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Abstract

We study random assignment economies with expected-utility agents, each of them eventually obtaining a single object. Inspired on Hylland and Zeckhauser's (1979) Pseudomarket mechanism (PM) and on a serial dictatorship, we introduce the Sequential Pseudomarket (SP) where groups of agents are called turn by turn and participate in a pseudomarket for the remaining objects. To measure efficiency, we focus on the set of ex-ante Pareto-optimal (PO) random assignments and on the ex-ante weak core (CO). We find: 1) PM $\subset$ PO but the converse is not always true (there is no Second Theorem of Welfare Economics), 2) PO $\subset$ SP but the converse is not always true, 3) SP $\subset$ CO

Keywords: Random assignment, ex-ante efficiency, weak core, sequential pseudomarket
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1 Introduction

We study random assignment economies. In a random assignment, each agent is provided with a probability distribution over the set of object types. Agents have preferences over their assigned distributions according to the expected utility form. No monetary transfers

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are allowed. Hylland and Zeckhauser's (1979) seminal paper suggests that a pseudomarket can be constructed in which each agent is endowed by some artificial income with which she can buy assignment probabilities. Each object type is given a nonnegative price and each agent buys a proper probability distribution (probabilities add up to 1) over them. Given the endowment vector, there is at least one equilibrium price vector yielding a feasible random assignment as an outcome. Moreover, this random assignment is ex-ante Pareto-efficient, in a sort of First Theorem of Welfare Economics for random assignment economies.

In this note, we prove that there is *no Second Theorem of Welfare Economics* for random assignment economies. That is, if a random assignment is ex-ante Pareto-optimal, there might not be a vector of artificial incomes and a vector of non-zero prices such that a pseudomarket equilibrium exists yielding that random assignment as its outcome.

Proving that a quasiequilibrium exists (where an agent's preferred probability distribution cannot be cheaper) that yields the ex-ante Pareto-optimal assignment follows an almost standard technique. But a quasiequilibrium is not always an equilibrium. We construct an example in which the only quasiequilibrium (up to affine transformations) is not an equilibrium. For one agent, the assigned object is the cheapest one. She prefers another object that should be more expensive in equilibrium. Yet the preferences of the remaining agents force the latter two prices to be equal.

We next present the *Sequential Pseudomarket* (SP) mechanism. Groups of agents are called in turns that attend the pseudomarket for the remaining objects. A SP-equilibrium is a sequence of pseudomarket equilibria turn by turn. It is easy to see that SP is a combination of serial dictatorship and pseudomarkets. We show that every ex-ante Pareto-optimal random assignment can be the outcome of a SP-equilibrium. However, not every SP-equilibrium yields an ex-ante Pareto-optimal random assignment. First comers to the pseudomarket may be indifferent about remaining objects after they leave while followers might not be indifferent at all.

Next, we define the ex-ante weak core as the set that contains all feasible random assignments such that, for each one, there is no coalition of agents and redistribution of probabilities across them in which all agents in the coalition are ex-ante strictly better-off. We finally show that the ex-ante weak core contains the set of random assignments generated by SP-equilibria.

The closest reference to this note is Mas-Colell's (1992) equilibrium theory approach to the assignment of indivisible goods. He provides a general result for every kind of assignment economy with possibly satiated preferences. For any ex-ante Pareto-optimal allocation, he shows the existence of a Walrasian equilibrium with slack supporting it.

More than thirty years after the seminal paper by Hylland and Zeckhauser, pseudomarkets are attracting increasing interest both in finite and continuum economies.[1] Examples of recent papers are Budish and Azevedo (2012) on strategy-proofness in the large that applies to pseudomarkets, Budish, Che, Kojima and Milgrom (2012) on pseudomarket mechanisms for multidimensional assignment, and He, Miralles and Yan (2012) on reconciling discriminatory priorities with pseudomarket mechanisms. The present note contributes to this recent literature in that it clarifies the relation among the sets of pseudomarket equilibrium outcomes, ex-ante Pareto-optimal random assignments, sequential pseudomarket equilibrium outcomes and the weak core. Each one is a subset of the next, while the opposite may not be true.

2 The model

In this economy there is a finite set of agents $N = \{1, ..., n\}$. The notation $x, y...$ is used for a generic element of N. There is a set of object types $S = \{1, ..., s\}$. The notation $i, j...$ serves to indicate a generic element of S. For each object type j there is a number of identical copies $\eta^j \in \mathbb{N}$. $\eta = (\eta^1, ..., \eta^s)$ is the supply in this economy. We have enough supply in the sense that $\sum_{j\in S} \eta^j \geq n$.[2]

A random assignment is a $n \times s$ matrix Q whose generic element $q_x^j \geq 0$ is the probability that agent x obtains a copy of object type j. This matrix is stochastic: $\sum_{j\in S} q_x^j = 1$ for any $x \in N$. Agent x's random assignment is the probability distribution $q_x = (q_x^1, ..., q_x^s) \in \Delta^s$ (Δ^s is the $s-1$ dimensional simplex). A random assignment is a pure assignment if each of its elements is either 1 or 0. A random assignment is feasible if $Q' \cdot 1_n \leq \eta$ (where 1_n is a vector of n ones and $'$ denotes the transpose of a matrix). A feasible random assignment can be expressed as a lottery over feasible pure assignments.

Let $V \in \mathbb{R}_+^{ns}$ denote a $n \times s$ matrix of nonnegative von Neumann-Morgenstein valuations, whose generic element v_x^j indicates agent x's valuation for object type j. A generic agent x's valuation vector is $v_x = (v_x^1, ..., v_x^s)$. She values her random assignment q_x as the vectorial product $u_x(q_x) = v_x \cdot q_x$. Each agent x has a set of most-preferred object types $M_x = \arg\max_{j\in S} v_x^j$. An economy is a triple $E = (N, \eta, V)$.

Let $\mathcal{F}_E$ denote the set of feasible random assignments in an economy E. A feasible random assignment Q^{PO} is *ex-ante Pareto-optimal*

[1] See Thomson and Zhou (1993) for a result on efficient and fair allocations in continuum economies.

[2] Notice that the weak inequality allows for an easy inclusion of an outside option for every agent.

at an economy E if for any random assignment Q, $diag(VQ') > diag(VQ'_{PO}) \implies Q \notin \mathcal{F}_E$. ($diag$ denotes the diagonal of a matrix, and $>$ indicates that the inequality is strict for at least one element).

Considering a feasible random assignment Q^{CO}, let a blocking coalition $C \subset N$ be defined as follows: $\exists Q$ such that a) $q_x^{CO} \cdot v_x < q_x \cdot v_x$ for all $x \in C$ and b) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \setminus C} q_x^{CO}$. A feasible random assignment Q^{CO} belongs to the *weak core* of an economy E if its unique blocking coalition is $C = \varnothing$.

A price vector is notated as $P \in \mathbb{R}_+^s$. A price vector P^* constitutes a pseudomarket *quasiequilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \implies P^* \cdot q_x \geq P^* \cdot q_x^*$. A price vector P^* constitutes a pseudomarket *equilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \implies P^* \cdot q_x > P^* \cdot q_x^*$.

3 No Second Welfare Theorem

Theorem 1 *For a finite economy E, let Q^* be an ex-ante Pareto-optimal random assignment. There may not exist a price vector P^* that constitutes a pseudomarket equilibrium for this economy E with associated random assignment Q^*.*

Proof. Consider an example with $N = \{x_1, x_2, y_1, y_2, z\}$, object types $S = \{1, ..., 4\}$, supply vector $\eta = (1, 1, 1, 2)$, and preferences as follows: $v_x = (3, 2, 1, 0)$; $v_y = (3, 2, 0, 1)$; $v_z = (0, 0, 2, 1)$ (agents x_1 and x_2 have identical preferences v_x; likewise for y-type agents). Consider the feasible random assignment Q such that: $q_x = (1/4, 1/4, 1/2, 0)$; $q_y = (1/4, 1/4, 0, 1/2)$; $q_z = (0, 0, 0, 1)$ (agents x_1 and x_2 have identical assignment q_x; likewise for y-type agents).

Q is ex-ante Pareto-optimal. z would be willing to give $\alpha \in (0, 1)$ units of object type 4 in exchange of no less than $\alpha/2$ units of object type 3 (and $1 - \alpha/2$ units of the other object types). Yet no $x-$type nor $y-$type agent (nor any coalition of them) would accept any such trade. There is no mutually beneficial trade among $x-$type and $y-$type agents either. For them object types 3 and 4 could be regarded as a unique object type 3' from which each agent picks half a unit. All these agents have identical preferences (and identical random assignments) over object types 1, 2 and 3'.

We show that there is no pseudomarket equilibrium supporting Q. Since q_x is interior for three object types, the equilibrium prices for these objects must be an affine transformation of the $x-$agents' preferences over these objects (otherwise some strictly preferred distribution would be cheaper). Then: $P^1 = a_x + 3b_x$; $P^2 = a_x + 2b_x$; $P^3 = a_x + b_x$, with $a_x \in \mathbb{R}, b_x > 0$. Analogously for $y-$type agents:

$P^1 = a_y + 3b_y$; $P^2 = a_y + 2b_y$; $P^4 = a_y + b_y$, with $a_y \in \mathbb{R}, b_y > 0$. From the prices for object types 1 and 2 we deduce that $b_x = b_y$ and $a_x = a_y$. Hence $P^3 = P^4$. But this cannot be an equilibrium price vector since z strictly prefers object type 3 to her assignment and it is not more expensive. ■

We can see that the price vector in this counterexample constitutes a quasiequilibrium with Q as associated random assignment, for agent z does not have a more-preferred distribution that is at the same time cheaper than the assigned distribution. Indeed, it can be shown that any ex-ante Pareto-optimal random assignment can be supported by a quasiequilibrium price vector. The existence of a quasiequilibrium does not provide much information, though. For instance, a vector with equal prices across object types is always a quasiequilibrium. For this reason, we need to provide a different characterization of ex-ante Pareto-optima.

We go back to the counterexample. We can observe that the quasiequilibrium price vector was fixed by $x-$type and $y-$type agents' preferences. In a reduced economy without z, this vector would indeed constitute a pseudomarket equilibrium. It is as if z was "waiting for her turn" and picking the remaining object, a copy of object type 4, after the other agents had received their pseudomarket equilibrium random assignments.

4 Sequential pseudomarkets

Let the set N be partitioned into disjoint ordered sets $N_1, ..., N_\pi$ with $\pi \leq n$. Start with a reduced economy with N_1 on the demand side and $\eta_1 = \eta$ as the supply side. Calculate a pseudomarket equilibrium allocation Q_1^* for this reduced economy. For $t = 2, .., s$, calculate the remaining supply $\eta_t = \eta_{t-1} - Q_{t-1}^{*\prime} \cdot 1_{|N_{t-1}|}$ and use N_t on the demand side to calculate a new pseudomarket equilibrium allocation $Q_t^*$for the agents in N_t. The vertical composite matrix $Q^* = [Q_1^*, ..., Q_\pi^*]$ constitutes a Sequential Pseudomarket (SP) equilibrium random assignment given the ordered partition $N_1, ..., N_\pi$.[3]

When $\pi = n$ we have a Serial Dictatorship, whereas on the other extreme we have a Pseudomarket equilibrium outcome if $\pi = 1$. SP is indeed a combination of these two mechanisms.

The following result states that any ex ante Pareto-optimal assignment can be supported by a SP-equilibrium.

Theorem 2 *For a finite economy E, let Q^* be an ex-ante Pareto-optimal random assignment. Then there is an ordered partition*

[3] Agents could be WLOG labeled in a way that the matrices Q^* and V are consistent (i.e. each row refers to the same agent in both matrices).

$N_1, ..., N_\pi$ *of the set* N *such that* Q^* *is a Sequential Pseudomarket equilibrium random assignment given the ordered partition* $N_1, ..., N_\pi$.

Proof. It follows a recursive argument. We explain the first iteration, which is afterwards repeated with the "continuation economy" (we define it below) until all agents are removed. We start this iteration by considering a reduced economy $E^r = (N^r, \eta^r, V^r)$ that is resulting from removing all agents x who obtain a most-preferred object type: $N^M = \{x \in N : \sum_{j \in M_x} q_x^{j*} = 1\}$. We also remove their assignments from the supply vector, obtaining η^r. The remaining assignment is denoted as $Q^r = (q_x^*)_{x \in N^r}$. This is without loss of generality since any price vector would meet the competitive equilibrium condition for these agents. We also skip the simple case in which everyone obtains a most-preferred assignment.

For any agent $x \in N^r$ there exists a non-empty convex set of strictly preferred probability distributions $U_x = \{q \in \Delta^s : u_x(q) > u_x(q_x^*)\}$. Likewise, the set $U = \sum_{x \in N^r} U_x$ is well-defined and convex. Naturally, $U \subset |N^r| \cdot \Delta^s$ (since $\sum_{x \in N^r} q_x = |N^r|$). Let us define $Y = \prod_{j \in S} [0, \eta_j^r]$ which is also convex. Since Q^* is ex-ante Pareto-optimal (and so is Q^r for E^r) we have $U \cap Y = \varnothing$. Applying the separating hyperplane theorem *to the amplified simplex* $|N^r| \cdot \Delta^s$, there exists a price vector $P \in \mathbb{R}_+^s / \{(p, ..., p) : p \geq 0\}$ and a number $w \in \mathbb{R}$ such that $P \cdot a \geq w \geq P \cdot b$, for any $a \in U, b \in Y$. We get rid of price vectors with all equal elements since those would not divide the amplified simplex. The object types with excess supply $\left(\sum_{x \in N^r} q_x^{rj} < \eta^{rj}\right)$ would have a zero price component in any such vector P $(P^j = 0)$.

Let M be a $n \times s$ random assignment matrix (with generic element m_x^j) such that $\sum_{j \in M_x} m_x^j = 1$ for every $x \in N$. Consider a random assignment Q such that $diag(VQ') \geq diag(VQ^{*\prime})$. Consider a number $\alpha \in (0, 1)$ and build the random assignment $Q^\alpha = \alpha Q + (1 - \alpha) M$. Since $q_x^\alpha \in U_x$ for every $x \in N^r$, we have $P \cdot \sum_{x \in N^r} q_x^\alpha \geq w$. Taking the limit, since $\lim\limits_{\alpha \to 1} Q^\alpha = Q$, we have $P \cdot \sum_{x \in N^r} q_x \geq w$.

The same applies to the case $Q = Q^* : P \cdot \sum_{x \in N^r} q_x^* \geq w$. But we know that $\sum_{x \in N^r} q_x^* \in Y$ because Q^* is feasible, implying $P \cdot \sum_{x \in N^r} q_x^* \leq w$. We conclude $P \cdot \sum_{x \in N^r} q_x^* = w$. For, this reason, if we consider $q_x \in U_x$ for any agent $x \in N^r$, we have $P \cdot \left(q_x + \sum_{y \in N^r \setminus \{x\}} q_y^*\right) \geq w = P \cdot \left(q_x^* + \sum_{y \in N^r \setminus \{x\}} q_y^*\right)$. Consequently we have $P \cdot q_x \geq P \cdot q_x^*$, proving that P constitutes a pseudomarket quasiequilibrium for this economy E with associated random assignment Q^*.

For each agent $x \in N^r$ such that there exists a probability distribution $\bar{q}_x$ meeting $P \cdot \bar{q}_x < P \cdot q_x^*$, P is indeed a pseudomarket equilibrium price vector. This follows a standard argument. Suppose $q_x \in U_x$ and $P \cdot q_x = P \cdot q_x^*$. Take a number $\alpha \in (0, 1)$ and build the random assignment $q_x^\alpha = \alpha\bar{q}_x + (1-\alpha)q_x$, which meets $P \cdot q_x^\alpha < P \cdot q_x^*$. But for α close to 0, $q_x^\alpha \in U_x$, and this would contradict the fact that P constitutes a quasi-equilibrium. Therefore we must have $P \cdot q_x > P \cdot q_x^*$, proving that P constitutes an equilibrium price vector for these agents.

We then focus on the agents for which there is no such probability distribution $\bar{q}_x$. If there is no $q_x \in U_x$ such that $P \cdot q_x = P \cdot q_x^*$, then P is indeed a quasi-equilibrium vector for this agent x. So define $N^c = \{x \in N : \exists q_x \in U_x,\ P \cdot q_x = P \cdot q_x^* = \min_{j \in S} P^j\}$. If $N^c = \varnothing$ we are done since the quasiequilibrium price vector actually constitutes an equilibrium. Thus we assume $N^c \neq \varnothing$.

We claim that our partition starts by setting $N_1 = N \backslash N^c$ (the set for which P is actually an equilibrium price vector with associated random assignment $Q_1^* = [q_x^*]_{x \in N_1}$) and $N_2 \cup ... \cup N_\pi = N^c$. For this we just need to show that N_1 is not empty. If N^M is not empty, we are done. If it is, we know that there exists an "expensive" object type i such that $P^i > \min_{j \in S} P^j$ (since $P \notin \{(p, ..., p) : p \geq 0\}$). If no agent x gets $q_x^{*i} > 0$, then the object type has excess supply implying $P^i = 0$, contradicting $P^i > \min_{j \in S} P^j$. Therefore, some agent $x \in N$ gets $q_x^{*i} > 0$, and consequently $x \notin N^c$. Then $N \backslash N^c \neq \varnothing$ as we wanted to show.

For the next iteration, the "continuation economy" would consist of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and N^c. We proceed as in the first iteration to find, subsequently, nonempty disjoint sets $N_2, ..., N_\pi$. For some iteration $\pi \leq n$ we have $N_1 \cup ... \cup N_\pi = N$ since N is finite, and we are done. ■

We ideally want to fully characterize the set of ex-ante Pareto-optimal random assignments. Unfortunately, the set of SP-equilibria outcomes may not coincide with the set of ex-ante Pareto-optimal assignments. A simple example with two agents x and y and two objects i and j illustrates this fact. x is indifferent between the objects whereas y strictly prefers object i. If $N_1 = \{x\}$ and $N_2 = \{y\}$ there exists a SP-equilibrium such that x picks i and y picks the remaining object j, which is not Pareto-optimal. However, the following theorem brings us better news.

Theorem 3 *For each ordered partition $N_1, ..., N_\pi$ of N, every associated Sequential Pseudomarket equilibrium outcome Q^* belongs to the weak core.*

Proof. Inspired on He, Miralles and Yan (2012), proposition 1 on weak efficiency. It follows a recursive argument. Let a blocking coalition $C \subset N$ be defined as follows: $\exists Q$ such that a) $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in C$ and b) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \backslash C} q_x^*$. We show that it must be the case that $C = \varnothing$.

We claim that $N_1 \cap C = \varnothing$. If not, there must be a nonempty subset $\tilde{N} \subset N_1$ and an alternative feasible random assignment Q

such that $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in \tilde{N}$ and $q_x^* = q_x$ for all $x \in N_1 \backslash \tilde{N}$. The SP-equilibrium (with price vector P_1^* associated to N_1) implies $P_1^* \cdot \sum_{x \in N_1} q_x^* < P_1^* \cdot \sum_{x \in N_1} q_x$, and therefore $\sum_{x \in N_1} q_x^{*j} < \sum_{x \in N_1} q_x^j$ for some object type j such that $P_1^{*j} > 0$. Since this price is strictly positive, there is no excess supply in the reduced economy with N_1 on the demand side and η as the supply side. We must have $\sum_{x \in N_1} q_x^{*j} = \eta^j$ and thus $\sum_{x \in N_1} q_x^j > \eta^j$. This constitutes a contradiction as Q is not feasible.

Consequently, $N_1 \cap C = \varnothing$. We focus on the "continuation economy" consisting of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and $N \backslash N_1$. Using the same argument in each "continuation economy", we recursively see that $N_2 \cap C = \varnothing$, $N_3 \cap C = \varnothing$... Since $N = \cup_{t=1}^{\pi} N_t$, we conclude that $C = \varnothing$. ■

The weak core is a superset of the set of ex-ante Pareto-optimal allocations, therefore the latter concept of efficiency is finer. However, we consider that the weak core is a nice concept of ex-ante efficiency because there are no monetary transfers in this economy. A potential coalition that needs to attract an indifferent agent to the coalition in order to make all of its previous members better-off has no means inside a random assignment economy to attract the indifferent agent.

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