Construction of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes for each allowable value of the rank and dimension of the kernel

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Abstract

This work deals with Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes, which are binary codes after a Gray map from a subgroup of a direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8 groups, where Q_8 is the quaternionic group. In a previous work, these kind of codes were classified in five shapes. In this paper we analyze the allowable range of values for the rank and dimension of the kernel, which depends on the particular shape of the code. We show that all these codes can be represented in a standard form, from a set of generators, which help to understand the characteristics of each shape. The main results we present are the characterization of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes as a quotient of a semidirect product of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and, on the other hand, the construction of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes with each allowable pair of values for the rank and dimension of the kernel.

Index Terms

Combinatorial mathematics, Dimension of the kernel, error-correcting codes, Hadamard codes, rank, $\mathbb{Z}_2\mathbb{Z}_4$-codes, $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes.

I. Introduction

Non-linear codes (like $\mathbb{Z}_4$-linear, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes) have received a great deal of attention since [6]. The codes this paper deals with can be characterized as the image of a subgroup, by a suitable Gray map, of an algebraic group like a direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8, the quaternionic group of order 8 [10].

Hadamard matrices with a subjacent algebraic structure have been deeply studied, as well as the links with other topics in algebraic combinatorics [7]. We quote just a few papers about this subject [8], [5], [2], where we can find beautiful equivalences between Hadamard groups, 2-cocyclic matrices and relative difference sets. On the other hand, from the side of coding theory, it is desirable that the algebraic structures we are dealing with preserves the Hamming distance. This is the case, for example, of the $\mathbb{Z}_2\mathbb{Z}_4$-linear codes which has been intensively studied during the last years [6], [4]. More generally, the propelinear codes and, specially those which are translation invariant,

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are particularly interesting because the subjacent group structure has the property that both, left and right product, preserve the Hamming distance. Translation invariant propelinear codes has been characterized as the image of a subgroup by a suitable Gray map of a direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8 [10].

In this paper we analyze codes that have both properties, being Hadamard and $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. These codes were previously studied and classified [3] in five shapes. The aim of this paper is to go further. First of all by giving an standard form for a set of generators of the code, depending on the parameters, which helps to understand the characteristics of each shape and then by putting the focus in the exact computation of the values of the rank and dimension of the kernel. One of the main results is to characterize the $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes as a quotient of a semidirect product of Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. The second main result is to construct, using the above characterization, Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes such that the values for the rank and dimension of the kernel are each allowable pair previously chosen.

The structure of the paper is as follows. Section II introduces the notation, the basic classification from [3] and preliminary concepts; Section III shows the standard form of generators that allows to represent any Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code in a unique way, this section finishes with two important theorems characterizing a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code as a quotient of a semidirect product of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (Theorems III.2 and III.4); Section IV studies the values of the rank and dimension of the kernel, depending on the shape and parameters of the $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code and in Section V we give the constructions of $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes fulfilling the requirements for the prefixed values of the dimension of the kernel and rank. We finish this last section with Theorem V.2, a compendium of the results reached in this section, and a couple of examples about the constructions of codes with all allowable pair for the values of rank and dimension of the kernel.

II. Preliminaries

Almost all definitions and concepts below, in these preliminaries, can be found in [3].

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ denote the binary field and the ring of integers modulo 4, respectively. Any non-empty subset of $\mathbb{Z}_2^n$ is called a binary code and a linear subspace of $\mathbb{Z}_2^n$ is called a *binary linear code* or a $\mathbb{Z}_2$*-linear code*. Let $\mathrm{wt}(v)$ denote the *Hamming weight* of a vector $v \in \mathbb{Z}_2^n$ (i.e., the number of its nonzero components), and let $d(v,u) = \mathrm{wt}(v+u)$, the *Hamming distance* between two vectors $v, u \in \mathbb{Z}_2^n$.

Let Q_8 be the *quaternionic group* on eight elements. The following equalities provides a presentation and the list of elements of Q_8:

$$Q_8 = \langle \mathbf{a}, \mathbf{b} \,:\, \mathbf{a}^4 = \mathbf{a}^2\mathbf{b}^2 = \mathbf{1}, \mathbf{b}\mathbf{a}\mathbf{b}^{-1} = \mathbf{a}^{-1} \rangle = \{\mathbf{1}, \mathbf{a}, \mathbf{a}^2, \mathbf{a}^3, \mathbf{b}, \mathbf{ab}, \mathbf{a}^2\mathbf{b}, \mathbf{a}^3\mathbf{b}\}.$$

Given three non-negative integers k_1, k_2 and k_3, denote as $\mathcal{G}$ the group $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$. Any element of $\mathcal{G}$ can be represented as a vector where the first k_1 components belong to $\mathbb{Z}_2$, the next k_2 components belong to $\mathbb{Z}_4$ and the last k_3 components belong to Q_8.

We use the multiplicative notation for $\mathcal{G}$ and we denote by $\mathbf{e}$ the identity element of the group and by $\mathbf{u}$ the element of order two, so $\mathbf{e} = (0, \overset{k_1+k_2}{\ldots}, 0, \mathbf{1}, \overset{k_3}{\ldots}, \mathbf{1})$ and $\mathbf{u} = (1, \overset{k_1}{\ldots}, 1, 2, \overset{k_2}{\ldots}, 2, \mathbf{a}^2, \overset{k_3}{\ldots}, \mathbf{a}^2)$, respectively.

We call a *Gray map* the function Φ:

$$\Phi: \mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3} \longrightarrow \mathbb{Z}_2^{k_1+2k_2+4k_3},$$

acting componentwise in such a way that over the binary part is the identity, over the quaternary part acts as the usual Gray map, so $0 \to (00)$, $1 \to (01)$, $2 \to (11)$, $3 \to (10)$ and over the quaternionic part acts in the following way [3]: $\mathbf{1} \to (0,0,0,0)$, $\mathbf{b} \to (0,1,1,0)$, $\mathbf{a} \to (0,1,0,1)$, $\mathbf{ab} \to (1,1,0,0)$, $\mathbf{a}^2 \to (1,1,1,1)$, $\mathbf{a}^2\mathbf{b} \to (1,0,0,1)$, $\mathbf{a}^3 \to (1,0,1,0)$, $\mathbf{a}^3\mathbf{b} \to (0,0,1,1)$.

Note that $\Phi(\mathbf{e})$ is the binary all zero vector and $\Phi(\mathbf{u})$ is the binary all one vector.

Binary codes $C = \Phi(\mathcal{C})$ are called $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. In the specific case $k_3 = 0$, C is called $\mathbb{Z}_2Z_4$-linear code. In this last case, note that $\mathcal{C} = \mathbb{Z}_2^{\gamma} \times \mathbb{Z}_4^{\delta} \subset \mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2}$. We will say that $\mathcal{C}$ is of type $2^{\gamma}4^{\delta}$ [6].

We are interested in Hadamard binary codes $C = \Phi(\mathcal{C})$ where $\mathcal{C}$ is a subgroup of $\mathcal{G} = \mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ of length $n = 2^m$. All through the paper we are assuming it.

The *kernel* of a binary code C of length n is $K(C) = \{z \in \mathbb{Z}_2^n : C + z = C\}$. The dimension of $K(C)$ is denoted by $k(C)$ or simply k. The *rank* of a binary code C is the dimension of the linear span of C. It is denoted by $r(C)$ or simply r.

A *Hadamard matrix* of order n is a matrix of size $n \times n$ with entries ± 1, such that $HH^T = nI$. Any two rows (columns) of a Hadamard matrix agree in precisely $n/2$ components. If $n > 2$ then any three rows (columns) agree in precisely $n/4$ components. Thus, if $n > 2$ and there is a Hadamard matrix of order n then n is multiple of 4.

Two *Hadamard matrices* are *equivalent* if one can be obtained from the other by permuting rows and/or columns and multiplying rows and/or columns by -1. With the last operations we can change the first row and column of H into $+1$'s and we obtain an equivalent Hadamard matrix which is called normalized. If $+1$'s are replaced by 0's and -1's by 1's, the initial Hadamard matrix is changed into a (binary) Hadamard matrix and, from now on, we will refer to it when we deal with Hadamard matrices. The binary code consisting of the rows of a (binary) Hadamard matrix and their complements is called a (binary) *Hadamard code*, which is of length n, with $2n$ codewords, and minimum distance $n/2$.

Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes were studied in [3] and a classification in five shapes was given. Set $|T(\mathcal{C})| = 2^{\sigma}$, $|Z(\mathcal{C})/T(\mathcal{C})| = 2^{\delta}$ and $|\mathcal{C}/Z(\mathcal{C})| = 2^{\rho}$, where $T(\mathcal{C})$ is the subgroup of elements of order two, $Z(\mathcal{C})$ is the center of $\mathcal{C}$ and $m = \sigma + \delta + \rho - 1$. A normalized generator set in [3] has the form $\mathcal{C} = \langle x_1, \dots, x_{\sigma};\, y_1, \dots, y_{\delta};\, z_1, \dots, z_{\rho}\rangle$, where x_i are elements of order two generating $T(\mathcal{C}) = \langle x_1 ... x_{\sigma}\rangle$ and $Z(\mathcal{C}) = \langle x_1, \dots, x_{\sigma};\, y_1, \dots, y_{\delta}\rangle$ is the center of $\mathcal{C}$. In summary, the five shapes found in [3] are:

- Shape 1: $\rho = 0$.
- Shape 2: $\delta = 0$, $z_1^2 = z_2^2 = [z_1, z_2] = \mathbf{u}$, $[z_i, z_j] = z_j^2$ and $[z_j, z_k] = \mathbf{e}$ for every $i \in \{1, 2\}$ and $3 \le j, k \le \rho$.
- Shape 3: $\delta = 0$, $z_1^2 = \mathbf{u} \notin \langle z_2^2, \dots, z_{\rho}^2\rangle \cong \mathbb{Z}_2^{\rho-1}$, $[z_1, z_i] = z_i^2$ and $[z_i, z_j] = \mathbf{e}$, for every $i \neq j$ in $\{2, \dots, \rho\}$.
- Shape 4: $\delta \le 1$ and $z_1^2 = z_2^2 = [z_1, z_2] \neq \mathbf{u}$.
- Shape 5: $\delta = 0$, $\rho = 4$, $z_1^2 = z_2^2 = [z_1, z_2] = \mathbf{u} \neq z_3^2 = z_4^2 = [z_3, z_4]$ and $[z_i, z_j] \in \langle z_j^2\rangle$ for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

However, in the current paper we distinguish the case when $\mathbf{u}$ is not the square of some element of order four (this is equivalent to $\delta = 0$) and the case when $\mathbf{u}$ is the square of some element of order four (this is equivalent to $\delta = 1$). Hence, we will use two more shapes. Shape 1^* and 4^* will distinguish the cases of shape 1 and 4, respectively, when $\mathbf{u}$ is the square of some element of order four (this is equivalent to $\delta = 1$).

Two elements a and b of $\mathcal{C}$ commutes if and only if $ab = ba$. As an extension of this concept, the *commutator* of a and b is defined as the element $[a, b]$ such that $ab = [a, b]ba$. Note that all commutators belong to $T(\mathcal{C})$ and any element of $T(\mathcal{C})$ commutes with all elements of $\mathcal{C}$.

We say that two elements a and b of $\mathcal{C}$ *swap* if and only if $\Phi(ab) = \Phi(a) + \Phi(b)$. As an extension of this concept, we define the *swapper* of a and b as the element $(a\colon b)$ such that $\Phi((a\colon b)ab) = \Phi(a) + \Phi(b)$. Note that all swappers belong to $T(\mathcal{G})$ but they can be out of $\mathcal{C}$.

Both, commutators and swappers can be obtained as a component-wise expression, if $a = (a_1, \ldots, a_l)$ and $b = (b_1, \ldots, b_l)$ then $(a\colon b) = ((a_1\colon b_1), \ldots, (a_l\colon b_l))$ and $[a, b] = ([a_1, b_1], \ldots, [a_l, b_l])$. Table I and Table II describes the values of all swappers and commutators, respectively, in $\mathbb{Z}_4$ and Q_8 (the value in $\mathbb{Z}_2$ is always 0).

	0,2	1,3
0,2	0	0
1,3	0	2

	$\mathbf{1,a^2}$	$\mathbf{a,a^3}$	$\mathbf{b,a^2b}$	$\mathbf{ab,a^3b}$
$\mathbf{1,a^2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{a,a^3}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{a^2}$	$\mathbf{1}$
$\mathbf{b,a^2b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{a^2}$
$\mathbf{ab,a^3b}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{1}$	$\mathbf{a^2}$

TABLE I

SWAPPERS IN $\mathbb{Z}_4$ AND Q_8

	0,2	1,3
0,2	0	0
1,3	0	0

	$\mathbf{1,a^2}$	$\mathbf{a,a^3}$	$\mathbf{b,a^2b}$	$\mathbf{ab,a^3b}$
$\mathbf{1,a^2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{a,a^3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{a^2}$
$\mathbf{b,a^2b}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{1}$	$\mathbf{a^2}$
$\mathbf{ab,a^3b}$	$\mathbf{1}$	$\mathbf{a^2}$	$\mathbf{a^2}$	$\mathbf{1}$

TABLE II

COMMUTATORS IN $\mathbb{Z}_4$ AND Q_8

It is known [3] the following relationship between swappers, the kernel and the linear span of C. For any element a of $\mathcal{C}$ we have $\Phi(a) \in K(C)$ if and only if, for every $b \in \mathcal{C}$, all swappers $(a\colon b) \in \mathcal{C}$. Moreover, the linear span of C can be seen as $\Phi(\langle \mathcal{C} \cup S(\mathcal{C}) \rangle)$, where $\langle \mathcal{C} \cup S(\mathcal{C}) \rangle$ is the group generated by $\mathcal{C}$ and $S(\mathcal{C})$, the set of swappers of the elements in $\mathcal{C}$.

Using Table-I and Table-II the next lemma can be easily verified.

Lemma II.1. *For any $a, b, c \in \mathcal{G}$:*

1) $[a, b] = [b, a]$. *Note it is not always true that $(a\colon b) = (b\colon a)$.*

2) $(ab\colon c)=(a\colon c)(b\colon c)$ *and* $(c\colon ab)=(c\colon a)(c\colon b)$
3) $[ab,c]=[a,c][b,c]$.
4) $(a\colon b)(b\colon a)=[a,b]$
5) $(a\colon a)=a^2$
6) *if* $a^2=\mathbf{e}$ *then* $[a,b]=(a\colon b)=(b\colon a)=\mathbf{e}$.
7) *if* $a^2=\mathbf{u}$ *and* $[a,b]=\mathbf{e}$ *then* $(a\colon b)=(b\colon a)=b^2$.

Definition II.2. *For any* $x\in T(\mathcal{G})$ *define* $M(x)$ *as the set of components where* x *has, as entry, an element of order two,* $\varnothing\subseteq M(x)\subseteq M(\mathbf{u})$.

Example: let $x=(1,a^2,a^2,1,1,a^2)$ then $M(x)=\{1,2,5\}$, where we enumerate the 0th component as the first one.

The next results in this section are technical lemmas, which prove to be useful later.

Lemma II.3. *Let* $x,y\in\mathcal{G}$, *then*

1) $M(\,(x\colon y)\,)\subseteq M(x^2)\cap M(y^2)$ *and* $M([x,y])\subseteq M(x^2)\cap M(y^2)$.
   In the specific case when $[x,y]=\mathbf{e}$ *we have* $M((x\colon y))=M(x^2)\cap M(y^2)$ *and* $M([x,y])=\varnothing$.
2) *if* $[x,y]=\mathbf{e}$ *then* $\mathrm{wt}((xy)^2)=\mathrm{wt}(x^2y^2)=\mathrm{wt}(x^2)+\mathrm{wt}(y^2)-2\mathrm{wt}((x\colon y))$.

Proof. These items follow straightforwardly from Tables I and II. □

Lemma II.4. *Let* $\mathcal{C}$ *be a subgroup of* $Q_8^{k_3}$ *such that* $\Phi(\mathcal{C})$ *is a Hadamard code and let* $a,b\in T(\mathcal{C})$. *If* $a,b,ab\notin\{\mathbf{e},\mathbf{u}\}$, *then* $|M(a)\cap M(b)|=|M(a)\cap\overline{M(b)}|=|\overline{M(a)}\cap M(b)|=|\overline{M(a)}\cap\overline{M(b)}|=k_3/4$.

Proof. Straightforward. □

Lemma II.5. *Let* $\mathcal{C}$ *be a subgroup of* $\mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}\times Q_8^{k_3}$ *such that* $\Phi(\mathcal{C})$ *is a Hadamard code. Let* $a,b,c\in\mathcal{C}\setminus T(\mathcal{C})$.

1) *either* $a^2=\mathbf{u}$ *or* $[a,b]=[b,a]=\mathbf{e}$ *or* $[a,b]=[b,a]=a^2$.
2) *if* $a^2=u$ *and* $b^2=c^2=[b,c]\notin\{\mathbf{e},\mathbf{u}\}$ *then* $[a,b]=\mathbf{e}$ *or* $[a,c]=\mathbf{e}$ *or* $[a,bc]=\mathbf{e}$.
3) *if* $b^2=c^2=[b,c]$ *and* $[a,b]=[a,c]=\mathbf{e}$ *then* $(ab)^2=(ac)^2=\mathbf{u}$ *and* a^2, b^2, c^2 *are not equal to* $\mathbf{u}$.

Proof. The first item was already proven in [3, Lemma IV.6].

For the second item we will assume that the first two possibilities of the conclusion are false. Using the first item in this Lemma we have $[a,b]=[a,c]=b^2=c^2$, so $[a,bc]=[a,b][a,c]=b^2c^2=e$. This proves the second item.

For the third item note that $(bc)^2=b^2c^2[b,c]=b^2=c^2$, thus $M(b^2)=M(c^2)=M((bc)^2)$. Taken into account that $[a,b]=[a,c]=[a,bc]=\mathbf{e}$, by Lemma II.3 we have $M((a\colon b))=M((a\colon c))=M((a\colon bc))$. Hence, $(a\colon b)=(a\colon c)=(a\colon bc)$. Moreover, $(a\colon bc)=(a\colon b)(a\colon c)=(a\colon b)^2=\mathbf{e}$ and so $(a\colon b)=(a\colon c)=\mathbf{e}$. Now, using again Lemma II.3, $\mathrm{wt}(a^2b^2)=\mathrm{wt}(a^2)+\mathrm{wt}(b^2)-2\mathrm{wt}((a\colon b))=\mathrm{wt}(a^2)+\mathrm{wt}(b^2)$. As we are working with elements of a Hadamard code, the weights must be equal to n, $n/2$ or 0. The last possibility has been discarded when we state that they do not belong to $T(\mathcal{C})$, and so the only remainder possibility is $\mathrm{wt}(a^2)=\mathrm{wt}(b^2)=\mathrm{wt}(c^2)=n/2$

and wt$(a^2b^2) = n$, proving in this way that a^2, b^2, c^2 are not equal to $\mathbf{u}$ and $a^2b^2 = \mathbf{u}$. The same argumentation leads to $a^2c^2 = \mathbf{u}$. $\square$

III. The standard form for the generator set of a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code

To know the shape of a given Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code we need to begin with a normalized generator set of $\mathcal{C}$ [3]. Now, in this section, we present a new point of view which lead to us to construct a standard generator set which will allow to decide the classification of a given subgroup in a more efficient way.

The next theorem shows that a subgroup $\mathcal{C}$, such that $\phi(\mathcal{C})$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code, has an abelian maximal subgroup $\mathcal{A}$ which is normal in $\mathcal{C}$ and $\mathcal{C}/\mathcal{A}$ is an abelian group of order 2^a, for $a \in \{0, 1, 2\}$. We begin by a technical lemma.

Lemma III.1. *Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $\phi(\mathcal{C}) = C$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code. Let $\mathcal{A}$ be any subgroup of $\mathcal{C}$ containing $T(\mathcal{C})$, the subgroup of the elements of order two in $\mathcal{C}$. Then $\mathcal{A}$ is normal in $\mathcal{C}$.*

Proof. We want to show that $c^{-1}ac \in \mathcal{A}$ for every $a \in \mathcal{A}, c \in \mathcal{C}$. We have $c^{-1}ac = a[a, c]$ and all commutators belong to $T(\mathcal{C}) \subseteq \mathcal{A}$, so the statement follows. $\square$

Theorem III.2. *Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $\phi(\mathcal{C}) = C$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code. Then $\mathcal{C}$ has an abelian maximal subgroup $\mathcal{A}$ which is normal in $\mathcal{C}$ and $|\,\mathcal{C}/\mathcal{A}\,| \in \{1, 2, 4\}$. Futher, $\mathcal{C}$ may be expressed as a quotient of a semidirect product of $\mathcal{A}$.*

Proof. A normalized generator set in [3] has the form $\mathcal{C} = \langle x_1, \ldots, x_\sigma;\ y_1, \ldots, y_\delta;\ z_1,\ \ldots, z_\rho\rangle$, where x_i are elements of order two that generates $T(\mathcal{C}) = \langle x_1 ... x_\sigma\rangle$ and $Z(\mathcal{C}) = \langle x_1, \ldots, x_\sigma;\ y_1, \ldots, y_\delta\rangle$ is the center of $\mathcal{C}$. Throughout this proof we will use a new generator set for $\mathcal{C}$, which will be called *standardized generator set*: $\mathcal{C} = \langle x_1, \ldots, x_\sigma, r_1, \ldots, r_\tau, s_1, \ldots, s_\upsilon\rangle$ and we always define the subgroup $\mathcal{A} = \langle x_1, \ldots, x_\sigma, r_1, \ldots, r_\tau\rangle$, which is normal in $\mathcal{C}$ by Lemma III.1.

For the case when $\mathcal{C}$ is of shape 1 or shape 1^* we have that the whole group $\mathcal{C}$ is abelian, so $\mathcal{A} = \mathcal{C}$ and $|\,\mathcal{C}/\mathcal{A}\,| = 1$.

For the case when $\mathcal{C}$ is of shape 2 we have [3] $\delta = 0$, $z_1^2 = z_2^2 = [z_1, z_2] = \mathbf{u}$, $[z_i, z_j] = z_j^2$ and $[z_j, z_k] = \mathbf{e}$ for every $i \in \{1, 2\}$ and $3 \leq j, k \leq \rho$. We define the standardized generator set taking $x_1, \ldots, x_\sigma$; $r_1 = z_1z_2$, $r_i = z_{i+1}$ for every $2 \leq i \leq \tau$; $s_1 = z_1$. Now we want to show that $\mathcal{A}$ is abelian and maximal in $\mathcal{C}$ and $\mathcal{C}/\mathcal{A} = \langle s_1\rangle$. Indeed, for every $2 \leq i, j \leq \tau$, $[r_1, r_i] = [z_1z_2, z_{i+1}] = [z_1, z_{i+1}][z_2, z_{i+1}] = z_{i+1}^2 z_{i+1}^2 = \mathbf{e}$ and $[r_i, r_j] = [z_{i+1}, z_{j+1}] = \mathbf{e}$. Hence $\mathcal{A}$ is abelian. To prove the maximality of $\mathcal{A}$ in $\mathcal{C}$ we show that $[s_1, r_1] = [z_1, z_1z_2] = [z_1, z_2] = \mathbf{u} \neq \mathbf{e}$. In addition, we see that $r_1^2 = (z_1z_2)^2 = z_1^2z_2^2[z_1, z_2] = \mathbf{u}$ and $s_1^2 = z_1^2 = \mathbf{u}$ and so $\mathcal{C} = \mathcal{A} \rtimes \langle s_1\rangle/(\mathbf{u}, s_1^2)$, with $r_1^2 = \mathbf{u}$.

For the case when $\mathcal{C}$ is of shape 3 we have [3] $\delta = 0$, $z_1^2 = \mathbf{u} \notin \langle z_2^2, \ldots, z_\rho^2\rangle$, $[z_1, z_i] = z_i^2$ and $[z_i, z_j] = \mathbf{e}$, for every $i \neq j$ in $\{2, \ldots, \rho\}$. We define the standardized generator set taking $r_i = z_{i+1}$ for every $1 \leq i \leq \tau = \rho - 1$; $s_1 = z_1$. Now we want to show that $\mathcal{A}$ is abelian and maximal in $\mathcal{C}$ and $\mathcal{C}/\mathcal{A} = \langle s_1\rangle$. Indeed, for every $1 \leq i, j \leq \tau$, $[r_i, r_j] = [z_{i+1}, z_{j+1}] = \mathbf{e}$. Hence $\mathcal{A}$ is abelian. To prove the maximality of $\mathcal{A}$ in $\mathcal{C}$ we show that

$[s_1, r_1] = [z_1, z_2] = z_2^2 \neq \mathbf{e}$. In addition, we note that $\mathbf{u} \notin \langle r_1^2 \ldots r_\tau^2 \rangle$ and $s_1^2 = z_1^2 = \mathbf{u}$ and so $\mathcal{C} = \mathcal{A} \rtimes \langle s_1 \rangle / (\mathbf{u}, s_1^2)$, with $r_1^2 \neq \mathbf{u}$.

For the case when $\mathcal{C}$ is of shape 4 with $\delta = 0$ we have $\delta = 0$, $\rho = 2$ and $z_1^2 = z_2^2 = [z_1, z_2] \notin \{\mathbf{e}, \mathbf{u}\}$. We define the standardized generator set taking $r_1 = z_1$, $s_1 = z_2$ and define $\mathcal{A} = \langle x_1, \ldots, x_\sigma; r_1 \rangle$. In this case, $r_1 \neq \mathbf{u}$ and $\upsilon = 1$. As all generators, except one, belong to $T(\mathcal{C})$ it is immediate that $\mathcal{A}$ is abelian and $\mathcal{C}/\mathcal{A} = \langle s_1 \rangle$. For the maximality, see that $[r_1, s_1] = [z_1, z_2] = z_1^2 \neq \mathbf{e}$. Note $r_1^2 = s_1^2 \neq \mathbf{u}$ and $\mathcal{C} = \mathcal{A} \rtimes \langle s_1 \rangle / (r_1^2, s_1^2)$.

For the case when $\mathcal{C}$ is of shape 4^* we have $\rho = 2$ and $z_1^2 = z_2^2 = [z_1, z_2] \notin \{\mathbf{e}, \mathbf{u}\}$. The element y_1 commutes with both z_1, z_2 and so, by item 3 of Lemma II.5 we have $y_1^2 \neq \mathbf{u}$ and $(y_1 z_1)^2 = (y_1 z_2)^2 = \mathbf{u}$. We define the standardized generator set taking $r_1 = y_1 z_1$, $r_2 = z_1$, $s_1 = z_2$. In this case, $r_1 = \mathbf{u}$ and $\upsilon = 1$. We have $[r_1, r_2] = [y_1 z_1, z_1] = \mathbf{e}^2 = \mathbf{e}$ and so $\mathcal{A}$ is abelian. For the maximality, see that $[r_1, s_1] = [y_1 z_1, z_2] = [z_1, z_2] = z_1^2 \neq \mathbf{e}$. In addition, $r_1^2 = (y_1 z_1)^2 = \mathbf{u} \neq r_2^2 = z_1^2$ and $s_1^2 = z_2^2 \neq \mathbf{u}$ and $\mathcal{C} = \mathcal{A} \rtimes \langle s_1 \rangle / (r_1^2, s_1^2)$.

For the case when $\mathcal{C}$ is of shape 5 we have $\delta = 0$ and $\rho = 4$. We have: $z_1^2 = z_2^2 = [z_1, z_2] = \mathbf{u} \neq z_3^2 = z_4^2 = [z_3, z_4]$ and $[z_i, z_j] \in \langle z_j^2 \rangle$ for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$. We define the standardized generator set taking $r_1 = z_1$, $r_2 = f(z_1)$, $s_1 = z_2$, $s_2 = f(z_2)$, where:

$$f(z) = \begin{cases} z_3 & \text{if } [z, z_3] = e, \\ z_4 & \text{if } [z, z_4] = e, \\ z_3 z_4 & \text{otherwise.} \end{cases}$$

From Lemma II.5 it is easy to check that in the following matrix

$$\begin{pmatrix} [z_1, z_3] & [z_1, z_4] & [z_1, z_3 z_4] \\ [z_2, z_3] & [z_2, z_4] & [z_2, z_3 z_4] \\ [z_1 z_2, z_3] & [z_1 z_2, z_4] & [z_1 z_2, z_3 z_4] \end{pmatrix}$$

there is one and only one element in each row or column equal to $\mathbf{e}$, being the other two elements equals to $z_3^2 = z_4^2$. Therefore, $[z_1, f(z_1)] = [z_2, f(z_2)] = \mathbf{e}$ and $[z_1, f(z_2)] = [z_2, f(z_1)] = [f(z_1), f(z_2)] = z_3^2 = z_4^2$.

We have $\mathcal{A} = \langle r_1, r_2 \rangle$ and $\mathcal{C}/\mathcal{A} = \langle s_1, s_2 \rangle$. In particular $[r_1, r_2] = [z_1, f(z_1)] = \mathbf{e}$, hence $\mathcal{A}$ is abelian. For the maximality, see that $[r_1, s_1] = [z_1, z_2] \neq \mathbf{e}$ and $[r_2, s_2] = [f(z_1), f(z_2)] \neq \mathbf{e}$. In addition, note $r_1^2 = s_1^2 = \mathbf{u} \neq r_2^2 = s_2^2$ and $\mathcal{C} = \mathcal{A} \rtimes \langle s_1, s_2 \rangle / (r_1^2, s_1^2)(r_2^2, s_2^2)$. □

The next corollary summarize the most relevant properties of the standardized set of generators we just defined.

Corollary III.3. *Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $C = \Phi(\mathcal{C})$ is a Hadamard code and let $\{x_1, \ldots, x_\sigma;\ r_1, \ldots, r_\tau;\ s_1, s_\upsilon\}$ be a standard set of generators of $\mathcal{C}$.*

- *The elements x_i are of order two and generate $T(\mathcal{C})$.*
- *The elements r_i are of order four and commute with each other, $[r_i, r_j] = \mathbf{e}$ for every $1 \leq i, j \leq \tau$. When $\mathbf{u} \in \langle r_1 \ldots r_\tau \rangle$ we will take $\mathbf{u} = r_1^2$ and we have $r_1^2 = \mathbf{u} \notin \langle r_2^2 \ldots r_\tau^2 \rangle$.*
- *The cardinal υ of the set $\{s_1, s_\upsilon\}$ is in $\{0, 1, 2\}$ and when $\upsilon = 2$ we have $s_1^2 = \mathbf{u} \neq s_2^2$, and $[s_1, s_2] = \mathbf{e}$. Moreover, when $r_1^2 = s_1^2 = \mathbf{u}$ then $[r_1, s_1] = \mathbf{u}$.*

- *Any element $c \in \mathcal{C}$ can be written in a unique way as*

$$c = \prod_{i=1}^{\sigma} x_i^{a_i} \prod_{j=1}^{\tau} r_j^{b_j} \prod_{k=1}^{\upsilon} s_k^{c_k}, \text{ where } a_i, b_j, c_k \in \{0,1\}.$$

There are a few more facts that we want to emphasize and that we use later in the next section about constructions. First of all, if there exists $x \in \mathcal{C}$ such that $x^2 = \mathbf{u}$ then $k_1 = 0$. This is the reason why the shapes 2, 3, 4^* and shape 5 must have $k_1 = 0$. Secondly, if $x, y \in \mathcal{C}$ and $[x, y] = z \neq \mathbf{e}$ then the components in $M(z)$ does not correspond to $\mathbb{Z}_4$. This is the reason why the shape 2 and 5 (where $r_1^2 = s_1^2 = [r_1, s_1] = \mathbf{u}$) have $k_2 = 0$.

For shape 4 we have that $x^2 \neq \mathbf{u}$, for all $x \in \mathcal{C}$. Hence, since $[r_1, s_1] = r_1^2 = s_1^2 \neq \mathbf{u}$, we have that all $\mathbb{Z}_4$-components in r_1 and in s_1 are 0 or 2. The other generator elements in $\mathcal{C}$ (apart from r_1 and s_1) are of order two. We claim that in this case $k_2 = 0$. Otherwise, after the Gray map, the corresponding Hadamard matrix would have repeated columns, those corresponding the two binary components of each $\mathbb{Z}_4$, and this contradicts that we had a Hadamard matrix. We can go further, for the same reason it can not exist any Q_8-component of order two in r_1 or s_1 and, since $[r_1, s_1]$ has weight $4k_3$ which must be $n/2$. We conclude that $k_1 = 4k_3$.

Finally, for shape 3 and shape 4^*, the elements $x \in \mathcal{C}$ such that $x^2 = \mathbf{u}$ should have their $\mathbb{Z}_4$-components belonging to $\{1, 3\}$. The rest of elements x, so with $x^2 \neq \mathbf{u}$, have their $\mathbb{Z}_4$-components belonging to $\{0, 2\}$ since, for any $i \in \{1, \ldots, \tau\}$, $[r_i, s_1] = r_i^2 \neq \mathbf{u}$ for shape 3 and $[r_i, s_1] = s_1^2 \neq \mathbf{u}$ for shape 4^*. The same argumentation as in the previous paragraph lead us to say that $k_2 = 2k_3$ for shape 4^*. We summarize these results in Table III.

The next theorem characterizes the maximal abelian subgroup $\mathcal{A}$ and makes possible all constructions of these kind of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes.

Theorem III.4. *Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $\phi(\mathcal{C}) = C$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code and $\mathcal{A}$ the abelian maximal subgroup in $\mathcal{C}$. Then $\phi(\mathcal{A})$ can be described as a duplication of a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code when $\upsilon = 1$ or as a quadruplication of a Hadamard $\mathbb{Z}_4$-linear code, if $\upsilon = 2$.*

Proof. Let $\mathcal{A} = \langle x_1, \ldots, x_\sigma, r_1, \ldots, r_\tau \rangle$ and $\mathcal{C} = \langle \mathcal{A}, s_1, \ldots, s_\upsilon \rangle$. We know from Theorem III.2 that $|\mathcal{C}/\mathcal{A}| \in \{1, 2, 4\}$. If $|\mathcal{C}/\mathcal{A}| = 1$ then there is nothing to prove, $\phi(\mathcal{A})$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code.

Let us assume that $\upsilon = 1$, so $|\mathcal{C}/\mathcal{A}| = 2$. Code $\mathcal{A}$ is additive and has length $2^m = 2^{\sigma+\tau+\upsilon-1}$. Let M be the matrix with the rows given by $x_1, \ldots, x_\sigma, r_1, \ldots, r_\tau$. Matrix M has k_1 binary columns, k_2 quaternary columns and k_3 quaternionic columns and is a generator matrix for $\mathcal{A}$. Let $\overline{M}$ the matrix M extended with one more row given by s_1. Also, let N be the matrix where the rows are all elements in $\mathcal{A}$ and $\overline{N}$ the matrix N extended adding the remainder rows of $\mathcal{C}$. Columns in N could be considered as binary columns after a Gray map of the original elements. First of all we claim that there are not three repeated binary columns in M. Deny the claim. Let a, b, c be the repeated binary columns in N and a', b', c' the corresponding extension to the second part of matrix $\overline{N}$. Since $\phi(\mathcal{C})$ is a Hadamard code any two columns of $\overline{N}$ agree in precisely half of components. Hence, we should have $b'^{(c)} = a'$, where $b'^{(c)}$ means the complementary of b'. Also we should have $c'^{(c)} = a'$ and $c'^{(c)} = b'$ obtaining $b'^{(c)} = b'$ which could not happen. So, in N there are not three or more repeated binary columns. It is known that the parity check matrix M of and additive code $\mathcal{A} = \langle x_1, \ldots, x_\sigma, r_1, \ldots, r_\tau \rangle$ has at most $k_1 + k_2$ different (up to sign) $\mathbb{Z}_2\mathbb{Z}_4$-columns [9], where $k_1 + 2k_2 = 2^{\sigma+\tau-1}$, and this maximum corresponds to a Hadamard code.

Therefore, since matrix M has not three repeated binary columns and has length $2(k_1+2k_2)$ we conclude that M has exactly k_1+k_2 different $\mathbb{Z}_2\mathbb{Z}_4-$columns (up to sign) each one repeated twice and $\mathcal{A}$ is a duplicated additive Hadamard code.

Finally, let us assume $\upsilon=2$, so $|\mathcal{C}/\mathcal{A}|=4$. In this case we known that $k_1=k_2=0$ (Table III). Matrix M has repeated binary columns. Indeed, each element is in $\langle \mathbf{a}\rangle \subset Q_8$ and so, each binary column is repeated twice. We claim that we can not have five repeated binary columns, so three Q_8-columns. Deny the claim. Let a_1,a_2,a_3,a_4,a_5 the five repeated binary columns in N and a'_1,a'_2,a'_3,a'_4,a'_5 the corresponding extension to the second part of matrix $\overline{N}$. We know that in a Hadamard matrix, any three columns agree in precisely a fourth part of the components. Since any three different columns a_{i_1},a_{i_2},a_{i_3} coincide we have that any three different columns $a'_{i_1},a'_{i_2},a'_{i_3}$ does not agree, simultaneously, in any component. This could not happen if we have five or more repeated columns a_1,a_2,a_3,a_4,a_5. Therefore, since matrix M has length $2^{\sigma+\tau}$ and has no five repeated binary columns, so M has no three repeated Q_8-columns, we conclude that M has duplicated all Q_8-columns. Hence, M is a quatriplication of a $\mathbb{Z}_4$-linear hadamard code. □

From Theorem III.4 and the rest of results of this section we see that there are two big classes of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. Despite all codes contains the all one vector $\mathbf{u}$, there are codes where there exist an element r_1, of order four, such that $r_1^2=\mathbf{u}$ (codes of shape 1^*, 2, 4^* and 5) and there are codes where $\mathbf{u}$ is not the square of any other element of order four (codes of shape 1, 3 and 4). We will define the new parameter $\overline{\tau}=\tau-1$ in the first case ($r_1^2=\mathbf{u}$) and $\overline{\tau}=\tau$ in the second case ($r_1^2\neq\mathbf{u}$). The existence conditions for Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes we included in Table III easily come from Theorem III.4 and [9], where it was stated the existence conditions for Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

IV. Rank and kernel dimension of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes

In this section we show the conditions that s_1 and s_2 must fulfill in order to compose a code with specific values for the dimension of the kernel and the rank. Later, in the next section, our focus will be the construction of codes $\mathcal{C}$, by adding the generator s_1 and, optionally, s_2 to a previous subgroup $\mathcal{A}(\mathcal{C})$.

Recall that in the proof of Theorem III.2 we saw that $r_1^2=\mathbf{u}$ for codes of shapes $1^*,2,4^*,5$ and $\mathbf{u}\notin\langle r_1^2\dots r_\tau^2\rangle$ for the other shapes.

Let $\mathcal{A}(\mathcal{C})=\langle x_1,\dots,x_\sigma,r_1,\dots,r_\tau\rangle$ and let $\mathcal{R}(\mathcal{C})$ be defined by

$$\begin{cases} \mathcal{R}(\mathcal{C})=\langle x_1...x_\sigma, r_2...r_\tau\rangle; & \text{if } r_1^2=\mathbf{u}\\ \mathcal{R}(\mathcal{C})=\mathcal{A}(\mathcal{C}); & \text{if } r_1^2\neq\mathbf{u}\end{cases}$$

With this definition, we have the following technical lemmas.

Lemma IV.1. *Let $a,b\in\mathcal{R}(\mathcal{C})\backslash T(\mathcal{C})$ which are not in the same coset of $T(\mathcal{C})$, so $ba^{-1}\notin T(\mathcal{C})$ then:*

1) $a^2,b^2,(ab)^2\notin\{\mathbf{e},\mathbf{u}\}$ *and* $\mathrm{wt}(a^2)=\mathrm{wt}(b^2)=\mathrm{wt}((ab)^2)=n/2$
2) $\mathrm{wt}((a\!:\!b))=n/4$ *and so* $(a\!:\!b)\notin\mathcal{C}$.
3) *With the same hypothesis as for a,b, let a',b' a different pair, such that the different elements in $\{a,b,a',b'\}$ are pairwise not in the same coset of $T(\mathcal{C})$. Then $(a\!:\!b)\neq(a'\!:\!b')$.*

shape	$\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$			$\mathcal{C}$	existence
	k_1	k_2	k_3		
1^* $(r_1^2 = \mathbf{u}, \overline{\tau} = \tau - 1, \upsilon = 0)$	0	$2^{\sigma+\tau-2}$	0	$\mathcal{A}$	$\forall\tau \leq \lfloor \frac{m+1}{2} \rfloor$; $\sigma = m - \tau + 1$
1 $(r_1^2 \neq \mathbf{u}, \overline{\tau} = \tau, \upsilon = 0)$	$2^{\sigma-1}$	$(2^\tau - 1)2^{\sigma-2}$	0	$\mathcal{A}$	$\forall\tau \leq \lfloor \frac{m}{2} \rfloor$; $\sigma = m - \tau + 1$
2 $(r_1^2 = \mathbf{u} = s_1^2, \overline{\tau} = \tau - 1, \upsilon = 1)$	0	0	$2^{\sigma+\tau-2}$	$\mathcal{A} \rtimes \mathbb{Z}_4/(\mathbf{u}, s_1^2)$	$\forall\tau \leq \lfloor \frac{m}{2} \rfloor$; $\sigma = m - \tau$
3 $(r_1^2 \neq \mathbf{u} = s_1^2, \overline{\tau} = \tau, \upsilon = 1)$	0	$2^{\sigma-1}$	$(2^\tau - 1)2^{\sigma-2}$	$\mathcal{A} \rtimes \mathbb{Z}_4/(\mathbf{u}, s_1^2)$	$\forall\tau \leq \lfloor \frac{m-1}{2} \rfloor$; $\sigma = m - \tau$
4 $(r_1^2 \neq \mathbf{u} \neq s_1^2, \overline{\tau} = \tau, \upsilon = 1)$	$2^{\sigma-1}$	0	$2^{\sigma-3}$	$\mathcal{A} \rtimes \mathbb{Z}_4/(r_1^2, s_1^2)$	m even; $\tau = 1$; $\sigma = \frac{m}{2} + 1$
4^* $(r_1^2 = \mathbf{u} \neq s_1^2, \overline{\tau} = \tau - 1, \upsilon = 1)$	0	2^σ	$2^{\sigma-1}$	$\mathcal{A} \rtimes \mathbb{Z}_4/(r_2, s_1^2)$	m even; $\tau = 2$; $\sigma = \frac{m}{2} - 1$
5 $(r_1^2 = \mathbf{u}, \overline{\tau} = \tau - 1, \upsilon = 2)$	0	0	$2^{\sigma+1}$	$\mathcal{A} \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_4)/(r_1^2, s_1^2)(r_2^2, s_2^2)$	$\tau = 2$; $\sigma = m - 3$

TABLE III

EXISTENCE CONDITIONS AND PARAMETERS k_1, k_2, k_3 DEPENDING ON THE SHAPE OF HADAMARD $\mathbb{Z}_2\mathbb{Z}_4Q_8$-CODES OF LENGTH $n = 2^m$, WHERE $m = \sigma + \tau + \upsilon - 1$

Proof.

- Elements a, b are not in $T(\mathcal{C})$ so their square is not $\mathbf{e}$. Also, the construction of $\mathcal{R}(\mathcal{C})$ explicitly excludes any element with square equal to $\mathbf{u}$. The product ab is also an element of $\mathcal{R}(\mathcal{C})$), thus their square can not be $\mathbf{u}$. Moreover, if $(ab)^2 = \mathbf{e}$ then $a = bT(\mathcal{C})$ which contradicts the hypothesis. This proves the first item.
- As the elements a, b commute, we have from Lemma II.3 $n/2 = \text{wt}((ab)^2) = \text{wt}(a^2b^2) = \text{wt}(a^2) + \text{wt}(b^2) - 2\text{wt}((a\!:\!b)) = n/2 + n/2 - 2\text{wt}((a\!:\!b))$. Hence, $\text{wt}((a\!:\!b)) = n/4$. This proves the second item.
- Suppose $(a\!:\!b) = (a'\!:\!b')$. Since $\text{wt}((a\!:\!b)) = \text{wt}((a'\!:\!b')) = n/4$ there are some components (for a total weight of $n/8$) where all a, b, a', b' share an entry of order four. The rest of components of order four (for a total weight of $n/8$) in each a, b, a', b' is not shared at all, since the elements are pairwise not in the same coset of $T(\mathcal{C})$. This situation is not possible in the case where all a, b, a', b' are different, for we obtain a vector of length $5n/4$. If, without loss of generality, we suppose $b = a'$ we obtain $a^2b^2b'^2 = \mathbf{u} \in \mathcal{R}(\mathcal{C})$, a contradiction.

□

Lemma IV.2. *Let $\mathcal{C}$ be a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of shape 2 and length n with a standard set of generators $\mathcal{C} = \langle x_1 \dots x_\sigma; r_1 \dots r_\tau; s_1 \rangle$ where, by definition of shape 2, $r_1^2 = s_1^2 = [r_1, s_1] = \mathbf{u}$. Let r be and element of order four in $\mathcal{A}(\mathcal{C})$ such that $r^2 \neq \mathbf{u}$ and let $x \in T(\mathcal{C})$ be an element which is not the square of any other element of $\mathcal{C}$. If $(s_1 : r_1) = x$ then*

1) $|M((s_1\!:\!r))| = k_3/4$; $\text{wt}((s_1\!:\!r)) = n/4$ *and* $(s_1\!:\!r) \notin \mathcal{C}$.
2) *for any element $r \in \mathcal{R}(\mathcal{C})$ we have:*

a) $(s_1 : r) \neq (a \colon b)$, *for any* $a, b \in \mathcal{R}(\mathcal{C}) \backslash T(\mathcal{C})$ *such that* $ba^{-1} \in T(\mathcal{C})$.

b) $(s_1 : r) \neq (s_1 \colon a)$, *for any* $a \in \mathcal{R}(\mathcal{C}) \backslash T(\mathcal{C})$ *such that* $s_1 a^{-1} \in \langle T(\mathcal{C}), r_1 \rangle$.

Proof. Taken into account that $[r_1, r] = \mathbf{e}$ and $r_1^2 = \mathbf{u} \neq r^2$, from Lemma II.3 we have $M((s_1 : r)) = M((s_1 : r_1)) \cap M(r^2) = M(x) \cap M(r^2)$ and from Lemma II.4 $|M(x) \cap M(r^2)| = k_3/4$. This proves the first part of the first item. Since for codes of shape 2 all non-zero components of order two has binary length four, the second part of the first item comes. As all elements of $\mathcal{C}$ must have weight in $\{n, n/2, 0\}$, we conclude that $(s_1 : r) \notin \mathcal{C}$ for any r. The first item is done.

For the second item, let r_2, r_3, r_4 elements of $\mathcal{R}(\mathcal{C}) \backslash T(\mathcal{C})$, pairwise not in the same coset of $T(\mathcal{C})$. First assume $(s_1 : r_2) = (s_1 : r_3)$. In this case we have $(s_1 : r_2 r_3) = \mathbf{e} \in \mathcal{C}$ which contradicts Lemma IV.1. Now assume $(s_1 : r_2) = (r_3 : s_1)$ then, from Lemma II.5, $(s_1 : r_2 r_3) = (s_1 : r_2)(s_1 : r_3) = r_3^2 (r_3 : s_1)^2 = r_3^2 \in \mathcal{C}$, against the above item in this lemma. Finally, assume $(s_1 : r_2) = (r_3 : r_4)$. Since $M((s_1 : r_2)) = M((s_1 : r_1)) \cap M(r_2^2) = M(x) \cap M(r_2^2)$ and $M((r_3 : r_4)) = M(r_3^2) \cap M(r_4^2)$, we obtain $M(x) \cap M(r_2^2) = M(r_3^2) \cap M(r_4^2)$. The above equality means that $M(x)$, $M(r_2^2)$, $M(r_3^2)$, $M(r_4^2)$ have $k_3/4$ elements in common, while the other $k_3/4$ elements in each one of these sets are disjoint from each other. This is not possible, for a total of $5k_3/4$ components is needed for the above composition. The second item is proved. □

Now, we can enumerate a list of cases depending on τ, $\overline{\tau}$ and υ from which we obtain later, in the next section, a code with an specific dimension of the kernel and rank.

Proposition IV.3. *Let* $\mathcal{C}$ *be a subgroup of* $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ *such that* $C = \Phi(\mathcal{C})$ *is a Hadamard code generated by* $\langle \mathcal{A}(\mathcal{C}), s_1, s_\upsilon \rangle$. *The values of the dimension of the kernel and rank depends on* τ, $\overline{\tau}$ *and* υ *according to the following cases:*

1) *In the case* $\upsilon = 0$ *(*$\mathbb{Z}_2\mathbb{Z}_4$*-code) we have*
   a) *if* $\overline{\tau} \leq 1$ *(that is,* $\tau \leq 2$ *and* $r_1^2 = \mathbf{u}$ *or* $\tau \leq 1$ *and* $r_1^2 \neq \mathbf{u}$*) the code is linear and* $k = r = \sigma + \tau$*;*
   b) *if* $\tau \geq 3$ *and* $r_1^2 = \mathbf{u}$ *then* $k = \sigma + 1$, $r = \sigma + \tau + \binom{\tau-1}{2}$ *and* C *is a* $\mathbb{Z}_2\mathbb{Z}_4$*-linear code of type* $2^{\sigma-\tau}4^{\tau}$*;*
   c) *if* $\tau \geq 2$ *and* $r_1^2 \neq \mathbf{u}$ *then* $k = \sigma$, $r = \sigma + \tau + \binom{\tau}{2}$ *and* C *is a* $\mathbb{Z}_4$*-linear code of type* $2^{\sigma-\tau}4^{\tau}$.
2) *In the case* $\tau = 1, \upsilon = 1$ *we have*
   a) *if* $(s_1 : r_1) \in \mathcal{C}$ *then* C *is linear and* $k = r = \sigma + 2$*;*
   b) *if* $(s_1 : r_1) \notin \mathcal{C}$ *then* $k = \sigma$ *and* $r = \sigma + 3$.
3) *In the case* $\tau = 2, \overline{\tau} = 1, \upsilon = 1$*, consider the swappers* $(s_1 : r_1)$*,* $(s_1 : r_2)$ *and* $(s_1 : r_1 r_2)$*:*
   a) *if these three swappers are in* $\mathcal{C}$ *then* $k = r = \sigma + 3$*;*
   b) *if one of them is in* $\mathcal{C}$ *then* $k = \sigma + 1$*,* $r = \sigma + 4$*;*
   c) *if none of them is in* $\mathcal{C}$ *then* $k = \sigma, r = \sigma + 5$.
4) *In the case* $\tau \geq \overline{\tau} \geq 2, \upsilon = 1$ *we have*
   a) *if* $(bs_1 : a) \in \mathcal{C}$ *for all* $a \in \mathcal{A}(\mathcal{C})$ *and some* $b \in \mathcal{R}(\mathcal{C})$*, then* $k = \sigma + \tau - \overline{\tau} + 1$*,* $r = \sigma + \tau + 1 + \binom{\overline{\tau}}{2}$*;*
   b) *if the previous condition is not satisfied but* $(r_1 : s_1) \in \mathcal{C}$ *and* $r_1^2 = \mathbf{u}$ *then* $k = \sigma + 1$ *and* $r = \sigma + \tau + 1 + \binom{\overline{\tau}+1}{2}$*;*

c) *if none of the previous conditions is satisfied then* $k = \sigma$ *and*

$$\sigma+\tau+\upsilon+\binom{\tau-1}{2} \le r \le \sigma+\tau+\upsilon+\binom{\tau}{2}+1 \text{ when } \tau=\overline{\tau}+1, (r_1^2=\mathbf{u})$$

$$\sigma+\tau+\upsilon+\binom{\tau}{2}+1 \le r \le \sigma+\tau+\upsilon+\binom{\tau+1}{2}, \text{ when } \tau=\overline{\tau}, (r_1^2\neq\mathbf{u})$$

5) *In the case* $\tau = 2$ *and* $\upsilon = 2$*, consider the two swappers* $(r_2 : s_2)$ *and* $(r_1r_2 : s_1s_2)$*.*

a) *if both swappers are in* $\mathcal{C}$ *then* C *is linear;*

b) *if only one of the two swappers is in* $\mathcal{C}$ *then* $k = \sigma + 2$ *and* $r = \sigma + 5$

c) *if none of the two swappers is in* $\mathcal{C}$ *then* $k = \sigma$ *and* $r = \sigma + 6$

Proof.

1) In the case $\upsilon = 0$ we have that $\mathcal{C}$ is abelian and the shape is 1.
If $r_1^2 = \mathbf{u}$, any element $c \in \mathcal{C}$ can be written as $c = xr_1^i r$ with $x \in T(\mathcal{C})$, $r \in R(\mathcal{C})$ and $i \in \{0, 1\}$. The element $\phi(c)$ belongs to $K(C)$ if the swapper of c with every element in $\mathcal{C}$ is still in $\mathcal{C}$. Hence, $\phi(c) \in K(C)$ if and only if $(c : r_1) \in \mathcal{C}$ and $(c : r) \in \mathcal{C}$. From Lemma II.1 we see that $(r : r_1) = (r : r) = r^2$ and for all $\overline{r} \in R(\mathcal{C})$ with $\overline{r} \neq rT(\mathcal{C})$ we have $(r : \overline{r}) \notin \mathcal{C}$. Hence, we conclude with $K(C) = \langle T(\mathcal{C}), r_1\rangle$. From Lemma IV.1, all swappers of two elements in $R(\mathcal{C})$ are different. Therefore, if $\overline{\tau} > 1$ and $r_1^2 = \mathbf{u}$ we have $k = \sigma + 1$ and $r = \sigma + \tau + \binom{\tau-1}{2}$. With the same argumentation, if $\overline{\tau} > 1$ and $r_1^2 \neq \mathbf{u}$ we have $k = \sigma$ and $r = \sigma + \tau + \binom{\tau}{2}$. In both cases, if $\overline{\tau} \le 1$ we have $r - k \le 1$ and so the code is linear and $k = r = \sigma + \tau$.

2) The case $\tau = 1$ and $\upsilon = 1$ means that $\mathcal{C} = \langle T(\mathcal{C}), r_1, s_1\rangle$. Since the linear span of C is generated by $\mathcal{C}$ and all swappers of pairs of elements in $\mathcal{C}$ we conclude that $\langle C\rangle$ is the binary image by the Gray map of $\langle \mathcal{C}, (r_1 : s_1)\rangle$. Hence, either $(s_1 : r_1) \in \mathcal{C}$ and the code is linear or $k = \sigma$ and $r = \sigma + \tau + \upsilon + 1 = \sigma + 3$.

3) In the case $\tau = 2$ and $\upsilon = 1$ and $\overline{\tau} = 1$ the code $\mathcal{C}$ is of shape 2 with $r_1^2 = s_1^2 = \mathbf{u} \neq r_2^2$ or is of shape 4^* with $r_1^2 = \mathbf{u} \neq s_1^2 = r_2^2$. Any element $c \in \mathcal{C}$ can be written as $c = xr_1^i r_2^j s_1^k$ where $x \in T(\mathcal{C})$ and $i, j, k \in \{0, 1\}$. It belongs to $K(\mathcal{C})$ if the swapper of c with every element in $\mathcal{C}$ is still in $\mathcal{C}$. Hence, from Lemma II.1, $c \in K(C)$ if and only if $(s_1^k : r_1), (s_1^k : r_2) \in \mathcal{C}$ (recall that when $(s_1 : r_1) \in \mathcal{C}$ and $(s_1 : r_2) \in \mathcal{C}$ then also $(s_1 : r_1r_2) \in \mathcal{C}$). If all swappers above are in $\mathcal{C}$ then code C is linear and $K(C) = \langle T(\mathcal{C}), r_1, r_2, s_1\rangle$. If some swapper does not belong to $\mathcal{C}$, for instance, $(s_1 : r_1) \in \mathcal{C}$ and $(s_1 : r_2) \notin \mathcal{C}$ then $(s_1 : r_1r_2) = (s_1 : r_1)(s_1 : r_2) \notin \mathcal{C}$, hence $K(C) = \langle T(\mathcal{C}), r_1\rangle$ and $\langle C\rangle = \phi(\langle T(\mathcal{C}), r_1, r_2, s_1, (s_1 : r_2)\rangle)$. The same argumentation works for the other instances proving the statement.
If none of the swappers belong to $\mathcal{C}$ then $\langle C\rangle = \phi(\langle T(\mathcal{C}), r_1, r_2, s_1, (s_1 : r_2), (s_1 : r_1)\rangle)$ and $K(\mathcal{C}) = T(\mathcal{C})$.
Note that if $s_1^2 \neq \mathbf{u}$ then $M((r_1r_2)^2) \cap M(s_1^2) = \varnothing$, thus by Lemma II.3, $(r_1r_2 : s_1) = \mathbf{e}$.

4) In the case $\tau \ge \overline{\tau} \ge 2$ and $\upsilon = 1$ the code is of shape 2 with $r_1^2 = s_1^2 = \mathbf{u}$, or shape 3 with $s_1^2 = \mathbf{u} \notin \langle r_1^2 \ldots r_\tau^2\rangle$.
Assume $r_1^2 = \mathbf{u}$ and $(bs_1 : a) \in \mathcal{C}$ for all $a \in \mathcal{A}(\mathcal{C})$ and some $b \in \mathcal{R}(\mathcal{C})$. We can assert that $(s_1 : r_1) \in \mathcal{C}$ because, from Lemma II.1, $(bs_1 : r_1) = (b : r_1)(s_1 : r_1) = b^2(s_1 : r_1) \in \mathcal{C}$. In this way we have $K(C) = \langle T(\mathcal{C}), r_1, bs_1\rangle$ and the linear span is generated by $\mathcal{C}$ and the swappers of pairs in $\mathcal{R}(\mathcal{C})$ (from Lemma-IV.1 we known there are a total amount of $\binom{\overline{\tau}}{2}$ swappers of this kind to be included in the generator set of the

linear span). If $r_1^2 \neq \mathbf{u}$ we have $K(\mathcal{C}) = \langle T(\mathcal{C}), bs_1\rangle$ and the linear span is generated as before. Hence, $k = \sigma + 1 + \tau - \overline{\tau}$, $r = \sigma + \tau + \upsilon + \binom{\overline{\tau}}{2}$.

Assume now $r_1^2 = \mathbf{u}$ and $(s_1 : r_1) \in \mathcal{C}$ (but not the previous condition about $(bs_1 : a)$). If $(s_1 : r_1) = b^2$ where b is an element of $\mathcal{A}(\mathcal{C})$, then $(bs_1 : r_1) = (b : r_1)(s_1 : r_1) = b^2b^2 = \mathbf{e}$. So $(bs_1 : a) = \mathbf{e}$ for all $a \in \mathcal{A}(\mathcal{C})$ and some $b \in \mathcal{R}(\mathcal{C})$, and the previous condition is fulfilled, a contradiction. Thus, without lost of generality, we can assert $(s_1 : r_1) = x$ with $x \in T(\mathcal{C})$ and x is not the square of any other element of $\mathcal{C}$. In this case $K(\mathcal{C}) = \langle T(\mathcal{C}), r_1\rangle$ and the linear span of C is generated by $\mathcal{C}$, the swappers of pairs in $\mathcal{R}(\mathcal{C})$ (from Lemma-IV.1, $\binom{\overline{\tau}}{2}$ swappers), and the swappers of s_1 with the elements in $R(\mathcal{C})$ (from Lemma IV.2, $\overline{\tau}$ swappers). Hence, $k = \sigma + 1$, $r = \sigma + \tau + \upsilon + \binom{\overline{\tau}}{2} + \overline{\tau} = \sigma + \tau + \upsilon + \binom{\overline{\tau}+1}{2}$.

When $(s_1 : r_1) \notin \mathcal{C}$ and $r_1^2 = \mathbf{u}$, we have $\overline{\tau} = \tau - 1$ and $K(\mathcal{C}) = T(\mathcal{C})$. The linear span is generated by $\mathcal{C}$, the swappers of pairs in $\mathcal{R}(\mathcal{C})$ (from Lemma-IV.1, $\binom{\overline{\tau}}{2}$ swappers), the swapper $(r_1 : s_1)$ and the swappers of s_1 with elements of $\mathcal{R}(\mathcal{C})$ that do not belong to $\mathcal{C}$. Hence $k = \sigma$, $\sigma + \tau + \upsilon + \binom{\overline{\tau}}{2} \leq r \leq \sigma + \tau + \upsilon + \binom{\overline{\tau}}{2} + \tau = \sigma + \tau + \upsilon + \binom{\overline{\tau}+1}{2} + 1$.

When $(s_1 : r_1) \notin \mathcal{C}$ and $r_1^2 \neq \mathbf{u}$, we have $\tau = \overline{\tau}$. Now $K(\mathcal{C}) = T(\mathcal{C})$ and the linear span is generated by $\mathcal{C}$, the swapper of elements in $\mathcal{R}(\mathcal{C})$ and the swappers of s_1 with elements of $\mathcal{R}(\mathcal{C})$ that do not belong to $\mathcal{C}$ (at least one of these last swappers must not belong to $\mathcal{C}$, otherwise we are in the previous case 4a). Hence, $\sigma + \tau + \upsilon + \binom{\overline{\tau}}{2} + 1 \leq r \leq \sigma + \tau + \upsilon + \binom{\overline{\tau}}{2} + \tau = \sigma + \tau + \upsilon + \binom{\overline{\tau}+1}{2}$.

5) In the case $\tau = 2$ and $\upsilon = 2$ the code $\mathcal{C}$ is of shape 5 with $r_1^2 = s_1^2 = \mathbf{u} \neq r_2^2 = s_2^2$, $\mathcal{C} = \langle T(\mathcal{C}); r_1, r_2; s_1, s_2\rangle$. We have $\mathcal{C} = \langle T(\mathcal{C}); r_1, r_2; s_1, s_2\rangle = \langle T(\mathcal{C}); r_1r_2, r_2; s_1s_2, s_2\rangle$), so it is enough to analyze the swappers $(r_1r_2 : r_2)$, $(r_1r_2 : s_1s_2)$, $(r_1r_2 : s_2)$, $(r_2 : s_1s_2)$, $(r_2 : s_2)$ and $(s_1s_2 : s_2)$.

From Lemma II.1, $(r_1r_2 : r_2) = (r_1 : r_2)(r_2 : r_2) = r_2^2r_2^2 = \mathbf{e}$, and $(s_1s_2 : s_2) =(s_1 : s_2)(s_2 : s_2) = s_2^2s_2^2 = \mathbf{e}$. Moreover, we have $(r_1r_2)^2 = (s_1s_2)^2 = \mathbf{u}r_2^2 = \mathbf{u}s_2^2$, so $\mathrm{Supp}((r_1r_2)^2)$ is disjoint from $\mathrm{Supp}(s_2^2)$ and also from $\mathrm{Supp}(r_2^2)$. The same for $\mathrm{Supp}((s_1s_2)^2)$, which is disjoint from $\mathrm{Supp}(r_2^2)$ and $\mathrm{Supp}(s_2^2)$. Hence, $(r_1r_2 : s_2) = (s_1s_2 : r_2) = (r_1r_2 : r_2) = (s_1s_2 : s_2) = \mathbf{e}$. Now, only swappers $(r_1r_2 : s_1s_2)$ and $(r_2 : s_2)$ are left in our analysis. When $(r_1r_2 : s_1s_2) \in \mathcal{C}$ and $(r_2 : s_2) \in \mathcal{C}$ we have the linear case $K(C) = S(C) = \mathcal{C}$; if only one of these swappers belongs to $\mathcal{C}$ we have the case where $k = \sigma + 2$ and the linear span is generated by $\mathcal{C}$ and the swapper that does not belong to $\mathcal{C}$. Finally, if none of these two swappers belongs to $\mathcal{C}$ we have $K(\mathcal{C}) = T(\mathcal{C})$ and the linear span is generated by $\mathcal{C}$ and the two swappers.

□

From the above results in Proposition IV.3 we can specify a little more for what shapes we obtain the values for the rank and dimension of the kernel. Tables IV and V are a summary. We write in the right top corner of each value the corresponding shape and, separate by a colon, the item of Proposition IV.3 where the case is studied. Note that we do not include the additive cases which were studied in [3]. Table IV covers all items in Proposition IV.3, except 4) and Table V covers the codes in item 4) of Proposition IV.3.

$m+1-k$	$r-(m+1)$		
4	–	–	$2^{[5;5c]}$
3	–	$2^{[2,4^*;3c]}$	–
2	$1^{[2,3,4;2b]}$	$1^{[2,4^*;3b]}$	$1^{[5;5b]}$
0	$0^{[2,4^*;2a]}$	$0^{[2,4^*;3a]}$	$0^{[5;5a]}$

TABLE IV

RANK AND DIMENSION OF THE KERNEL FOR THE CODES FULFILLING PROPOSITION IV.3, EXCEPT ITEM 4)

$m+1-k$	$r-(m+1)$	
$\tau+1$	$\binom{\tau-1}{2}\cdots\binom{\tau}{2}+1^{[2;4c]}$	$\binom{\tau}{2}+1\cdots\binom{\tau+1}{2}^{[3;4c]}$
τ	$\binom{\tau}{2}^{[2;4b]}$	$\binom{\tau}{2}^{[3;4a]}$
$\tau-1$	$\binom{\tau-1}{2}^{[2;4a]}$	–

TABLE V

RANK AND DIMENSION OF THE KERNEL FOR THE CODES FULFILLING PROPOSITION IV.3, ITEM 4)

V. CONSTRUCTION OF HADAMARD $\mathbb{Z}_2\mathbb{Z}_4Q_8$-CODES

In this section it is shown how to construct Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes with each allowable pair of values for the rank and the dimension of the kernel. We follow the entries of Proposition IV.3 explaining, in each case, how to construct the desired Hadamard code. After the constructions, as a summary, we include Theorem V.2, where it is described what are the allowable parameters for the dimension of the kernel and, for each one of these values, the corresponding range of values for the rank. For each one of the possible pair of allowable values for the dimension of the kernel and rank, we construct a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code fulfilling it. As an illustration of the constructions we include two examples at the end of the section.

We can take as starting point a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-code $\mathcal{D}$ of type $2^{\sigma-\tau}4^{\tau}$, which can be constructed using the methods described in [9], [1]. Recall that an element x with square equal to $\mathbf{u}$ is included in $\mathcal{D}$ if and only if $\mathcal{D}$ is a $\mathbb{Z}_4$-code.

We define three basic homomorphisms:

$$\begin{aligned} \chi_1: &\quad \mathbb{Z}_2 \to \mathbb{Z}_4 &&\text{such that } \chi_1(x)=2x,\\ \chi_2: &\quad \mathbb{Z}_4 \to Q_8 &&\text{such that } \chi_2(x)=\mathbf{a}^x,\\ \chi_3: &\quad A \to A\times A &&\text{such that } \chi_3(x)=(x,x), \text{ where } A\in\{\mathbb{Z}_2,\mathbb{Z}_4\}. \end{aligned} \tag{1}$$

The next theorem is one of the main results in the current paper. For a given pair of allowable parameters r, k we construct a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code with these parameters.

Theorem V.1. *Let C a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of length 2^m and $|T(\mathcal{C})|=2^{\sigma}$, where $T(\mathcal{C})$ is the subgroup of elements in $\mathcal{C}$ of order two; $|\mathcal{C}/\mathcal{A}(\mathcal{C})|=2^{\upsilon}$; $|\mathcal{A}(\mathcal{C})/T(\mathcal{C})|=2^{\tau}$; $|\mathcal{C}/T(\mathcal{C})|=2^{\tau+\upsilon}$ and $m+1=\sigma+\tau+\upsilon$. Then, for any two allowable values of the rank r and dimension of the kernel k (Proposition IV.3) of a putative Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code, we construct this code.*

Proof. First, we describe the elementary constructions based on the previously defined homomorphisms (1). After that, for each of all possibilities given in Lemma IV.3, we show how to construct the putative code.

The first step is the construction of the subgroup $\mathcal{A}(\mathcal{C})$ applying (1) to some Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-code $\mathcal{D}$, of type $2^{\sigma-\tau}4^{\tau}$, which can be constructed using the methods described in [9], [1]. After obtaining this code $\mathcal{D}$, from Theorem III.4 we will duplicated it (or quatruplicate it). One of the following ways must be used:

- From Table III we know that codes of shape 1 or shape 1^* have $k_3 = 0$, so they are Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-codes, $\mathcal{C} = \mathcal{A}(\mathcal{C}) = \mathcal{D}$.
- In a code of shape 2, from Table III we have that $k_1 = k_2 = 0$. Also $r_1^2 = \mathbf{u}$ and $\mathcal{D}$ is a $\mathbb{Z}_4$-linear Hadamard code. So $\mathcal{A}(\mathcal{C})$ can be obtained applying χ_2 component-wise to $\mathcal{D}$.
- In a code of shape 3, since $r_1^2 \neq \mathbf{u}$, $\mathcal{D}$ must be a $\mathbb{Z}_2\mathbb{Z}_4$-code. Moreover, from Table III, $k_1 = 0$ and $\mathcal{A}(\mathcal{C})$ can be obtained applying χ_1 component-wise to $\mathbb{Z}_2$ components of $\mathcal{D}$ and χ_2 to $\mathbb{Z}_4$ ones. This means that the rate between the number of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ components in a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-code is extended to theses codes of shape 3.
- In a code of shape 4, since $r_1^2 \neq \mathbf{u}$ we have that $\mathcal{D}$ must be a $\mathbb{Z}_2\mathbb{Z}_4$-code. As $\tau = 1$, this code must be of type $2^{2k}4^k$. From Table III $k_2 = 0$, so $\mathcal{A}(\mathcal{C})$ can be obtained applying χ_3 component-wise to $\mathbb{Z}_2$ components of $\mathcal{D}$ and χ_2 to the $\mathbb{Z}_4$ components.
- In a code of shape 4^*, since $r_1^2 = \mathbf{u}$ we have that $\mathcal{D}$ must be a $\mathbb{Z}_4$-code. $\mathcal{A}(\mathcal{C})$ can be obtained applying χ_3 component-wise to half of the $\mathbb{Z}_4$ components of $\mathcal{D}$ and χ_2 to the rest of components.
- In a code of shape 5, since $r_1^2 = \mathbf{u}$, $\mathcal{D}$ must be a $\mathbb{Z}_4$-code. From Table III, $k_1 = k_2 = 0$, so $\mathcal{A}(\mathcal{C})$ can be obtained applying χ_3 component-wise to all $\mathbb{Z}_4$ components of $\mathcal{D}$ followed by χ_2 to obtain the desired $\mathcal{A}(\mathcal{C})$.

The code $\mathcal{C}$ is obtained adding one generator (two in case of shape 5) to $\mathcal{A}(\mathcal{C})$. The components of these added generators must follow some restrictions to be sure that a Hadamard code is obtained. In addition, several values remains free of choice. These values are used to select the target rank and dimension of the kernel.

Specifically, we follow the list of all possibilities given in Proposition IV.3 and, for each one of them, we show how to construct the putative code.

For codes in item 2) of Proposition IV.3 we can construct a code of shape 2 (it is also possible to construct codes of shape 3 or shape 4). Code $\mathcal{D}$ must be in $\mathbb{Z}_4^{m-2}$ with $r_1^2 = \mathbf{u}$, length 2^{m-1}, $\sigma = m-1$ and $\tau = 1$. This code is of type $2^{\sigma-1}4^1$ [9]. Now, we add one new generator s_1 to $\mathcal{A}(\mathcal{C})$ with its components in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ or $\{\mathbf{ab}, \mathbf{a}^3\mathbf{b}\}$ obtaining a code $\mathcal{C}$ with the following generator matrix.

$$\mathcal{C} = \left(\begin{array}{ccc} \mathcal{A}(\mathcal{C}) & = & \chi_2(\mathcal{D}) \\ \hline s_1^{(1)} & \dots & s_1^{(2^{m-2})} \end{array} \right)$$

If we take all $s_1^{(i)}$ components in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ then $(s_1 : r_1) \in \mathcal{C}$ then $\mathcal{C}$ is in the subcase 2a) of Proposition IV.3 where it is proven that the obtained code is linear. If we fill one of the components of s_1 with one value in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ and the remainder ones with values in $\{\mathbf{ab}, \mathbf{a}^3\mathbf{b}\}$ then $(r_1 : s_1) \notin \mathcal{C}$ (if $m > 3$), we obtain the subcase 2b) where it is proven that $k = \sigma$, $r = \sigma + 3$.

For codes in item 3) of Proposition IV.3 the shape is 3. We begin by taking a Hadamard $\mathbb{Z}_4$-linear code $\mathcal{D}$ with $r_1^2 = \mathbf{u}$ and length 2^{m-1}, $\tau = 2$ and $\sigma = m-2$. This code is of type $2^{\sigma-2}4^2$ and the number of $\mathbb{Z}_4$ components is 2^{m-2} [9]. If s_1 is constructed with all components in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ then all swappers of s_1 with elements of $\mathcal{A}(\mathcal{C})$ are in $\mathcal{C}$ and we have a linear code according to sub-case 3a) of Proposition IV.3. If s_1 takes values in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ for all components where r_2 has order four, plus one more component (which can be randomly selected), and we take the rest of the components of s_1 in $\{\mathbf{ab}, \mathbf{a}^3\mathbf{b}\}$, then $(r_1 : s_1) \notin \mathcal{C}$, $(r_2 : s_1) \in \mathcal{C}$ and we are fulfilling item 3b) of Proposition IV.3, where it is proven that $k = \sigma + 1$ and $r = \sigma + 4$. Finally, if we fill one of the components of s_1 with a value in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ and the remainder components with values in $\{\mathbf{ab}, \mathbf{a}^3\mathbf{b}\}$ then $(r_1 : s_1) \notin \mathcal{C}$, $(r_2 : s_1) \notin \mathcal{C}$, $(r_1 r_2 : s_1) \notin \mathcal{C}$ and we obtain the subcase 3c) of Proposition IV.3, where it is proven that $k = \sigma$, $r = \sigma + 5$.

For codes in item 4) of Proposition IV.3 the shape is 2 or 3. We start by taking a Hadamard $\mathbb{Z}_4$-linear code $\mathcal{D}$ with $r_1^2 = \mathbf{u}$ and length 2^{m-1}, $\sigma + \tau = m-2$. This code is of type $2^{\sigma-\tau}4^{\tau}$ [9]. If we select all components of s_1 in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ then $(s_1 : r_1) = \mathbf{e}$, we are in the case 4a), where it is proven that $k = \sigma + \tau - \overline{\tau} + 1$, $r = \sigma + \tau + 1 + \binom{\overline{\tau}}{2}$. If we select an element $x \in T(\mathcal{C})$ which is not the square of any other element in $\mathcal{A}(\mathcal{C})$ and we replace its zero components by $\mathbf{b}$ and the components $\mathbf{a}^2$ by $\mathbf{ab}$, then $(s_1 : r_1) = x$ and so we are in the case 4b) where it is proven that $k = \sigma + 1$ and $r = \sigma + \tau + 1 + \binom{\overline{\tau}+1}{2}$. To reach the upper bound for the rank in the case 4c), $k = \sigma$ and $r = \sigma + \tau + 1 + \binom{\tau}{2} + 1$, we need the following construction.

Split all components in two sets taken into account if the value of r_2 either has order four or not. Split each one of these two sets according if the value of r_3 is either of order four or not. Repeat the process again and again for each r_i until r_τ. Since $\langle r_1^2, r_2^2, \ldots, r_\tau^2 \rangle$ is a linear subspace of the Hadamard code we obtain $2^{\tau-1}$ sets with $k_3/2^{\tau-1}$ components in each one. Now, construct the element s_1 with the value $\mathbf{b}$ in all components, except for one component in each one of the previous sets, where we put the value $\mathbf{ab}$. In this way $(s_1, r_i) \notin \mathcal{C}$ for any $2 \geq i \geq \tau$, which assure to obtain the maximum rank $\sigma + \tau + 1 + \binom{\tau-1}{2} + \tau = \sigma + \tau + 1 + \binom{\tau}{2} + 1$.

A value of the rank equal to one less than the above maximum can be reached if in the constructed s_1 we put the value $\mathbf{b}$ in all components where r_2 has order four. Repetitively, we can decrease by one the value of the previous rank by putting the value $\mathbf{b}$ in all components where r_2 or r_3 has order four and so on. The lower rank we obtain is $\sigma + \tau + 1 + \binom{\tau-1}{2} + 1$. To obtain the lower limit for the rank, $r = \sigma + \tau + 1 + \binom{\tau-1}{2}$, we take the value $\mathbf{b}$ in a component of s_1 if some of the generators $r_2 \ldots r_\tau$ has the respective component of order four. Otherwise the value $\mathbf{ab}$. Example V.3 shows these constructions.

Now, we deal with codes in item 4, c) of Proposition IV.3, of shape 3. The starting point is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\mathcal{D}$ with $r_1^2 \neq \mathbf{u}$ and length 2^{m-1}, $\sigma + \tau = m-2$. This code is of type $2^{\sigma-\tau}4^{\tau}$ and has $2^{\sigma-1}$ binary components and $(2^{\tau} - 1)2^{\sigma-2}$ quaternary components [9]. After applying χ_1, χ_2 to the binary and quaternary components, respectively, we define s_1 taking in all the quaternary components de value 1 and using the same technique as before, splitting the quaternionic components, to decide the values in these components. We obtain $(2^{\tau} - 1)$ sets with $2^{\sigma-2}$ components in each one. The maximum rank we obtain is $\sigma + \tau + 1 + \binom{\tau}{2} + \tau = \sigma + \tau + 1 + \binom{\tau+1}{2}$. The minimum is not as before, but $\sigma + \tau + \upsilon + \binom{\tau}{2} + 1$. Indeed, when the rank is $\sigma + \tau + \upsilon + \binom{\tau}{2}$, the constructed s_1 belongs to the kernel and so $k = \sigma + 1$ (this corresponds to the case 4a). Example V.4 shows these constructions.

For codes in item 5) of Proposition IV.3 we must construct a code of shape 5. We begin by taking a Hadamard

$\mathbb{Z}_4$-linear code $\mathcal{D}$ with $r_1^2 = \mathbf{u}$, length 2^{m-2}, $\tau = 2$ and $\sigma = m-4$. This code is of type $2^{\sigma-3}4^2$ and the number of quaternary components is 2^{m-3} [9]. Now, as we said before, we can obtain $\mathcal{A}(\mathcal{C})$ as $\chi_2(\chi_3(\mathcal{D}))$ and $\mathcal{C} = \langle \mathcal{A}(\mathcal{C}), s_1, s_2 \rangle$. The following matrix is a generator matrix for C:

$$\left(\begin{array}{cccccccccc} \multicolumn{10}{c}{\mathcal{A}(\mathcal{C}) = \chi_2(\chi_3(\mathcal{D}))} \\ \hline s_1^{(1)} & & \dots & & s_1^{(2^{m-3})} & 1 & \mathbf{a}^2 & \overset{2^{m-3}}{\dots} & 1 & \mathbf{a}^2 \\ 1 & \mathbf{a}^2 & \overset{2^{m-3}}{\dots} & 1 & \mathbf{a}^2 & s_2^{(2^{m-3}+1)} & & \dots & & s_2^{(2^{m-2})} \end{array}\right)$$

If all components of order four of s_1 and s_2 are in $\{\mathbf{b}, \mathbf{a}^2\mathbf{b}\}$ then all swappers of s_1 and s_2 with elements of $\mathcal{A}(\mathcal{C})$ are in $\mathcal{C}$, so we have the linear case according to item 5a) of Proposition IV.3. Say that the first component of r_1 is of order four, but the first component of r_2 is of order at most two. Take $s_1^{(1)} = \mathbf{ab}$ and the rest of components of order four of s_1 and s_2 are equal to $\mathbf{b}$ then $(s_1 s_2 : r_1 r_2) \notin \mathcal{C}$, $(s_2 : r_2) \in \mathcal{C}$ and item 5b) of Proposition IV.3 is fulfilled, reaching a code with $k = \sigma + 2$ and $r = \sigma + 5$. Finally, if $s_1^{(1)} = s_2^{(2^{m-3}+1)} = \mathbf{ab}$ and the rest of components of order four of s_1 and s_2 are equal to $\mathbf{b}$ then $(s_1 s_2 : r_1 r_2) \notin \mathcal{C}$, $(s_2 : r_2) \notin \mathcal{C}$ and item 5c) of Proposition IV.3 is fulfilled, reaching a code with $k = \sigma$ and $r = \sigma + 6$. $\square$

For a generic Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code, the range of values for the rank as well as the range of values given by the dimension of the kernel depends on the specific shape of the code. However, summing up the Proposition IV.3 and all results in this section about constructions, we can establish a tight upper and lower bound for the values of the rank and dimension of the kernel for Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. The next theorem gives these bounds, which improve the ones previously given in [3]. Further, in this section we have given constructions of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes covering all allowable values for the pair rank, dimension of the kernel.

Theorem V.2. *Let C a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of length 2^m and $|T(\mathcal{C})| = 2^\sigma$, where $T(\mathcal{C})$ is the subgroup of elements in $\mathcal{C}$ of order two; $|\mathcal{C}/\mathcal{A}(\mathcal{C})| = 2^\upsilon$; $|\mathcal{A}(\mathcal{C})/T(\mathcal{C})| = 2^\tau$; $|\mathcal{R}(\mathcal{C})/T(\mathcal{C})| = 2^{\overline{\tau}}$; $|\mathcal{C}/T(\mathcal{C})| = 2^{\tau+\upsilon}$ and $m+1 = \sigma+\tau+\upsilon$. Then the rank r and the dimension of the kernel k of C satisfy the following conditions.*

1) *The values of the dimension of the kernel are $1 \neq m+1-k \in \{0, 4, \tau-1, \tau, \tau+1\}$. The specific case $m+1-k = 0$ is obtained in codes where $\overline{\tau} \leq 1$ or in codes of shape 5. The specific case $m+1-k = 4$ is obtained in codes of shape 5.*
2) a) *If $m+1-k = 0$ then we have $r-(m+1) = 0$,*
   b) *If $m+1-k = 4$ and $\upsilon = 2$ then we have $r-(m+1) = 2$,*
   c) *If $m+1-k = \tau-1 \geq 2$ then we have $r-(m+1) = \binom{\tau-1}{2}$,*
   d) *If $m+1-k = \tau \geq 2$ then we have $r-(m+1) = \binom{\tau}{2}$,*
   e) *If $m+1-k = \tau+1$ and $\overline{\tau} <= 1$ then we have $r-(m+1) = \tau$.*
   f) *If $m+1-k = \tau+1$ and $\overline{\tau} = \tau-1 \geq 2$ then we have $r-(m+1) \in \{\binom{\tau-1}{2}, \dots \binom{\tau}{2}+1\}$.*
   g) *If $m+1-k = \tau+1$ and $\overline{\tau} = \tau \geq 2$ then we have $r-(m+1) \in \{\binom{\tau}{2}+1, \dots \binom{\tau+1}{2}\}$.*

Example V.3. The following example shows constructions of codes of length $n = 2^m = 2^7 = 128$, with $\tau = 3 \geq \overline{\tau} = 2 \geq 2, \upsilon = 1$ (item 4 of Proposition IV.3) and $\sigma = 4$. The resulting codes are of shape 2 and, before the Gray map, subgroups of Q_8^{32}. All possible pairs of rank and dimension of the kernel are presented:

Let $\overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{x_1} \in Q_8^{16}$ be the vectors:

$$\begin{aligned}
\overline{r_1} &= (\mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a})\\
\overline{r_2} &= (\mathbf{a} \quad \mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{a}^3 \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{a}^3 \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{a}^2)\\
\overline{r_3} &= (\mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{a} \quad \mathbf{a}^3 \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{1} \quad \mathbf{a}^2)\\
\overline{x_1} &= (\mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1})\\
\overline{x_2} &= (\mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2 \quad \mathbf{a}^2)
\end{aligned}$$

and now, take the following vectors in Q_8^{32}:

$$\begin{aligned}
r_1 &= (\overline{r_1}, \overline{r_1})\\
r_2 &= (\overline{r_2}, \overline{r_2})\\
r_3 &= (\overline{r_3}, \overline{r_3})\\
x_1 &= (\overline{x_1}, \overline{x_2})
\end{aligned}$$

Let $y_1, y_2, y_3, y_4, y_5, y_6 \in Q_8^{16}$ be the vectors:

$$\begin{aligned}
y_1 &= (\mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b})\\
y_2 &= (\mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab})\\
y_3 &= (\mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab})\\
y_4 &= (\mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab})\\
y_5 &= (\mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab})\\
y_6 &= (\mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{b} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab} \quad \mathbf{ab})
\end{aligned}$$

The codes with all possible pairs of values rank,dimension of the kernel are generated by r_1, r_2, r_3 and s_1 which is taken following Theorem V.1. We show the vector s_1 and the values of the pair rank, dimension of the kernel.

When $s_1 = (y_1, y_1)$ the constructed code has $k = 6, r = 9$.

When $s_1 = (y_2, y_1)$ the constructed code has $k = 5, r = 11$.

When $s_1 = (y_3, y_3)$ the constructed code has $k = 4, r = 12$.

When $s_1 = (y_4, y_4)$ the constructed code has $k = 4, r = 11$.

When $s_1 = (y_5, y_5)$ the constructed code has $k = 4, r = 10$.

When $s_1 = (y_6, y_6)$ the constructed code has $k = 4, r = 9$.

Example V.4. The following example shows constructions of codes of length 32, with $\tau = \overline{\tau} = 2$, $\upsilon = 1$ (item 4 of Proposition IV.3) and $\sigma = 3$. The resulting codes are of shape 3 and, before the Gray map, subgroups of $\mathbb{Z}_4^4 Q_8^6$. All possible pairs of rank and dimension of the kernel are presented.

Take the following vectors in $\mathbb{Z}_4^4 Q_8^6$:

$$\begin{aligned}
r_1 &= (0 \quad 2 \quad 0 \quad 2 \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{a})\\
r_2 &= (0 \quad 0 \quad 2 \quad 2 \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{1} \quad \mathbf{a}^2 \quad \mathbf{a} \quad \mathbf{a}^3)
\end{aligned}$$

The codes with all possible pairs of values rank, dimension of the kernel are generated by r_1, r_2 and s_1 which is taken following Theorem V.1. We show the vector s_1 and the values of the pair rank, dimension of the kernel.

When $s_1 = (1, 1, 1, 1, \mathbf{b}, \mathbf{b}, \mathbf{b}, \mathbf{b}, \mathbf{b}, \mathbf{b})$ the constructed code has $k = 4, r = 7$.

When $s_1 = (1, 1, 1, 1, \mathbf{b}, \mathbf{ab}, \mathbf{b}, \mathbf{ab}, \mathbf{b}, \mathbf{ab})$ the constructed code has $k = 3, r = 9$.

When $s_1 = (1, 1, 1, 1, \mathbf{b}, \mathbf{ab}, \mathbf{b}, \mathbf{b}, \mathbf{b}, \mathbf{b})$ the constructed code has $k = 3, r = 8$.

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