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THE INTEGER CHEBYSHEV CONSTANT OF FAREY INTERVALS

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Abstract

We obtain new bounds for the integer Chebyshev constant of intervals $[p/q, r/s]$ where p, q, r and s are non-negative integers such that $q\,r - p\,s = 1$. As a consequence of the methods used, we improve the known lower bound for the trace of totally positive algebraic integers.

1. Introduction

Let $I \subset \mathbb{R}$ be a closed interval. The classical Chebyshev problem asks for the monic polynomial of degree N with real coefficients of minimal uniform norm on I. When $I = [-1, 1]$ the solution is given by the Chebyshev polynomial of degree N

$$T_N(x) = 2^{1-N} \cos(N \arccos x).$$

A linear change of variables gives the solution on an arbitrary interval.

The integer Chebyshev problem asks for the polynomial of degree N with integer coefficients of minimal uniform norm on I. Given $N \in \mathbb{N}$, define

$$t_N(I) = \min \left\{ \sup_{x\in I} |P(x)|^{1/\partial P} : P \in \mathbb{Z}[x],\ \partial P \leq N,\ P \neq 0 \right\} \tag{1}$$

and

$$t_{\mathbb{Z}}(I) = \inf\{t_N(I) : N \in \mathbb{N}\}, \tag{2}$$

where as usual ∂P denotes the degree of P. It is easy to see that in fact $t_{\mathbb{Z}}(I) = \lim_{N\to\infty} t_N(I)$. The constant $t_{\mathbb{Z}}(I)$ is known as the *integer Chebyshev constant* or the *integer transfinite radius* of the interval I. If $|I| \geq 4$ (where $|I|$ denotes the length of I), then $t_{\mathbb{Z}}(I) = |I|/4$. No exact

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value of $t_{\mathbb{Z}}(I)$ is known for any interval of length less than 4, but upper and lower bounds have been obtained among others by Aparicio [**Ap1**], [**Ap2**], Amoroso [**Am**], Flammang [**F1**], Borwein and Erdélyi [**BE**], Flammang, Rhin and Smyth [**FRS**] and Pritsker [**P**]. In this paper we obtain bounds for the integer transfinite radius of intervals whose end-points are consecutive numbers in a Farey sequence, that we call Farey intervals. We improve the known upper bounds, and for a large class of intervals, also the lower bounds.

Since $t_{\mathbb{Z}}(I) \leq t_N(I)$ for all N, upper bounds can be obtained by computation. To achieve this in a manner valid for any Farey interval I, we use a fractional linear transformation to take I into $[0, \infty)$, and convert the original problem in a semi-infinite optimization problem for an appropriate auxiliary function. Such functions have been used for the computation of different measures of totally positive algebraic integers, like the Mahler measure [**F2**] and the trace [**F1**], [**ABP**], and the same optimization methods can be applied.

There are more relationships between the integer Chebyshev problem and the Schur-Siegel problem on the trace of totally positive algebraic integers as explained in [**ABP**], [**BE**], [**FRS**]. In this last problem one is lead to estimate the quantity

$$\mathcal{K} = \sup_{Q\in\mathbb{Z}[x],\, Q\neq 0,\, t>0} \left\{ \inf_{x>0} \left(x - \frac{t}{\partial Q} \log|Q(x)| \right) \right\}. \tag{3}$$

The constant $\mathcal{K}$ appears in our estimates from below of $t_{\mathbb{Z}}(I)$, and we shall prove in Corollary 2 that

$$\lim_{m\to\infty} \left(\frac{1}{t_{\mathbb{Z}}([1, 1/m])} - m \right) = \mathcal{K}.$$

This gives a partial answer to question number 5 in the open problems section of [**BE**]. To give a complete solution, $\mathcal{K}$ must be computed exactly. As for the integer Chebyshev problem, only bounds on $\mathcal{K}$ are known. The best bounds are, as far as we know,

$$1.783622 < \mathcal{K} < 1.898302.$$

The lower bound will be proved in Theorem 3, and is an improvement over the one obtained in [**ABP**]. The upper bound is due to J. P. Serre (see the note added in proof in [**Sm**]).

A polynomial $P \in \mathbb{Z}[x]$ of degree N such that $t_N(I) = \sup_{x\in I} |P(x)|^{1/\partial P}$ is called a N-th Chebyshev polynomial on I. The structure of such polynomials has been studied extensively by Aparicio [**Ap2**], Borwein and Erdélyi [**BE**], Habsieger and Salvy [**HS**] and Pritsker [**P**]. Habsieger and Salvy determine all integer Chebyshev polynomials on $[0,1]$ up to degree 75. All their irreducible factors except one have all their roots in $(0,1)$, and after a change of variable have small trace. These polynomials, together with others having all their roots in $(0,\infty)$ and small trace are used in the computations to derive the upper bounds of the Chebyshev constant of Farey intervals.

2. The integer Chebyshev problem for Farey intervals

We call Farey interval an interval $[p/q, r/s]$ where p, q, r and s are non-negative integers such that $q\,r - p\,s = 1$. This in particular implies that p, q, r and s are pairwise coprime, unless $p = 0$, in which case $q = r = 1$. Given coprime integers q and s such that $1 \le q \le s$ (allowing for the case $q = s = 1$), there exist unique integers p and r such that $I_{q,s} = [p/q, r/s]$ is a Farey interval contained in $[0,1]$. The interval $J_{q,s} = [(q-p)/q, (s-r)r/s]$, the symmetric of $I_{q,s}$ with respect to $x = 1/2$, is also a Farey interval, and the substitution $x \to 1-x$ shows that $t_{\mathbb{Z}}(I_{q,s}) = t_{\mathbb{Z}}(J_{q,s})$. All other Farey intervals with the same denominators are the integer translates of $I_{q,s}$ or of $J_{q,s}$, and all have the same integer Chebyshev constant.

As a first step in studying $t_{\mathbb{Z}}(I_{q,s})$, we make a change of variable which transforms the original problem into another one in $(0,\infty)$. The fractional linear transformation

$$\phi(x) = \frac{p\,x + r}{q\,x + s} \tag{4}$$

is a bijection between $(0,\infty)$ and $(p/q, r/s)$. Associated to it is a map $\Phi\colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ defined by

$$(\Phi P)(x) = (q\,x + s)^{\partial P} P(\phi(x)) \quad \forall\, P \in \mathbb{Z}[x].$$

Then

$$\sup_{x\in I_{q,s}} |P(x)|^{1/\partial P} = \sup_{x>0} (q\,x+s)^{-1} |(\Phi P)(x)|^{1/\partial P}.$$

The image of $(q\,x - p)^k$ is the constant polynomial 1 for any positive integer k, so that in general $\partial(\Phi P) \le \partial P$. But the fact that $q\,r - p\,s = 1$ implies that Φ is surjective. It follows that

$$\begin{aligned} t_{\mathbb{Z}}(I_{q,s}) &= \inf_{Q\in\mathbb{Z}[x],\, Q\neq 0\; k\ge 0} \left\{ \sup_{x>0} (q\,x+s)^{-1} |Q(x)|^{1/(k+\partial Q)} \right\} \\ &= \frac{1}{q} \cdot \inf_{Q\in\mathbb{Z}[x],\, Q\neq 0,\, 0<t<1} \left\{ \sup_{x>0} \left(x + \frac{s}{q} \right)^{-1} |Q(x)|^{t/\partial Q} \right\}. \end{aligned}$$

We define the function $\rho\colon [1,\infty) \to \mathbb{R}$ by

$$\rho(\sigma) = \sup_{Q\in\mathbb{Z}[x],\, Q\neq 0,\, 0<t<1} \left\{ \inf_{x>0} \left(\log(x+\sigma) - \frac{t}{\partial Q} \log|Q(x)| \right) \right\}.$$

Then

$$t_{\mathbb{Z}}(I_{q,s}) = \frac{1}{q} e^{-\rho(s/q)}.$$

The function

$$\log(x+\sigma) - \frac{t}{\partial Q} \log|Q(x)|, \quad x > 0,$$

is an instance of an auxiliary function. Changing $\log(x+\sigma)$ by $\log_+ x$ or x, the corresponding function can be used to provide estimates on the Mahler measure and on the trace of totally positive algebraic integers respectively.

It is convenient to define the function $\lambda(\sigma)$ such that

$$\rho(\sigma) = \log(\sigma + \lambda(\sigma)) \quad \text{and} \quad t_{\mathbb{Z}}(I_{q,s}) = \frac{1}{q\,\lambda(s/q) + s}.$$

We are ready now to state our first result, which relates the function λ and the constant $\mathcal{K}$.

Theorem 1.

$$1 \le \lambda(\sigma) \le \mathcal{K} \quad \forall\, \sigma \ge 1. \tag{5}$$

$$\lim_{\sigma\to\infty} \lambda(\sigma) = \mathcal{K}. \tag{6}$$

Proof: Let $\sigma \ge 1$. From the easily verified inequality

$$\log(x+\sigma) - \frac{1}{\sigma+1} \log x \ge \log(\sigma+1) \quad \forall\, x > 0$$

it follows that

$$\rho(\sigma) \ge \log(\sigma+1) \quad \text{and} \quad \lambda(\sigma) \ge 1.$$

To get the upper bound in (5) we use the inequality

$$\log(x+\sigma) \le \log(\sigma+\mathcal{K}) + \frac{x-\mathcal{K}}{\sigma+\mathcal{K}} \quad \forall\, x > 0,$$

which holds because of the concavity of the logarithm. It follows that

$$\rho(\sigma) \le \log(\sigma+\mathcal{K}) \quad \text{and} \quad \lambda(\sigma) \le \mathcal{K}.$$

We turn now to the proof of (6). Given $\varepsilon > 0$ there exist $t \in (0,1)$ and a non constant $Q \in \mathbb{Z}[x]$ such that

$$x - t\log|Q(x)| \ge \mathcal{K} - \varepsilon \quad \forall\, x > 0.$$

Define

$$f(x,\sigma) = \log(x+\sigma) - \frac{t}{\sigma}\log|Q(x)|.$$

There exists $x_0 > 2\,\partial Q$ such that

$$\frac{x\,Q'(x)}{Q(x)} \le 2\,\partial Q \quad \forall\, x \ge x_0.$$

Let $\sigma_0 = 2\,\partial Q\, x_0/(x_0 - 2\,\partial Q)$. Then

$$\frac{1}{x+\sigma} - \frac{t}{\sigma}\frac{Q'(x)}{Q(x)} \ge 0 \quad \forall\, x \ge x_0, \quad \sigma \ge \sigma_0.$$

This implies that $f(x,\sigma)$ is increasing on $[x_0,\infty)$ as a function of x for each $\sigma > \sigma_0$. It follows that

$$\sigma > \sigma_0 \implies \inf_{x>0} f(x,\sigma) = \inf_{0<x\le x_0} f(x,\sigma).$$

Moreover, from the inequality $\log(1+x) \ge x - x^2/2$, it follows that

$$f(x,\sigma) \ge \log\sigma + \frac{1}{s}\bigl(x - t\log|Q(x)|\bigr) - \frac{x^2}{2\,\sigma^2} \quad \forall\, x,\sigma > 0.$$

If $\sigma > \sigma_0$, then

$$\rho(\sigma) \ge \inf_{0<x\le x_0} f(x,\sigma) \ge \log\sigma + \frac{\mathcal{K}-\varepsilon}{\sigma} - \frac{x_0^2}{2\,\sigma^2}.$$

Thus, for all $\sigma > \max\bigl(\sigma_0, x_0^2/(2\,\varepsilon)\bigr)$ we have

$$\rho(\sigma) \ge \log\sigma + \frac{\mathcal{K}-2\,\varepsilon}{\sigma}$$

and

$$\lambda(\sigma) \ge \sigma\bigl(e^{\frac{\mathcal{K}-2\varepsilon}{\sigma}} - 1\bigr) \ge \mathcal{K} - 2\,\varepsilon. \qquad \square$$

An immediate consequence is the following

Corollary 2.

$$\frac{1}{\mathcal{K}\,q+s} \le t_{\mathbb{Z}}(I_{q,s}) \le \frac{1}{q+s}. \tag{7}$$

$$\lim_{s\to\infty}\left(\frac{1}{t_{\mathbb{Z}}(I_{q,s})} - s\right) = q\,\mathcal{K}. \tag{8}$$

Remark. Numerical evidence suggest that the function λ is increasing and concave (see Figure 1).

Remark. In [**FRS**] the authors define a function $g(t)$, related to the functions defined in this paper by the identity

$$g(t) = \frac{t}{t^2 + \lambda(t^2)} \quad \forall\, t > 1.$$

It is clear from Theorem 1 and Corollary 2 that it is important to know the value of $\mathcal{K}$. The following theorem, announced in the introduction, gives bounds on its value.

Theorem 3.

$$1.783622 < \mathcal{K} < 1.898302. \tag{9}$$

Proof: As mentioned in the introduction, the upper bound is due to J. P. Serre. The lower bound is proved checking the inequality

$$x - \sum_i c_i \log |Q_i(x)| > 1.783622 \quad \forall\, x > 0, \tag{10}$$

where the values of i and c_i are given in Table 1, and the corresponding polynomials Q_i in Table 2. To get inequality (10) we applied several optimization techniques as explained in [**ABP**]. □

TABLE 1. Values of i and c_i in (10)

i	c_i	i	c_i
1	0.544760718417	17	0.013264681142
2	0.507054323911	20	0.004052954265
3	0.075640549036	22	0.002738162962
4	0.191387374021	26	0.001080633346
5	0.019056756653	31	0.005267836285
6	0.011380615544	32	0.000698262799
7	0.086001728874	33	0.001476081258
9	0.007972093676	34	0.001807257036
10	0.005954670447	35	0.002376358820
11	0.032795949532	36	0.003563976995
12	0.029898783554	37	0.003731485598
14	0.012200820630	38	0.003362999773
15	0.011116063090	41	0.002017653392
16	0.005459150854	42	0.001924042210

As a consequence of Theorem 3 we obtain the following result about the Schur-Siegel trace problem.

Theorem 4. *Let α be a totally positive algebraic integer of degree d with conjugates $\alpha_1 < \cdots < \alpha_d$ and such that $Q_i(\alpha) \neq 0$ for $i = 2, 4, 7, 11, 12$. Then*

$$\alpha_1 + \cdots + \alpha_d > 1.783622\, d. \tag{11}$$

3. Counterexamples to invariance properties of $t_{\mathbb{Z}}$

It is well known that $t_{\mathbb{Z}}$ is not invariant under translation, does not scale linearly under dilations and has no additive properties. We use Theorem 1 to give a family of counterexamples depending on an integer parameter.

3.1. Translation invariance. Let n be a positive integer and consider the intervals $I_{n,n+1}$ and $I_{1,n(n+1)}$. Both have the same length $1/(n(n+1))$, so that there exists $r \in \mathbb{Q}$ such that $I_{n,n+1} = r + I_{1,n(n+1)}$. However their integer transfinite radius is different:

$$t_{\mathbb{Z}}(I_{n,n+1}) = \frac{1}{\lambda(1+1/n)n + n + 1} \sim \frac{1}{(1+\lambda(1))n} \sim \frac{\sqrt{|I_{n,n+1}|}}{1+\lambda(1)},$$

while

$$t_{\mathbb{Z}}(I_{1,n(n+1)}) = \frac{1}{\lambda(n(n+1)) + n(n+1)} \sim \frac{1}{n^2} \sim |I_{1,n(n+1)}|.$$

More generally, given a positive integer N, consider all possible factorings $N = q\,s$ with $1 \leq q \leq s$ and $(q, s) = 1$. Then all the intervals $I_{q,s}$ have the same length $1/N$ but different integer transfinite radius $1/(\lambda(s/q)\, q + s)$. We see that $t_{\mathbb{Z}}(I_{q,s})$ depends not only on the size of the interval, but also on number theoretical properties of the denominators q and s.

3.2. Dilations. Consider the intervals $I_{1,m} = [0, 1/m]$ and $I_{1,nm} = [0, 1/(n\,m)]$ for integers $m \geq 1$ and $n \geq 2$. They satisfy $n\, I_{1,nm} = I_{1,m}$, but

$$n\, t_{\mathbb{Z}}(I_{1,nm}) - t_{\mathbb{Z}}(I_{1,m}) = \frac{n\,\lambda(m) - \lambda(n\,m)}{\big(\lambda(n\,m) + n\,m\big)\big(\lambda(m) + m\big)} > \frac{n - \mathcal{K}}{(\mathcal{K} + n\,m)(\mathcal{K} + m)}.$$

3.3. Additivity. Given $m \in \mathbb{N}$, consider the intervals $I_{1,m} = [1, 1/m]$, $I_{1,m+1} = [1, 1/(m+1)]$ and $J_{m,m+1} = [1/(m+1), 1/m]$. Then $I_{1,m} = I_{1,m+1} \cup J_{m,m+1}$, but

$$t_{\mathbb{Z}}(I_{1,m}) < t_{\mathbb{Z}}(I_{1,m+1}) + t_{\mathbb{Z}}(J_{m,m+1}).$$

For $m = 1$, 2, 3 and 4 we see that this is true from the data in Table 4. For $m > 4$, we have

$$\begin{aligned}&\big(t_{\mathbb{Z}}(I_{1,m+1}) + t_{\mathbb{Z}}(J_{m,m+1})\big) - t_{\mathbb{Z}}(I_{1,m})\\&\qquad = \frac{1}{(1+\lambda(1+1/m))m+1} - \frac{\lambda(m+1)-\lambda(m)+1}{(\lambda(m)+m)(\lambda(m+1)+m+1)}\\&\qquad > \frac{1}{(\mathcal{K}+1)\,m+1} - \frac{\mathcal{K}}{(m+1)(m+2)}.\end{aligned}$$

4. Upper bounds for $t_{\mathbb{Z}}(I_{q,s})$

Upper bounds of $t_{\mathbb{Z}}(I_{q,s})$ follow from lower bounds of $\rho(\sigma)$, or what is the same, lower bounds of $\lambda(\sigma)$. These are obtained by taking specific values of t and Q in the definition of $\rho(\sigma)$. Be begin with an estimate valid for all $\sigma \geq 1$.

Theorem 5.

$$\lambda(\sigma) \geq 1.4737 - \frac{0.7573}{2.186+\sigma} \geq 1.236 \quad \forall\, \sigma \geq 1. \tag{12}$$

$$t_{\mathbb{Z}}(I_{q,s}) \leq \frac{1}{q\left(1.4737 - \dfrac{0.7573}{2.186+s/q}\right)+s}. \tag{13}$$

Proof: Let

$$f(x,\sigma) = \log(x+\sigma) - \frac{0.709209}{0.773023+\sigma}\log x - \frac{0.897765}{3.48883+\sigma}\log|x-1|.$$

As a function of x, f has two critical points

$$\xi_{\pm}(\sigma) = \frac{0.111317 + 3.36047\,\sigma + 1.30349\,\sigma^2 \pm 0.994922\sqrt{\Delta}}{(-0.167035+\sigma)(2.82191+\sigma)},$$

where

$$\Delta = (0.00659344+\sigma)(0.65923+\sigma)(1.05636+\sigma)(2.72637+\sigma).$$

Both are local minima, and it is an easy matter to verify with a Computer Algebra System that $0 < \xi_-(s) < 1 < \xi_+(s)$ and $f(\xi_-(s),\sigma) < f(\xi_+(s),\sigma)$ for all $\sigma > 1$. Thus

$$\inf_{x>0} f(x,\sigma) = f(\xi_-(s),\sigma) \quad \text{and} \quad \rho(\sigma) \geq f(\xi_-(s),\sigma).$$

As σ goes to infinity we have

$$f(\xi_-(s),\sigma) = \log\sigma + \frac{1.47373}{\sigma} + O\left(\frac{1}{\sigma^2}\right),$$

and from here it is again easy to verify that

$$\lambda(\sigma) > 1.4737 - \frac{0.7573}{2.186 + \sigma} \quad \forall\, \sigma \geq 1. \qquad \square$$

Next, we bound from below $\lambda(\sigma)$ for the 32 values of σ in the leftmost column of Table 4, and derive upper bounds for the integer Chebyshev constant of the corresponding intervals. The values $\sigma = 5/4$, $4/3$, $3/2$, $5/3$, 2, $5/2$, 3, 4, 5, 6, 12, 15 and 20 were considered in [**F1**], and $\sigma = 200$ in [**BE**]. In all cases, we improve the upper bounds given in those papers.

Following the proof of Theorem 1 we begin with large values of sigma and the polynomials and coefficients, conveniently rescaled, used in [**ABP**] to bound $\mathcal{K}$ from below, together with the two polynomials of degree 10 and trace 18 which appear for the first time in [**MSm**]. As σ decreases, some of these polynomials are not useful anymore, and new ones have to be used.

In all, we use 45 polynomials given in Table 2. For each of the values of σ, we select a subset of them, defined by a set of integers $S(\sigma) \subset \{1, 2, \ldots, 45\}$. Then

$$\rho(\sigma) \geq \sup \left\{ \inf_{x>0} \left(\log(x+\sigma) - \sum_{j \in S(\sigma)} \frac{a_j}{\partial Q_j} \log |Q_j(x)| \right) \right\}, \tag{14}$$

where the sup is taken over all $a_j > 0$, $j \in S(\sigma)$, with $\sum_{j \in S(\sigma)} a_j \leq 1$. The right hand side of (14) is a semi-infinite minimax problem that can be solved by Reme's algorithm and other optimization techniques (see [**ABP**] for details). We should note that no calculations are carried out to estimate $\lambda(1)$, since $\lambda(1) = \sqrt{4 + \lambda(4)} - 1$.

The set of indexes $S(\sigma)$ are given in Table 3. The bounds of $\lambda(\sigma)$ and the corresponding bounds for $t_{\mathbb{Z}}(I_{q,s})$ are given in Table 4. They are represented graphically in Figure 1.

5. Lower bounds

Lower bounds of $t_{\mathbb{Z}}(I_{q,s})$ were obtained in [**F1**] by applying a lemma of Chudnovsky to a sequence of polynomials which generalizes the sequence of Gorškov-Wirsing polynomials in $[0, 1]$ to any Farey interval. We have computed the lower bounds with this method for various values of $\sigma = s/q$, and found that the lower bound given by Corollary 2 is better for $\sigma > 23$. Any improvement in the upper bound of $\mathcal{K}$ will result in better lower bounds of $t_{\mathbb{Z}}(I_{q,s})$. The lower bounds given in Table 4 coincide with those given in [**F1**] for the intervals corresponding to the values $\sigma = 5/4, 4/3, 3/2, 5/3, 2, 5/2, 3, 5, 6, 12, 15$ and 20; for $\sigma = 1, 4$, they follow from the results in [**P**].

6. Other intervals

The methods developed in previous sections can be used to obtain bounds of the integer Chebyshev constant for general intervals $[a, b] \subset [0, 1]$. There are only a finite number of Farey intervals containing $[a, b]$, since their length must be at least $b - a$. Let $I_{q,s}$ be minimal among them with respect to length, and let $\phi^{-1}(a) = \alpha$, $\phi^{-1}(b) = \beta$, where ϕ is the fractional linear transformation (4). Arguing as in Section 2 we get $t_{\mathbb{Z}}([a, b]) = e^{-\mu}/q$, where

$$\mu = \sup_{Q\in\mathbb{Z}[x],\, Q\neq 0,\, 0<t<1} \left\{ \inf_{\beta<x<\alpha} \left(\log(x+\sigma) - \frac{t}{\partial Q} \log|Q(x)| \right) \right\}.$$

More precise estimates can be obtained for small intervals with a fixed rational endpoint.

Theorem 6. *Let $0 \le p/q < 1$ be a rational point in lowest terms, and let $r/s > p/q$ be the adjacent fraction of p/q in the Farey sequence of greatest denominator q. Given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $0 < \delta < \delta_\varepsilon$, then*

$$\frac{\delta\, q}{1 + (\mathcal{K} + 1 - b)\delta\, q^2} \le t_{\mathbb{Z}} \left(\left[\frac{p}{q}, \frac{p}{q} + \delta \right] \right) \le \frac{\delta\, q}{1 + (\mathcal{K} - b - \varepsilon)\delta\, q^2}, \tag{15}$$

where b is the fractional part of $1/(\delta\, q^2) - s/q$.

If $b = 0$, the denominator of the fraction in the left hand side can be replaced by $1 + \mathcal{K}\,\delta\, q^2$.

Proof: If $\delta \le 1/(s\, q)$, then there is a unique non-negative integer k such that

$$\frac{r + (k+1)\, p}{s + (k+1)\, q} < \frac{p}{q} + \delta \le \frac{r + k\, p}{s + k\, q}$$

and

$$I_{q,s+(k+1)q} \subsetneq \left[\frac{p}{q}, \frac{p}{q} + \delta \right] \subset I_{q,s+kq}.$$

For this value of k we have

$$0 \le \frac{1}{\delta\, q^2} - \frac{s}{q} - k < 1,$$

so that

$$b = \frac{1}{\delta\, q^2} - \frac{s}{q} - k \quad \text{and} \quad s + k\, q = \frac{1}{\delta\, q} - b\, q.$$

To get the lower bound we observe that

$$t_{\mathbb{Z}}\left(\left[\frac{p}{q},\frac{p}{q}+\delta\right]\right) \geq \frac{1}{\lambda(k+1+s/q)\,q+s+(k+1)\,q}$$
$$\geq \frac{\delta\,q}{1+(\mathcal{K}+1-b)\delta\,q^2}.$$

We remark that this inequality is valid for all $\delta \in (0, 1/(q\,s)]$. As for the upper bound we have

$$t_{\mathbb{Z}}\left(\left[\frac{p}{q},\frac{p}{q}+\delta\right]\right) \leq \frac{1}{\lambda(k+s/q)\,q+s+k\,q} \leq \frac{\delta\,q}{1+(\lambda(k+s/q)-b)\delta\,q^2}.$$

The proof is finished by observing that as δ goes to zero, k goes to infinity and $\lambda(k+s/q)$ converges to $\mathcal{K}$.

If $b=0$, then

$$\left[\frac{p}{q},\frac{p}{q}+\delta\right] = I_{q,s+kq}$$

and

$$t_{\mathbb{Z}}\left(\left[\frac{p}{q},\frac{p}{q}+\delta\right]\right) = \frac{1}{\lambda(k+s/q)\,q+s+k\,q} \geq \frac{\delta\,q}{1+\mathcal{K}\,\delta\,q^2}. \qquad \square$$

Remark. The lower bound in Theorem 6 is an improvement over the corresponding bound in [**FRS**]. This is not always the case for the upper bound. However, the proof is much simpler.

7. Data

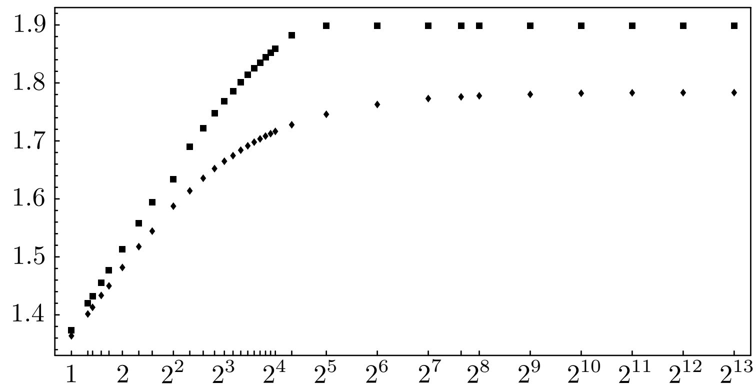

FIGURE 1. Computed bounds of λ. Horizontal axes in logarithmic scale

TABLE 2. Polynomials Q_i

i	Q_i
i	Q_i
1	x
2	$1-x$
3	$2-x$
4	$1-3\,x+x^2$
5	$1-4\,x+x^2$
6	$2-4\,x+x^2$
7	$1-6\,x+5\,x^2-x^3$
8	$1-5\,x+6\,x^2-x^3$
9	$1-8\,x+6\,x^2-x^3$
10	$1-9\,x+6\,x^2-x^3$
11	$1-7\,x+13\,x^2-7\,x^3+x^4$
12	$1-8\,x+14\,x^2-7\,x^3+x^4$
13	$1-7\,x+14\,x^2-8\,x^3+x^4$
14	$1-11\,x+29\,x^2-26\,x^3+9\,x^4-x^5$
15	$1-12\,x+31\,x^2-27\,x^3+9\,x^4-x^5$
16	$1-13\,x+32\,x^2-27\,x^3+9\,x^4-x^5$
17	$1-15\,x+35\,x^2-28\,x^3+9\,x^4-x^5$
18	$1-12\,x+45\,x^2-67\,x^3+42\,x^4-11\,x^5+x^6$
19	$1-13\,x+47\,x^2-68\,x^3+42\,x^4-11\,x^5+x^6$
20	$1-14\,x+51\,x^2-72\,x^3+43\,x^4-11\,x^5+x^6$
21	$1-15\,x+59\,x^2-78\,x^3+44\,x^4-11\,x^5+x^6$
22	$1-18\,x+63\,x^2-79\,x^3+44\,x^4-11\,x^5+x^6$
23	$1-14\,x+66\,x^2-136\,x^3+131\,x^4-61\,x^5+13\,x^6-x^7$
24	$1-14\,x+67\,x^2-138\,x^3+132\,x^4-61\,x^5+13\,x^6-x^7$
25	$1-15\,x+71\,x^2-142\,x^3+133\,x^4-61\,x^5+13\,x^6-x^7$
26	$1-15\,x+71\,x^2-144\,x^3+136\,x^4-62\,x^5+13\,x^6-x^7$
27	$1-15\,x+72\,x^2-146\,x^3+137\,x^4-62\,x^5+13\,x^6-x^7$
28	$1-15\,x+73\,x^2-147\,x^3+137\,x^4-62\,x^5+13\,x^6-x^7$
29	$1-16\,x+75\,x^2-148\,x^3+137\,x^4-62\,x^5+13\,x^6-x^7$
30	$1-15\,x+75\,x^2-153\,x^3+142\,x^4-63\,x^5+13\,x^6-x^7$
31	$1-16\,x+78\,x^2-157\,x^3+143\,x^4-63\,x^5+13\,x^6-x^7$
32	$1-17\,x+81\,x^2-158\,x^3+143\,x^4-63\,x^5+13\,x^6-x^7$
33	$1-17\,x+82\,x^2-159\,x^3+143\,x^4-63\,x^5+13\,x^6-x^7$
34	$1-18\,x+89\,x^2-172\,x^3+150\,x^4-64\,x^5+13\,x^6-x^7$

TABLE 2. Polynomials Q_i (continued)

35	$1-15\,x+83\,x^2-220\,x^3+303\,x^4-220\,x^5+83\,x^6-15\,x^7+x^8$
36	$1-24\,x+194\,x^2-743\,x^3+1526\,x^4-1798\,x^5+1265\,x^6$ $-537\,x^7+134\,x^8-18\,x^9+x^{10}$
37	$1-24\,x+200\,x^2-766\,x^3+1560\,x^4-1822\,x^5+1273\,x^6$ $-538\,x^7+134\,x^8-18\,x^9+x^{10}$
38	$1-24\,x+206\,x^2-813\,x^3+1662\,x^4-1920\,x^5+1320\,x^6$ $-549\,x^7+135\,x^8-18\,x^9+x^{10}$
39	$1-22\,x+192\,x^2+873\,x^3-2287\,x^4+3618\,x^5-3535\,x^6$ $+2150\,x^7-805\,x^8+178\,x^9-21\,x^{10}+x^{11}$
40	$1-25\,x+248\,x^2-1290\,x^3+3922\,x^4+7362\,x^5+8783\,x^6$ $-6752\,x^7+3352\,x^8+1060\,x^9+205\,x^{10}-22\,x^{11}+x^{12}$
41	$1-29\,x+314\,x^2-1676\,x^3+5007\,x^4+9012\,x^5+10213\,x^6$ $-7474\,x^7+3561\,x^8+1092\,x^9+207\,x^{10}-22\,x^{11}+x^{12}$
42	$1-33\,x+377\,x^2-2009\,x^3+5846\,x^4+10166\,x^5+11134\,x^6$ $-7908\,x^7+3679\,x^8+1109\,x^9+208\,x^{10}-22\,x^{11}+x^{12}$
43	$1-26\,x+279\,x^2-1625\,x^3+5702\,x^4+12694\,x^5+18464\,x^6$ $-17848\,x^7+11549\,x^8+4980\,x^9+1401\,x^{10}-245\,x^{11}+24\,x^{12}-x^{13}$
44	$1-27\,x+308\,x^2-1965\,x^3+7812\,x^4-20404\,x^5+35986\,x^6$ $-43423\,x^7+35986\,x^8-20404\,x^9+7812\,x^{10}$ $-1965\,x^{11}+308\,x^{12}-27\,x^{13}+x^{14}$
45	$1-31\,x+413\,x^2-3141\,x^3+15261\,x^4-50187\,x^5+115410\,x^6$ $-189036\,x^7+222621\,x^8-189036\,x^9+115410\,x^{10}-50187\,x^{11}$ $+15261\,x^{12}-3141\,x^{13}+413\,x^{14}-31\,x^{15}+x^{16}$

TABLE 3. Polynomials used to bound $\lambda(\sigma)$ from above

σ	$J(\sigma)$
5/4	1, 2, 4, 5, 7, 8, 11, 13, 12, 23, 24, 31, 35, 43, 44, 45
4/3	1, 2, 4, 5, 7, 8, 11, 12, 13, 19, 23, 24, 31, 35, 43, 45
3/2	1, 2, 4, 5, 7, 8, 11, 12, 13, 14, 18, 23, 24, 35, 43, 45
5/3	1, 2, 4, 5, 7, 8, 11, 12, 14, 23, 24, 31, 35, 43, 44, 45
2	1, 2, 3, 4, 5, 7, 8, 11, 12, 14, 23, 25, 31, 35, 39, 43, 44, 45
5/2	1, 2, 3, 4, 5, 7, 8, 11, 12, 14, 18, 19, 23, 25, 31, 35, 39, 43, 45

Table 3. Polynomials used to bound $\lambda(\sigma)$ from above (continued)

3	1, 2, 3, 4, 5, 7, 8, 11, 12, 14, 18, 19, 23, 25, 29, 31, 35, 39
4	1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 16, 18, 19, 23, 24, 25, 28, 29, 31, 35, 39, 40, 43
5	1, 2, 3, 4, 5, 7, 11, 12, 14, 15, 16, 18, 19, 23, 25, 29, 31, 35, 39, 43
6	1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 16, 18, 19, 23, 25, 29, 31, 35, 39
7	1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 16, 19, 23, 25, 29, 31, 35
8	1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 16, 19, 23, 25, 29, 31, 35
9	1, 2, 3, 4, 5, 6, 7, 11, 12, 14, 15, 16, 17, 18, 19, 23, 25, 29, 31, 35,
10	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
11	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
12	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
13	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
14	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
15	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 23, 29, 31, 32, 35
16	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 29, 31, 32, 35, 38
20	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 29, 31, 32, 35, 38
32	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 20, 31, 32, 35
64	1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 14, 15, 16, 17, 19, 20, 21, 27, 31, 32, 37, 38
128	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 19, 21, 26, 31, 32, 35, 36, 37, 38
200	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 19, 21, 22, 31, 32, 33, 35, 36, 37, 38
256	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 19, 21, 22, 30, 31, 33, 35, 36, 37, 38
512	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 20, 31, 32, 33, 34, 36, 37, 38, 40, 41
1024	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 26, 31, 33, 34, 35, 36, 37, 38, 41, 42
2048	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 26, 31, 33, 34, 35, 36, 37, 38, 41, 42
4096	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 26, 31, 33, 34, 35, 36, 37, 38, 41, 42
8192	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 20, 22, 26, 31, 33, 34, 35, 36, 37, 38, 41, 42

TABLE 4. Bounds of λ and $t_{\mathbb{Z}}$

	$\lambda(\sigma)$			$t_{\mathbb{Z}}(I_{q,s})$	
σ	Lower	Upper	$I_{q,s}$	Lower	Upper
1	1.36377513	1.37361000	$[0,1]$	0.42130000	0.42305209
5/4	1.40149743	1.41966904	$[3/4,4/5]$	0.09364456	0.09428635
4/3	1.41298098	1.43230671	$[2/3,3/4]$	0.12052665	0.12137480
3/2	1.43351581	1.45559058	$[1/2,2/3]$	0.16917092	0.17044394
5/3	1.44993773	1.47660201	$[1/3,2/5]$	0.10604672	0.10695401
2	1.48172294	1.51318217	$[0,1/2]$	0.28464223	0.28721413
5/2	1.51761967	1.55791221	$[1/2,3/5]$	0.12321607	0.12445180
3	1.54441251	1.59402618	$[0,1/3]$	0.21767399	0.22005045
4	1.58743287	1.63380000	$[0,1/4]$	0.17750000	0.17897307
5	1.61389711	1.69027490	$[0,1/5]$	0.14947069	0.15119679
6	1.63564969	1.72199582	$[0,1/6]$	0.12950020	0.13096463
7	1.65226671	1.74744898	$[0,1/7]$	0.11431904	0.11557665
8	1.66493944	1.76839689	$[0,1/8]$	0.10237094	0.10346677
9	1.67459781	1.78598039	$[0,1/9]$	0.09271294	0.09368035
10	1.68400639	1.80097560	$[0,1/10]$	0.08473875	0.08558709
11	1.69156814	1.81393148	$[0,1/11]$	0.07804006	0.07879247
12	1.69789528	1.82524867	$[0,1/12]$	0.07233143	0.07300392
13	1.70348697	1.83522722	$[0,1/13]$	0.06740712	0.06801108
14	1.70832113	1.84409676	$[0,1/14]$	0.06311498	0.06366053
15	1.71251696	1.85203632	$[0,1/15]$	0.05934001	0.05983540
16	1.71644481	1.85918775	$[0,1/16]$	0.05599358	0.05644474
20	1.72782259	1.88189943	$[0,1/20]$	0.04569987	0.04602394
32	1.74589502	1.89830210	$[0,1/32]$	0.02950000	0.02963324
64	1.76283741	"	$[0,1/64]$	0.01517489	0.01520616
128	1.77306654	"	$[0,1/128]$	0.00769833	0.00770558
200	1.77600839	"	$[0,1/200]$	0.00495310	0.00495599
256	1.77783421	"	$[0,1/256]$	0.00387749	0.00387931
512	1.78024589	"	$[0,1/512]$	0.00194591	0.00194636
1024	1.78217140	"	$[0,1/1024]$	0.00097475	0.00097487
2048	1.78296173	"	$[0,1/2048]$	0.00048783	0.00048786
4096	1.78316338	"	$[0,1/4096]$	0.00024402	0.00024404
8192	1.78330416	"	$[0,1/8192]$	0.00012198	0.00012204

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