in a translation association scheme [7, 8]. In the special case of a binary Hamming scheme, that is, when the underlying Abelian group is of order 2^n , the additive codes coincide with the codes that are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. In order to distinguish them from additive codes over finite fields [3], they are called $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes [4]. Since $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, they can be seen as a generalization of binary (when $\beta = 0$) and quaternary (when $\alpha = 0$) linear codes. As for quaternary linear codes, $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes can also be seen as binary codes by considering the extension of the usual Gray map: $\Phi: \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \rightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, given by

$$\begin{aligned} \Phi(x, y) &= (x, \varphi(y_1), \dots, \varphi(y_\beta)) \\ \forall x \in \mathbb{Z}_2^\alpha, \forall y &= (y_1, \dots, y_\beta) \in \mathbb{Z}_4^\beta. \end{aligned}$$

If \mathcal{C} is a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, $C = \Phi(\mathcal{C})$ is called a $\mathbb{Z}_2\mathbb{Z}_4$ -linear code. Moreover, a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code \mathcal{C} is also isomorphic to an Abelian group $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, and we say that \mathcal{C} (or equivalently the corresponding $\mathbb{Z}_2\mathbb{Z}_4$ -linear code $C = \Phi(\mathcal{C})$) is of type $(\alpha, \beta; \gamma, \delta)$.

Let S_n be the symmetric group of permutations on the set $\{1, \dots, n\}$, and let $\text{id} \in S_n$ be the identity permutation. The group operation in S_n is the function composition, denoted by \circ . The composition $\sigma_1 \circ \sigma_2$ maps any element x to $\sigma_1(\sigma_2(x))$. A $\sigma \in S_n$ acts linearly on words of \mathbb{Z}_2^n or \mathbb{Z}_4^n by permuting the coordinates, $\sigma((c_1, \dots, c_n)) = (c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)})$.

Let \mathcal{C} be a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type $(\alpha, \beta; \gamma, \delta)$. We can assign a permutation $\pi_x \in S_n$ to each codeword $x = (x'_1, \dots, x'_\alpha, x_1, \dots, x_{2\beta}) \in C = \Phi(\mathcal{C})$, such that $\pi_x = \pi_{12} \circ \pi_{34} \circ \dots \circ \pi_{2\beta-1, 2\beta}$, where

$$\pi_{ij} = \begin{cases} \text{id} & \text{if } (x_i, x_j) = (0, 0) \text{ or } (1, 1) \\ (i, j) & \text{otherwise.} \end{cases}$$

Given two codewords of C , $x = (x', x_1, \dots, x_{2\beta})$ and $y = (y', y_1, \dots, y_{2\beta})$, define $x \star y = x + \pi_x(y)$. Then, we have that (C, \star) is an Abelian group [22], which is isomorphic to $(\mathcal{C}, +)$ since

$$\begin{aligned} x \star y &= (x' + y', \varphi(\varphi^{-1}(x_1, x_2) + \varphi^{-1}(y_1, y_2)), \\ &\quad \dots, \varphi(\varphi^{-1}(x_{2\beta-1}, x_{2\beta}) + \varphi^{-1}(y_{2\beta-1}, y_{2\beta}))) \\ &= \Phi(\Phi^{-1}(x) + \Phi^{-1}(y)). \end{aligned}$$

There are $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes in several important classes of binary codes. For example, $\mathbb{Z}_2\mathbb{Z}_4$ -linear perfect single error-correcting codes (or 1-perfect codes) are found in [22] and fully characterized in [6]. Also, in subsequent papers [13, 5, 14, 19, 20], $\mathbb{Z}_2\mathbb{Z}_4$ -linear extended perfect and Hadamard codes are studied and classified independently for $\alpha = 0$ and $\alpha \neq 0$. Finally, in [21, 23, 17], $\mathbb{Z}_2\mathbb{Z}_4$ -linear Reed-

Muller codes are also studied. Note that $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes have allowed to classify more binary nonlinear codes, giving them a structure as $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes.

A (*binary*) *Hadamard code* of length n is a binary code with $2n$ codewords and minimum distance $n/2$ [16]. The $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes such that, under the Gray map, give a Hadamard code are called $\mathbb{Z}_2\mathbb{Z}_4$ -*additive Hadamard codes* and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are called $\mathbb{Z}_2\mathbb{Z}_4$ -*linear Hadamard codes*, or just \mathbb{Z}_4 -*linear Hadamard codes* when $\alpha = 0$. The classification of $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes is given by the following results. For any integer $t \geq 3$ and each $\delta \in \{1, \dots, \lfloor (t+1)/2 \rfloor\}$, there is a unique (up to equivalence) \mathbb{Z}_4 -linear Hadamard code of type $(0, 2^{t-1}; t+1-2\delta, \delta)$, and all these codes are pairwise nonequivalent, except for $\delta = 1$ and $\delta = 2$, where the codes are equivalent to the linear Hadamard code, that is, the dual of the extended Hamming code [14]. Therefore, the number of nonequivalent \mathbb{Z}_4 -linear Hadamard codes of length 2^t is $\lfloor \frac{t-1}{2} \rfloor$ for all $t \geq 3$. On the other hand, for any integer $t \geq 3$ and each $\delta \in \{0, \dots, \lfloor t/2 \rfloor\}$, there is a unique (up to equivalence) $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code of type $(2^{t-\delta}, 2^{t-1}-2^{t-\delta-1}; t+1-2\delta, \delta)$. All these codes are pairwise nonequivalent, except for $\delta = 0$ and $\delta = 1$, where the codes are equivalent to the linear Hadamard code [5]. Therefore, the number of nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes of length 2^t with $\alpha \neq 0$ is $\lfloor t/2 \rfloor$ for all $t \geq 3$.

Two structural properties of binary codes are the rank and the dimension of the kernel. The *rank* of a code C , denoted by r , is simply the dimension of the linear span, $\langle C \rangle$, of C . The *kernel* of a code C is defined as $\text{Ker}(C) = \{x \in \mathbb{Z}_2^n : x+C = C\}$ [2]. If the all-zero vector belongs to C , $\text{Ker}(C)$ is a linear subcode of C . We denote by k the dimension of $\text{Ker}(C)$. In general, C can be written as the union of cosets of $\text{Ker}(C)$, and $\text{Ker}(C)$ is the largest linear code for which this is true [2]. The $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes can also be classified using either the rank or the dimension of the kernel, as it is proven in [14, 20], where these parameters are computed.

Two $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes \mathcal{C}_1 and \mathcal{C}_2 both of type $(\alpha, \beta; \gamma, \delta)$ are said to be *monomially equivalent*, if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain \mathbb{Z}_4 coordinates. Two $\mathbb{Z}_2\mathbb{Z}_4$ -additive or $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are said to be *permutation equivalent* if they differ only by a permutation of coordinates. The *monomial automorphism group* of a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code \mathcal{C} , denoted by $\text{MAut}(\mathcal{C})$, is the group generated by all permutations and sign-changes of the \mathbb{Z}_4 coordinates that preserves the set of codewords of \mathcal{C} , while the *permutation automorphism group* of \mathcal{C} or $C = \Phi(\mathcal{C})$, denoted by $\text{PAut}(\mathcal{C})$ or $\text{PAut}(C)$, respectively, is the group generated by all permutations that preserves the set of codewords [12].

The permutation automorphism group of a code is also an invariant, so it can help in the classification of some families of codes. Moreover, the automorphism group can also be used in decoding algorithms and to describe some other properties like the weight distribution. The permutation automorphism group of $\mathbb{Z}_2\mathbb{Z}_4$ -linear (extended) 1-perfect codes has been studied in [19, 15]. The permutation automorphism group of (nonlinear) binary 1-perfect codes has also been studied before, obtaining some partial results [11, 10, 1, 9]. Finally, the permutation automorphism group of $\mathbb{Z}_2\mathbb{Z}_4$ -additive Hadamard codes has been studied in [18].

2 Classification of $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes

In [14] and [5], $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes are classified independently for $\alpha = 0$ and $\alpha \neq 0$. In this section, we show that each $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code with $\alpha = 0$ is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code with $\alpha \neq 0$, so there are only $\lfloor \frac{t}{2} \rfloor$ nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes of length 2^t .

We say that a function f from $\mathbb{Z}_2^i \times \mathbb{Z}_4^j$ to $\mathbb{Z}_2^s \times \mathbb{Z}_4^t$ is *affine* if $f(\bar{0}) - f(x) - f(y) + f(x+y) = \bar{0}$ for every x and y from $\mathbb{Z}_2^i \times \mathbb{Z}_4^j$ (here and in what follows, $\bar{0}$ denotes the all-zero vector). Equivalently, $f(\cdot) - f(\bar{0})$ is a linear function, i.e., a group homomorphism. Let \mathcal{B} be the set of all affine functions from $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ to \mathbb{Z}_4 . These \mathbb{Z}_4 -valued functions on $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ can be considered as words of length $2^{\gamma+2\delta}$ over \mathbb{Z}_4 . Denote $D_{\gamma,\delta} = \{x : \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta \rightarrow \mathbb{Z}_2^2 : x(\cdot) = \varphi(g(\cdot)) \text{ for some } g \in \mathcal{B}\}$.

Lemma 1. $D_{\gamma,\delta}$ is a \mathbb{Z}_4 -linear Hadamard code of length $n = 2^{\gamma+2\delta+1}$ and type $(0, n/2; \gamma, \delta)$, where $\delta = \check{\delta} + 1$.

Proof. There are $4 \cdot 2^\gamma \cdot 4^\delta = 2n$ affine functions in \mathcal{B} . The set \mathcal{B} is closed under the addition over \mathbb{Z}_4 ; so after applying the Gray map, $D_{\gamma,\delta}$ is a \mathbb{Z}_4 -linear code of length $2^\gamma \cdot 4^\delta \cdot 2 = n$. Clearly, the minimum Hamming distance is $n/2$.

Define the function $\varphi^+ : \mathbb{Z}_4 \rightarrow \{0, 1\}$ by $\varphi^+(0) = \varphi^+(3) = 0$, $\varphi^+(1) = \varphi^+(2) = 1$. Again, the \mathbb{Z}_2 -valued or \mathbb{Z}_4 -valued functions on $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$ can be considered as words of length $2^{\check{\gamma}+2\delta}$ over \mathbb{Z}_2 or \mathbb{Z}_4 , respectively. Let \mathcal{A} be the set of all affine functions f from $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$ to \mathbb{Z}_4 that map the all-zero vector to 0 or 2: $f(\bar{0}) \in \{0, 2\}$. Denote $C_{\check{\gamma},\delta} = \{h : \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta \rightarrow \mathbb{Z}_2^2 : h(\cdot) = \varphi^+(f(\cdot)) \text{ for some } f \in \mathcal{A}\}$.

Lemma 2. $C_{\check{\gamma},\delta}$ is a $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code of length $n = 2^{\check{\gamma}+2\delta}$ and type $(\alpha, \beta; \gamma, \delta)$, where $\gamma = \check{\gamma} + 1$, $\alpha = 2^{\check{\gamma}+\delta}$ corresponding to the elements of order at most 2 of $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$, and $\beta = 2^{\check{\gamma}+\delta-1}(2^\delta - 1)$ corresponding to the pairs of opposite elements of order 4.

Proof. There are $2 \cdot 2^{\check{\gamma}} \cdot 4^\delta = 2n$ affine functions in \mathcal{A} . The set \mathcal{A} is closed under the addition over \mathbb{Z}_4 ; so the Gray map image $A = \Phi(\mathcal{A})$ can also be considered as a $\mathbb{Z}_2\mathbb{Z}_4$ -linear code with $2^{\check{\gamma}+\delta+1}$ coordinates over \mathbb{Z}_2 , which correspond to the elements of order at most 2 of $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$.

Now, we will see that the code A can be obtained from $C_{\check{\gamma},\delta}$ by repeating twice every coordinate. That is, strictly speaking, A is permutation equivalent to $\{(h, h) : h \in C_{\check{\gamma},\delta}\}$. Indeed, given $v \in \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$ of order 4 and an affine function $f \in \mathcal{A}$, the values $\varphi^+(f(v))$ and $\varphi^+(f(-v))$ of the corresponding codeword of $C_{\check{\gamma},\delta}$ each occurs both in $\varphi(f(v))$ and $\varphi(f(-v))$. If the order of $v \in \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^\delta$ is 2 or less, then $\varphi^+(f(v))$ is duplicated in $\varphi(f(v))$.

Finally, it is easy to check that the minimum Lee distance for the set of affine functions \mathcal{A} is $n = 2^{\check{\gamma}+2\delta}$; so the minimum Hamming distance of $C_{\check{\gamma},\delta}$ is the half of this value, that is, $n/2$.

Lemma 3. *Let $f : \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^{\check{\delta}} \rightarrow \mathbb{Z}_4$ be an affine function. Then $h(\cdot) = \varphi^+(f(\cdot))$ belongs to $C_{\check{\gamma}, \check{\delta}}$.*

Proof. In the case that $f(\bar{0}) \in \{0, 2\}$, $C_{\check{\gamma}, \check{\delta}}$ contains h by definition. On the other hand, if $f(\bar{0}) \in \{1, 3\}$, we will use that $\varphi^+(l) = \varphi^+(3-l)$ for $l \in \mathbb{Z}_4$. Then, $h(\cdot) = \varphi^+(f(\cdot)) = \varphi^+(3-f(\cdot))$. Since $3-f(\cdot)$ is an affine function and $3-f(\bar{0}) \in \{0, 2\}$, we obtain that $h \in C_{\check{\gamma}, \check{\delta}}$.

Theorem 1. *The \mathbb{Z}_4 -linear Hadamard code $D_{\check{\gamma}, \check{\delta}}$ of length n and type $(0, n/2; \check{\gamma}, \check{\delta})$ is permutation equivalent to the $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code $C_{\check{\gamma}+1, \check{\delta}}$ of type $(\alpha, \beta; \check{\gamma}+2, \check{\delta}-1)$ with $\alpha \neq 0$.*

Proof. Consider a function f in \mathcal{B} and the related function $g(v, e) = f(v) + 2ef(\bar{0})$, where $v \in \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^{\check{\delta}}$ and $e \in \mathbb{Z}_2$. We can see that

$$\begin{aligned}\varphi(f(v)) &= (\varphi^+(g(-v, 1)), \varphi^+(g(v, 0))), \\ \varphi(f(-v)) &= (\varphi^+(g(v, 1)), \varphi^+(g(-v, 0))).\end{aligned}$$

In order to check these equalities, it is convenient to represent $f(v)$ as $f_0(v) + f(\bar{0})$, where $f_0 : \mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^{\check{\delta}} \rightarrow \mathbb{Z}_4$ is a group homomorphism (in particular, $f_0(-v) = -f_0(v)$).

Since g is an affine function from $v \in \mathbb{Z}_2^{\check{\gamma}+1} \times \mathbb{Z}_4^{\check{\delta}}$ to \mathbb{Z}_4 , we can deduce from Lemma 3 that there is a fixed coordinate permutation that sends every codeword of $D_{\check{\gamma}, \check{\delta}}$ to a codeword of $C_{\check{\gamma}+1, \check{\delta}}$.

Corollary 1. *There are exactly $\lfloor \frac{l}{2} \rfloor$ nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard codes of length 2^l .*

3 The permutation automorphism group

Considering the representation of a code as the union of cosets of its kernel, it is possible to prove the following fact.

Proposition 1. *If $\delta \geq 2$, then the order of the automorphism group of C satisfies*

$$|\text{Aut}(C)| \leq p \cdot 2^{\frac{1}{2}\check{\gamma}(\check{\gamma}+1)+2\check{\gamma}\check{\delta}+\frac{3}{2}\check{\delta}(\check{\delta}+1)} \prod_{i=1}^{\check{\gamma}} (2^i - 1) \prod_{j=1}^{\check{\delta}} (2^j - 1),$$

where $p = 6$ if $\delta = 2$ and $p = 1$ if $\delta \geq 3$.

By Lemma 3, any nonsingular affine transformation of $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^{\check{\delta}}$ belongs to $\text{Aut}(C)$. Therefore, since for $\delta \geq 3$, the number of nonsingular affine transformations coincides with the upper bound given in Proposition 1, we obtain the following result:

Theorem 2. *The automorphism group of the $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code C of type $(\alpha, \beta; \check{\gamma} + 1, \delta)$, with $\delta \geq 3$, is the group of nonsingular affine transformations of $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^{\delta}$. Therefore, its order is*

$$|\text{Aut}(C)| = 2^{\frac{1}{2}\check{\gamma}(\check{\gamma}+1)+2\check{\gamma}\delta+\frac{3}{2}\delta(\delta+1)} \prod_{i=1}^{\check{\gamma}} (2^i - 1) \prod_{j=1}^{\delta} (2^j - 1).$$

In the case $\delta = 2$, there are non-affine permutations in the automorphism group, and the resulting formula again coincides with the upper bound of Proposition 1.

Theorem 3. *The automorphism group of the $\mathbb{Z}_2\mathbb{Z}_4$ -linear Hadamard code C of type $(\alpha, \beta; \check{\gamma} + 1, 2)$ consists of all permutations expressed as $\psi\alpha$, where α is a nonsingular affine transformations of $\mathbb{Z}_2^{\check{\gamma}} \times \mathbb{Z}_4^2$ and ψ is the identity permutation or one of five non-affine permutations. The order of the automorphism group is*

$$|\text{Aut}(C)| = 6 \cdot 2^{\frac{1}{2}\check{\gamma}(\check{\gamma}+1)+4\check{\gamma}+9} \cdot 3 \prod_{i=1}^{\check{\gamma}} (2^i - 1).$$

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References

1. Avgustinovich, S.V., Solov'eva, F.I., Heden, O.: On the structure of symmetry groups of Vasil'ev codes. *Probl. Inf. Transm.* **41**(2), 105–112 (2005). doi:10.1007/s11122-005-0015-5
2. Bauer, H., Ganter, B., F., H.: Algebraic techniques for nonlinear codes. *Combinatorica* **3**(1), 21–33 (1983). doi:10.1007/BF02579339
3. Bierbrauer, J.: *Introduction to Coding Theory. Discrete and Combinatorial Mathematics Series.* Chapman & Hall/CRC (2005)
4. Borges, J., Fernández-Córdoba, C., Pujol, J., Rifà, J., Villanueva, M.: $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes: Generator matrices and duality. *Des. Codes Cryptography* **54**(2), 167–179 (2010). doi:10.1007/s10623-009-9316-9
5. Borges, J., Phelps, K.T., Rifà, J.: The rank and kernel of extended 1-perfect \mathbb{Z}_4 -linear and additive non- \mathbb{Z}_4 -linear codes. *IEEE Trans. Inf. Theory* **49**(8), 2028–2034 (2003). doi:10.1109/TIT.2003.814490
6. Borges, J., Rifà, J.: A characterization of 1-perfect additive codes. *IEEE Trans. Inf. Theory* **45**(5), 1688–1697 (1999). doi:10.1109/18.771247
7. Delsarte, P.: An Algebraic Approach to Association Schemes of Coding Theory, *Philips Res. Rep., Supplement*, vol. 10 (1973)
8. Delsarte, P., Levenshtein, V.I.: Association schemes and coding theory. *IEEE Trans. Inf. Theory* **44**(6), 2477–2504 (1998). doi:10.1109/18.720545

9. Fernández-Córdoba, C., Phelps, K.T., Villanueva, M.: Involutions in binary perfect codes. *IEEE Trans. Inf. Theory* **57**(9), 5926–5932 (2011). doi:10.1109/TIT.2011.2162185
10. Heden, O.: On the symmetry group of perfect 1-error correcting binary codes. *J. Comb. Math. Comb. Comput.* **52**, 109–115 (2005)
11. Heden, O., Pasticci, F., Westerbäck, T.: On the existence of extended perfect binary codes with trivial symmetry group. *Adv. Math. Commun.* **3**(3), 295–309 (2009). doi:10.3934/amc.2009.3.295
12. Huffman, W.C., Pless, V.: *Fundamentals of Error-Correcting Codes*. Cambridge Univ. Press (2003)
13. Krotov, D.S.: \mathbb{Z}_4 -Linear perfect codes. *Diskretn. Anal. Issled. Oper., Ser.1* **7**(4), 78–90 (2000). In Russian. URL: <http://mi.mathnet.ru/eng/da281> translated at <http://arxiv.org/abs/0710.0198>
14. Krotov, D.S.: \mathbb{Z}_4 -linear Hadamard and extended perfect codes. In: D. Augot, C. Carlet (eds.) WCC2001, International Workshop on Coding and Cryptography, *Electron. Notes Discrete Math.*, vol. 6, pp. 107–112. Elsevier B.V. (2001). doi:10.1016/S1571-0653(04)00161-1
15. Krotov, D.S.: On the automorphism groups of the additive 1-perfect binary codes. In: J. Borges, M. Villanueva (eds.) 3rd International Castle Meeting on Coding Theory and Application, pp. 171–176. Universitat Autònoma de Barcelona. Servei de Publicacions, Bellaterra, Spain (2011)
16. MacWilliams, F.J., Sloane, N.J.A.: *The Theory of Error-Correcting Codes*. Amsterdam, Netherlands: North Holland (1977)
17. Pernas, J., Pujol, J., Villanueva, M.: Classification of some families of quaternary Reed–Muller codes. *IEEE Trans. Inf. Theory* **57**(9), 6043–6051 (2011). doi:10.1109/TIT.2011.2119465
18. Pernas, J., Pujol, J., Villanueva, M.: Characterization of the automorphism group of quaternary linear Hadamard codes. *Des. Codes Cryptography* **70**(1-2), 105–115 (2014). doi:10.1007/s10623-012-9678-2
19. Phelps, K.T., Rifà, J.: On binary 1-perfect additive codes: Some structural properties. *IEEE Trans. Inf. Theory* **48**(9), 2587–2592 (2002). doi:10.1109/TIT.2002.801474
20. Phelps, K.T., Rifà, J., Villanueva, M.: On the additive (\mathbb{Z}_4 -linear and non- \mathbb{Z}_4 -linear) Hadamard codes: Rank and kernel. *IEEE Trans. Inf. Theory* **52**(1), 316–319 (2006). doi:10.1109/TIT.2005.860464
21. Pujol, J., Rifà, J., Solov’eva, F.I.: Construction of \mathbb{Z}_4 -linear Reed–Muller codes. *IEEE Trans. Inf. Theory* **55**(1), 99–104 (2009). doi:10.1109/TIT.2008.2008143
22. Rifà, J., Pujol, J.: Translation-invariant propelinear codes. *IEEE Trans. Inf. Theory* **43**(2), 590–598 (1997). doi:10.1109/18.556115
23. Solov’eva, F.I.: On \mathbb{Z}_4 -linear codes with the parameters of Reed-Muller codes. *Probl. Inf. Transm.* **43**(1), 26–32 (2007). Translated from *Probl. Peredachi Inf.*, 43(1):32-38, 2007. doi:10.1134/S0032946007010048