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A PRIORI L^2-ERROR ESTIMATES FOR APPROXIMATIONS OF FUNCTIONS ON COMPACT MANIFOLDS

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ABSTRACT. Given a $\mathcal{C}^2$-function f on a compact riemannian manifold (X, g) we give a set of frequencies $L = L_f(\varepsilon)$ depending on a small parameter $\varepsilon > 0$ such that the relative L^2-error $\frac{\|f-f^L\|}{\|f\|}$ is bounded above by ε, where f^L denotes the L-partial sum of the Fourier series f with respect to an orthonormal basis of $L^2(X)$ constituted by eigenfunctions of the Laplacian operator Δ associated to the metric g.

1. Introduction

The origin of this work was to give an answer to the following quite naive question:

Given a 2π-periodic function $f(\theta)$ and a fixed $\varepsilon > 0$, is it possible to find an explicit subset of frequencies $L = L_f(\varepsilon) \subset \mathbb{Z}$ for which we have the following a priori bound for the relative L^2-error $\frac{\|f-f^L\|}{\|f\|} \leq \varepsilon$?

Here f^L denotes the partial sum $f^L(\theta) = \sum_{\ell \in L} f_\ell e^{i\ell\theta}$, $f_\ell = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-i\ell\theta}d\theta$ are the Fourier coefficients and $\|g\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(\theta)|^2 d\theta\right)^{\frac{1}{2}}$ is the L^2-norm of a function $g(\theta)$. It turns out that such a bound can be explicitly constructed using only the quantities $\|f\|$, $\|f'\|$, $\|f''\|$ and ε by an elementary application of the Chebyshev inequality in probability theory.

In fact, the context in which we was first interested was a little bit technically involved but heuristically analogous: we wanted to obtain a bound for the number of significant Fourier coefficients of an spherical function $f : \mathbb{S}^2 \to \mathbb{R}$ in terms of the spherical harmonics basis we need to compute, once a bound of the relative L^2-error is prescribed. The original motivation was to obtain a smooth parametrizations from a triangulations of star-shaped surfaces in $\mathbb{R}^3$ representing the left atrium of the heart of a sample of patients with atrial fibrillation, see [2].

The main result we present in this paper gives a complete theoretical answer to this question for every (real or complex) $\mathcal{C}^2$-function defined on any compact riemannian manifold (X, g) when we compute its Fourier coefficients with respect to a countable basis formed by eigenfunctions of the Laplacian operator Δ of (X, g). A continuous counterpart is also stated in the non-compact case $X = \mathbb{R}^n$ by using the Fourier transform instead of Fourier series.

In Section 3 we explain how to compute effectively these a priori bounds from discrete geometric data in two different contexts: closed curves in $\mathbb{R}^n$ and star-shaped surfaces in $\mathbb{R}^3$. In the second case, we illustrate the computations with an explicit real example.

Although motivated by the applications, our approach is mainly theoric and we do not consider the numerical errors coming from the discretizations of the continuous models that we consider in this paper.

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2. Main result

In order to fix the ideas, we fix an oriented compact riemannian manifold (X, g) and we consider the (scalar or hermitian) product in the space of (real or complex valued) square integrable functions on X defined by

$$\langle a, b\rangle = \int_X a \cdot \bar{b}\, dV,$$

where dV is the volume element and we denote by $A^0(X)$ its L^2-completation. The riemannian metric g over TX extends to every tensorial fiber bundle over X and in particular to the vector bundle $\Omega^k(X)$ of differential k-form. On the other hand, we recall that every scalar product on a real vector space V extends in a natural way to a hermitian product on its complixification $V \otimes \mathbb{C}$. In this way we can define a (scalar or hermitian) product in the space of (real or complex valued) differential k-forms on X by means of

$$\langle a, b\rangle = \int_X g(a, b)\, dV$$

and we can consider its corresponding L^2-completation, which will be denoted by $A^k(X)$.

We consider the exterior derivative operator $d : A^0(X) \to A^1(X)$ and its formal adjoint $d^* : A^1(X) \to A^0(X)$ with respect to the (scalar o hermitian) products introduced below. It is well known that the Laplacian operator $\Delta := d^*d + dd^* = d^*d$ over $A^0(X)$ is self-adjoint, positive definite and it has discrete spectrum. Consequently, there exists a countable orthonormal basis $\{\psi_\ell\}_{\ell\in\Lambda}$ of eigenfunctions of Δ. Thus, there exists a function $\lambda : \Lambda \to \mathbb{R}^+$, $\ell \mapsto \lambda_\ell$, such that $\Delta\psi_\ell = \lambda_\ell\psi_\ell$ for all $\ell \in \Lambda$. For every $f \in A^0(X)$ we consider its Fourier series $\sum_{\ell\in\Lambda} f_\ell\psi_\ell$ with $f_\ell = \langle f, \psi_\ell\rangle \in \mathbb{C}$. For every subset $L \subset \Lambda$ we define the partial sum of f over L as

$$f^L := \sum_{\ell\in L} f_\ell\psi_\ell. \tag{1}$$

Our main result is the following.

Theorem 1. *Let $f \in A^0(X)$ be a function such that $df \in A^1(X)$ and $\Delta f \in A^0(X)$ are well-defined. For each $\varepsilon > 0$ there exists a finite subset $L_f(\varepsilon) \subset \Lambda$ depending only on $\|f\|$, $\|df\|$, $\|\Delta f\|$ and ε such that*

$$\|f - f^{L_f(\varepsilon)}\| \le \varepsilon\|f\|. \tag{2}$$

In fact, $L_f(\varepsilon)$ can be chosen as the preimage by $\lambda : \Lambda \to \mathbb{R}^+ \subset \mathbb{R}$ of the compact interval $[L_f^-(\varepsilon), L_f^+(\varepsilon)]$, where

$$L_f^\pm(\varepsilon) = \frac{\|df\|^2 \pm \varepsilon^{-1}\sqrt{\|\Delta f\|^2\|f\|^2 - \|df\|^4}}{\|f\|^2}. \tag{3}$$

The proof is a direct application of the following two statements.

Lemma 2. *Consider the (non-bounded) linear operators $D_1 := d : A^0(X) \to A^1(X)$ and $D_2 := \Delta : A^0(X) \to A^0(X)$. Then for every $f \in A^0(X)$ for which $D_j(f)$, $j = 1, 2$, are defined the following relations hold:*

$$\left\| D_j\left(\sum_{\ell\in\Lambda} f_\ell\psi_\ell\right)\right\|^2 = \sum_{\ell\in\Lambda} |f_\ell|^2\lambda_\ell^j, \qquad j = 1, 2. \tag{4}$$

Moreover, $\|D_1 f\|^2 = \int_X \|\nabla_g f\|_g^2\, dV$, where ∇_g and $\|\cdot\|_g$ are the gradient operator and the norm with respect to the metric g. Furthermore the map $\lambda : \Lambda \to \mathbb{R}^+$ has discrete image and finite fibers.

Proof. Since $D_2\psi_\ell = \lambda_\ell\psi_\ell$ it follows that $D_2 f = \sum_{\ell\in\Lambda} f_\ell\lambda_\ell\psi_\ell$ and consequently $\|D_2 f\|^2 = \sum_{\ell\in\Lambda} |f_\ell|^2\lambda_\ell^2$. On the other hand, for all $\ell,\ell'\in\Lambda$ we have

$$\langle d\psi_\ell, d\psi_{\ell'}\rangle = \langle \psi_\ell, d^*d\psi_{\ell'}\rangle = \langle \psi_\ell, \Delta\psi_{\ell'}\rangle = \langle \psi_\ell, \lambda_{\ell'}\psi_{\ell'}\rangle = \lambda_{\ell'}\delta_{\ell\ell'}.$$

Since $D_1 f = \sum_{\ell\in\Lambda} f_\ell d\psi_\ell$ we deduce $\|D_1 f\|^2 = \sum_{\ell\in\Lambda} |f_\ell|^2\lambda_\ell$. The other expression for $\|D_1 f\|$ follows from the well known formula $g(df,df) = g(\nabla_g f, \nabla_g f)$. Finally, the last claim follows from the fact that the image of λ is the spectrum of the Laplacian Δ and every eigenvalue has finite multiplicity. □

Theorem 3. *Let A_0 be a separable Hilbert space with a countable orthonormal basis $\{\psi_\ell\}_{\ell\in\Lambda}$ and A_j, $j=1,2$, be Banach spaces. Consider two (not necessarily bounded) linear operators $D_j : A_0 \to A_j$, $j=1,2$. Assume that there exists a function $\lambda : \Lambda \to \mathbb{R}^+$ satisfying Relations (4). Then $L_f(\varepsilon) = \lambda^{-1}([L_f^-(\varepsilon), L_f^+(\varepsilon)])$ satisfies the error estimates (2), where*

$$L_f^{\pm}(\varepsilon) = \frac{\|D_1 f\|^2 \pm \varepsilon^{-1}\sqrt{\|D_2 f\|^2\|f\|^2 - \|D_1 f\|^4}}{\|f\|^2}. \tag{5}$$

Moreover, if λ has discrete image and finite fibers then the partial sum (1) corresponding to $L = L_f(\varepsilon)$ has only a finite number of terms for every $\varepsilon > 0$.

Proof. Given $f \in A_0$, consider a discrete random variable Z satisfying

$$P(Z = z) = \frac{1}{\|f\|^2}\sum_{\lambda_\ell = z} |f_\ell|^2,$$

whose moments of order 1 and 2 are respectively

$$E(Z) = \sum_z z\, P(Z=z) = \frac{1}{\|f\|^2}\sum_z z \sum_{\lambda_\ell=z}|f_\ell|^2 = \frac{1}{\|f\|^2}\sum_{\ell\in\Lambda}\lambda_\ell|f_\ell|^2 = \frac{\|D_1 f\|^2}{\|f\|^2},$$

$$E(Z^2) = \sum_z z^2\, P(Z=z) = \frac{1}{\|f\|^2}\sum_z z^2 \sum_{\lambda_\ell=z}|f_\ell|^2 = \frac{1}{\|f\|^2}\sum_{\ell\in\Lambda}\lambda_\ell^2|f_\ell|^2 = \frac{\|D_2 f\|^2}{\|f\|^2}$$

thanks to Relations (4). The standard deviation of Z is given by

$$\sigma(Z) = \sqrt{Var(Z)} = \frac{\sqrt{\|D_2 f\|^2\|f\|^2 - \|D_1 f\|^4}}{\|f\|^2}.$$

On the other hand, given any pair of real numbers $L^- \le L^+$ we have

$$\begin{aligned}\left\| f - f^{\lambda^{-1}([L^-,L^+])}\right\|^2 &= \sum_{\lambda_\ell\notin[L^-,L^+]}|f_\ell|^2 = \|f\|^2\sum_{z\notin[L^-,L^+]}\frac{1}{\|f\|^2}\sum_{\lambda_\ell=z}|f_\ell|^2 \\ &= \|f\|^2\sum_{z\notin[L^-,L^+]}P(Z=z) = \|f\|^2 P(Z\notin[L^-,L^+]).\end{aligned}$$

By definition of (5), it follows that $L_f^{\pm}(\varepsilon) = E(Z) \pm \varepsilon^{-1}\sigma(Z)$. We conclude the proof by applying Chebyshev's inequality

$$P\Big(|Z - E(Z)| > k\,\sigma(Z)\Big) \le \frac{1}{k^2}$$

with $k = \frac{1}{\varepsilon}$. □

Example 4. *For $X = \mathbb{S}^1$ we take the parametrization $\chi : [0,2\pi] \subset \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ given by $\chi(\varphi) = e^{i\varphi}$ and the metric $g = \frac{d\varphi^2}{4\pi^2}$ whose volume element is $dV = \frac{d\varphi}{2\pi}$. The Laplacian can we written as $\Delta = -\partial_\varphi^2$. An orthonormal eigenbasis of complex functions is given by $\psi_\ell : \mathbb{S}^1 \to \mathbb{C}$, $\psi_\ell(z) = z^\ell$, varying $\ell \in \Lambda := \mathbb{Z}$. We have that $\psi_\ell(\chi(\varphi)) = e^{i\ell\varphi}$, $\lambda_\ell = \ell^2$ and $\|D_j f\|^2 = \frac{1}{2\pi}\int_0^{2\pi}|\partial_\varphi^j f|^2\,d\varphi$.*

Example 5. *For $X=\mathbb{S}^2$ we take the parametrization $\chi:[0,2\pi]\times[0,\pi]\subset\mathbb{R}^2\to\mathbb{S}^2\subset\mathbb{R}^3$ given by*

$$\chi(\varphi,\theta)=(\cos\varphi\,\sin\theta,\sin\varphi\,\sin\theta,\cos\theta)$$

and the metric $g=\frac{1}{16\pi^2}\big(\sin^2\theta\,d\varphi^2+d\theta^2\big)$ induced by that of $\mathbb{R}^3$, whose volume element is $dV=\frac{\sin\theta}{4\pi}\,d\varphi\,d\theta$. We consider the orthonormal basis given by the harmonic spherical functions $\psi_{\ell m}:\mathbb{S}^2\to\mathbb{R}$ varying $(\ell,m)\in\Lambda:=\{(\ell,m)\in\mathbb{Z}^2,\ |m|\le\ell\}$, defined by

$$\psi_{\ell m}(\chi(\varphi,\theta))=Y_{\ell m}(\varphi,\theta)=\begin{cases}\sqrt{\frac{2(2\ell+1)(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)\cos(m\varphi) & \text{if}\quad m>0,\\[2ex] \sqrt{2\ell+1}P_\ell^0(\cos\theta) & \text{if}\quad m=0,\\[2ex] \sqrt{\frac{2(2\ell+1)(\ell+m)!}{(\ell-m)!}}P_\ell^{-m}(\cos\theta)\sin(-m\varphi) & \text{if}\quad m<0,\end{cases}$$

where $P_\ell^m(x)=\frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{\frac{m}{2}}\frac{d^{\ell+m}}{dx^{\ell+m}}\Big((x^2-1)^\ell\Big)$ are the associated Legendre polynomials, see for instance [1]. In this case $\Delta f=-\left[\frac{\partial_\theta(\sin\theta\,\partial_\theta f)}{\sin\theta}+\frac{\partial_\varphi^2 f}{\sin^2\theta}\right]$ and $\lambda_\ell=\ell(\ell+1)$. Consequently,

$$\begin{aligned}\|D_1f\|^2 &= \frac{1}{4\pi}\int_U\left[(\partial_\theta f)^2+\left(\frac{\partial_\varphi f}{\sin\theta}\right)^2\right]\sin\theta\,d\theta\,d\varphi,\\ \|D_2f\|^2 &= \frac{1}{4\pi}\int_U\left[\frac{\partial_\theta\,(\sin\theta\,\partial_\theta f)}{\sin\theta}+\frac{\partial_\varphi^2 f}{\sin^2\theta}\right]^2\sin\theta\,d\theta\,d\varphi,\end{aligned}$$

with $U=\{0\le\varphi\le 2\pi,\ 0\le\theta\le\pi\}$.

We can improve the choice of the set of frequencies $L_f(\varepsilon)$ for which the required estimate (2) is already fulfilled. One can proceed in the following way.

Theorem 6. *Assume that we have already computed the Fourier coefficients of f for a given subset $I\subset\Lambda$. Then the inequality (2) also holds for the new set of frequencies*

$$L_f(\varepsilon,I):=I\cup L_{f^{\Lambda\setminus I}}\left(\frac{\varepsilon\|f\|}{\|f^{\Lambda\setminus I}\|}\right).$$

Proof. Indeed, if we denote $D_0=\mathrm{Id}$, it follows from Parseval identity and Relations (4) that for each $j=0,1,2$ the following equalities hold

$$\|D_jf^{\Lambda\setminus I}\|^2=\|D_jf\|^2-\|D_jf^I\|^2\qquad\text{and}\qquad\|D_jf^I\|^2=\sum_{i\in I}|f_i|^2\lambda_i^j.$$

Since $\left\|f-f^{I\cup L_{f^{\Lambda\setminus I}}(\varepsilon_I)}\right\|=\left\|f^{\Lambda\setminus I}-(f^{\Lambda\setminus I})^{L_{f^{\Lambda\setminus I}}(\varepsilon_I)}\right\|\le\varepsilon_I\left\|f^{\Lambda\setminus I}\right\|=\varepsilon\|f\|$, taking $\varepsilon_I=\frac{\varepsilon\|f\|}{\|f^{\Lambda\setminus J}\|}$, it follows that $\|f-f^{L_f(\varepsilon,I)}\|\le\varepsilon\|f\|$. □

Notice that $L_f(\varepsilon,I)$ can be computed using only the norms $\{\|D_jf\|^2\}_{j=0}^2$ and the coefficients $\{f_i\}$ where the index i belongs to the first subset $I\subset\Lambda$. In practice, the set of frequencies $L_f(\varepsilon,I)$ is usually much smaller than the set of frequencies $L_f(\varepsilon)$ considered in the statement of Theorem 1. We illustrate this improvement in two academic cases in the following example. On the other hand, in Example 10 we use this procedure in an actual application.

Example 7. *Consider a function $f:\mathbb{S}^2\to\mathbb{R}$ whose expansion in spherical harmonic functions $f=\sum_{|m|\le\ell}f_{\ell m}Y_{\ell m}$ satisfies $\sum_{m=-\ell}^{\ell}|f_{\ell m}|^2=g(\ell)^2$ for each $\ell\ge 0$, where $g(t)=\sum_{i=1}^{n}a_ie^{-\frac{(t-\mu_i)^2}{2\sigma_i^2}}$ is a sum of n gaussian distributions with amplitudes a_i, means μ_i and standard deviations σ_i, see Figure 1.*

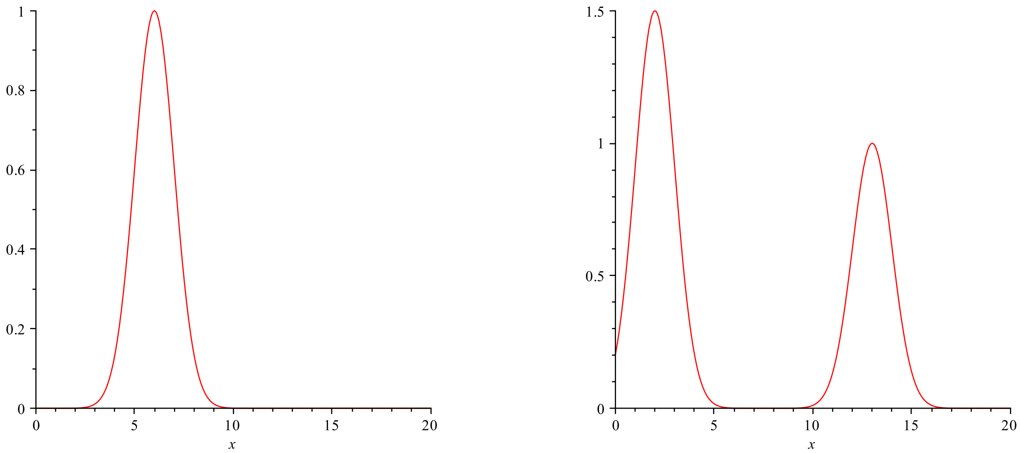

FIGURE 1. Unimodal distribution (left) with $a_1 = 1$, $\mu_1 = 6$ and $\sigma_1 = 1$ and bimodal distribution (right) with $a_1 = 1.5$, $a_2 = 1$, $\mu_1 = 2$, $\mu_2 = 13$ and $\sigma_1 = \sigma_2 = 1$.

We deal first with the unimodal case $n = 1$, $a_1 = 1$, $\mu_1 = 6$, $\sigma_1 = 1$. *We take* $I = \lambda^{-1}(J)$.

(ε, J)	$(0.5, \emptyset)$	$(0.1, [5,7])$	$(0.01, [3,8])$
$\|f - f^I\|/\|f\|$	*1.0*	*0.14*	*0.008*
$\lambda(L_f(\varepsilon, I) \setminus I)$	*[5,7]*	*[3,8]*	*[9,9]*
$\|f - f^{L_f(\varepsilon,I)}\|/\|f\|$	*0.14*	*0.008*	*0.0004*

Notice that the two first estimated intervals are approximately centered at $\mu_1 = 6$. *Now, we treat the bimodal case* $n = 2$, $a_1 = 1.5$, $a_2 = 1$, $\mu_1 = 2, \mu_2 = 13$, $\sigma_1 = \sigma_2 = 1$ *and* $I = \lambda^{-1}(J)$.

(ε, J)	$(0.5, \emptyset)$	$(0.1, [0,4])$	$(0.01, [0,4] \cup [11,15])$
$\|f - f^I\|/\|f\|$	*1.0*	*0.55*	*0.01*
$\lambda(L_f(\varepsilon, I) \setminus I)$	*[0,14]*	*[9,16]*	*[5,10]*
$\|f - f^{L_f(\varepsilon,I)}\|/\|f\|$	*0.06*	*0.006*	*0.004*

Notice that the first estimated interval contains $\mu_1 = 2$ *and the second one contains* $\mu_2 = 13$. *Moreover, the approximation* f^I *with* $I = [0,4] \cup [11,15]$ *provide the* 99% *of the norm of* f.

To finish the theoretical part of the paper, we point out that the compactness assumption on X is necessary to state Theorem 1 in its present form, but there exists an alternative statement on the simplest non-compact manifold $X = \mathbb{R}^n$:

Theorem 8. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function such that* f *itself, all its partial derivatives* $\frac{\partial f}{\partial x_j}$, $j = 1, \dots, n$ *and the Laplacian* $\Delta f := \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}$ *belong to* $L^2(\mathbb{R}^n)$. *For each* $\varepsilon > 0$ *there exists a compact subset* $L \subset \mathbb{R}^n$ *depending only on* $\|f\|, \|\nabla f\|, \|\Delta f\|$ *and* ε *such that the function*

$$f_L(x) := \int_L \hat{f}(\xi) e^{2i\pi x \cdot \xi} d\xi$$

belongs to $L^2(\mathbb{R}^n)$ *and it satisfies the following inequality*

$$\|f - f_L\| \le \varepsilon \|f\|.$$

The proof is completely analogous to the one of Theorem 1, by using the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi x \cdot \xi} dx$$

and its reconstruction formula

$$f(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2i\pi x\cdot\xi}d\xi$$

instead of Fourier series. In this case $\Lambda=\mathbb{R}^n$ and all the summations $\sum_{\ell\in\Lambda}a_\ell$ must be replaced by $\int_{\mathbb{R}^n}a(\xi)d\xi$. The analogues of Relations (4) follow from the well-known identity $\widehat{\frac{\partial f}{\partial x_j}}(\xi)=2i\pi\xi_j\hat{f}(\xi)$ using the map $\lambda:\mathbb{R}^n\to\mathbb{R}^+$ given by $\lambda(\xi)=4\pi|\xi|^2$, where $|\xi|$ denotes the euclidian norm of a vector $\xi\in\mathbb{R}^n$. In fact, as in the precedent version, $L=\lambda^{-1}(I)$ for some compact interval $I\subset\mathbb{R}$.

Remark 9. *The uncertainty principle for $f\in L^2(\mathbb{R}^n)$ asserts that $D_0(f)D_0(\hat{f})\geq C_n$, see* [3]*, where C_n is some explicit positive constant depending only on the dimension n and*

$$D_0(f)=\frac{1}{\|f\|^2}\int_{\mathbb{R}^n}|x|^2|f(x)|^2dx.$$

The uncertainty principle applied to f can be interpreted as a lower bound for the mid-point μ of the interval I:

$$\mu=\frac{\|\nabla f\|^2}{\|f\|^2}\geq 4\pi^2C_n\frac{\int_{\mathbb{R}^n}|f(x)|^2dx}{\int_{\mathbb{R}^n}|x|^2|f(x)|^2dx}.$$

Analogously, by applying the uncertainty principle to the partial derivatives $\frac{\partial f}{\partial x_j}$ for $j=1,\dots,n$, we obtain a lower bound for the width $\frac{2\sigma}{\varepsilon}$ of the interval I:

$$\left(\frac{\sigma}{\mu}\right)^2\geq\frac{4\pi^2C_n\|f\|^2}{\|\nabla f\|^4}\sum_{j=1}^{n}\frac{\left\|\frac{\partial f}{\partial x_j}\right\|^2}{D_0\left(\frac{\partial f}{\partial x_j}\right)}-1.$$

3. Application: smooth approximations of polyhedral objects

Theorem 1 can be applied to the problem of finding a smooth approximation of a geometric object, typically a curve or a surface, from which we only know a finite set of points.

3.1. **Closed Curves.** Let $\gamma(s)=(x_j(s))_{j=1}^n$ be a closed curve in $\mathbb{R}^n$ of class C^2 parametrized by arc length $s\in[0,L]$, where L is its total length. Assume we only know a finite number of consecutive points

$$\{p_k=(p_{kj})_{j=1}^n\}_{k=0}^N\subset\operatorname{Im}(\gamma),$$

with $p_0=p_N$, and we pretend to give an explicit parametrization $\hat{\gamma}(s)$ approximating $\gamma(s)$ with a relative error less than $\varepsilon>0$, i.e. $\|\gamma-\hat{\gamma}\|\leq\varepsilon\|\gamma\|$, using for this the hermitian L^2-product defined by $\langle f,g\rangle=\frac{1}{L}\int_0^L f\,\bar{g}\,ds$. We consider the orthonormal basis given by the functions $\psi_\ell(s)=e^{\frac{i2\pi\ell s}{L}}$, varying $\ell\in\mathbb{Z}$, and the corresponding Fourier series

$$x_j(s)=\sum_{\ell\in\mathbb{Z}}x_{j\ell}\psi_\ell(s)=x_{j0}+2\operatorname{Re}\left(\sum_{\ell>0}x_{j\ell}\psi_\ell(s)\right),\quad x_{j\ell}=\langle x_j,\psi_\ell\rangle\in\mathbb{C},\quad j=1,\dots,n.$$

Theorem 1 gives us the bound

$$\left\|x_j-\sum_{|\ell|=L_j^-(\varepsilon)}^{L_j^+(\varepsilon)}x_{j\ell}\psi_\ell\right\|\leq\varepsilon\|x_j\|,$$

with $L_j^{\pm}(\varepsilon)=\frac{\|x_j'\|^2\pm\varepsilon^{-1}\sqrt{\|x_j''\|^2\|x_j\|^2-\|x_j'\|^4}}{\|x_j\|^2}$. In order to obtain discrete counterparts of the continuous quantities involved in the precedent formula we proceed as follows. For each $k=1,\dots,N$ we define $ds_k=\|p_k-p_{k-1}\|$ and $s_k=s_{k-1}+ds_k$ taking also $s_0=0$. Then we can discretize the integrals involved above obtaining the following estimates for

(a) the length $L \simeq s_N$ and the Fourier coefficients $x_{j\ell} \simeq \frac{1}{s_N} \sum_{k=1}^{N} p_{kj} e^{-\frac{2i\pi\ell s_k}{s_N}} ds_k$;

(b) the squared norms $\|x_j^{(m)}\|^2 \simeq \frac{1}{s_N} \sum_{k=1}^{N} |p_{kj}^{(m)}|^2 ds_k$, $m = 0, 1, 2$, where

$$p_{kj}^{(0)} = p_{kj},\ p_{kj}^{(1)} = \frac{1}{2}\left(\frac{p_{k+1,j} - p_{k,j}}{ds_{k+1}} + \frac{p_{k,j} - p_{k-1,j}}{ds_k}\right),\ \frac{p_{kj}^{(2)}}{2} = \frac{\frac{p_{k+1,j}-p_{k,j}}{ds_{k+1}} - \frac{p_{k,j}-p_{k-1,j}}{ds_k}}{ds_{k+1} + ds_k}.$$

Besides the error ε given by considering only a finite number of Fourier terms, this procedure introduces two new sources of error, namely the approximations made in (a) and (b). Nevertheless, since the frequencies set Λ is discrete the method of choosing the subset $L_f(\varepsilon)$ is robust in the following sense. First, by perturbing slightly ε if necessary, we can assume that $L_f^{\pm}(\varepsilon)$ are not close to integer numbers. Then, if the distances ds_k between consecutive points are small enough then the set $L_f(\varepsilon)$, obtained by the discretization method described in (b), does not change.

3.2. **Star-shaped Surfaces.** Let $S \subset \mathbb{R}^3$ be a closed surface which is star-shaped with respect to the origin, i.e. for every $u \in \mathbb{S}^2$ the half line $\{\lambda u,\ \lambda \in \mathbb{R}^+\}$ cuts S in a unique point $r(u)u$ determined by the radial function $r : \mathbb{S}^2 \to \mathbb{R}^+$ which we can express as $r = \sum_{(\ell,m)\in\Lambda} r_{\ell m}\psi_{\ell m}$ according to the notations introduced in Example 5, where the coefficients

$$r_{\ell m} = \frac{1}{4\pi}\int_U r(\chi(\varphi,\theta))Y_{\ell m}(\varphi,\theta)\sin\theta\, d\theta\, d\varphi$$

can be approximately computed from a triangulation $\{T_i\}_{i=1}^N$ of S by means of

$$r_{\ell m} \simeq \frac{1}{4\pi}\sum_{i=1}^{N} \|\overline{T}_i\|^{1-\ell}\widehat{\psi}_{\ell m}(\overline{T}_i)\widehat{A}(T_i),$$

where $\overline{T}_i$ is the center of mass of the triangle T_i, $\widehat{\psi}_{\ell m} : \mathbb{R}^3 \to \mathbb{R}$ is a degree ℓ polynomial extension of $\psi_{\ell m} : \mathbb{S}^2 \to \mathbb{R}$ and $\widehat{A}(T_i)$ is the area of the spherical triangle obtained from T_i by radial projection onto $\mathbb{S}^2$. In the same vein, the squared norm $\|r\|^2 = \frac{1}{4\pi}\int_U r(\chi(\varphi,\theta))^2 \sin\theta\, d\theta\, d\varphi$ can be approximated by $\frac{1}{4\pi}\sum_{i=1}^{N}\|\overline{T}_i\|^2\widehat{A}(T_i)$. In order to obtain a discrete counterpart of

$$\|D_1 r\|^2 = \frac{1}{4\pi}\int_U \left[(\partial_\theta r)^2 + \left(\frac{\partial_\varphi r}{\sin\theta}\right)^2\right]\sin\theta\, d\theta\, d\varphi,$$

we need to consider the parametrization $\sigma(\varphi,\theta) = r(\chi(\varphi,\theta))\chi(\varphi,\theta)$ of S. A straightforward computation using that $\chi, \partial_\theta\chi$ and $\frac{\partial_\varphi \chi}{\sin\theta}$ is a direct orthonormal basis, we obtain that the outer normal unitary vector $N : S \to \mathbb{S}^2$ of S satisfies the equality

$$\sigma^* N \cdot \chi(\varphi,\theta) = \frac{r}{\sqrt{r^2 + (\partial_\theta r)^2 + \left(\frac{\partial_\varphi r}{\sin\theta}\right)^2}}$$

and consequently

$$\|D_1 r\|^2 \simeq \frac{1}{4\pi}\sum_{i=1}^{N}\|\overline{T}_i\|^2\left[\frac{\|\overline{T}_i\|^2}{(N_i\cdot\overline{T}_i)^2} - 1\right]\widehat{A}(T_i),$$

where N_i is the outer normal unitary vector to the triangle T_i, which can be easily computed from the given triangulation of S.

Finally, in order to compute a discrete counterpart of the squared norm of the spherical laplacian of r, we apply the following formula given in [4] for the discrete version of the

Laplacian of a function f defined in the vertex set of a triangulated surface $M \subset \mathbb{R}^3$:

$$\Delta_M f(p_i) = \frac{4 \sum\limits_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f(p_j) - f(p_i))}{\sum\limits_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \|p_j - p_i\|^2}.$$

Here $N(i)$ denotes the set indexing the vertex adjacent to p_i. If $j \in N(i)$ then α_{ij} and β_{ij} are the opposite angles to the edge $p_i p_j$. In our case we must take $M = \mathbb{S}^2$ and $f = r$. Thus,

$$\|D_2 r\|^2 \simeq \frac{1}{4\pi} \sum_{i=1}^{V} \left[\frac{4 \sum\limits_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(\|p_j\| - \|p_i\|)}{\sum\limits_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \|p_j - p_i\|^2} \right]^2 \frac{1}{3} \sum_{j \in N(i)} \widehat{A}(T_{ij}),$$

where $\{p_i\}_{i=1}^V$ is the set of vertex of the triangulation, T_{ij} is the unique triangle containing the oriented edge $p_i p_j$ and α_{ij} and β_{ij} are the opposite angles to the edge $p_i p_j$ after projecting the triangulation radially onto $\mathbb{S}^2$.

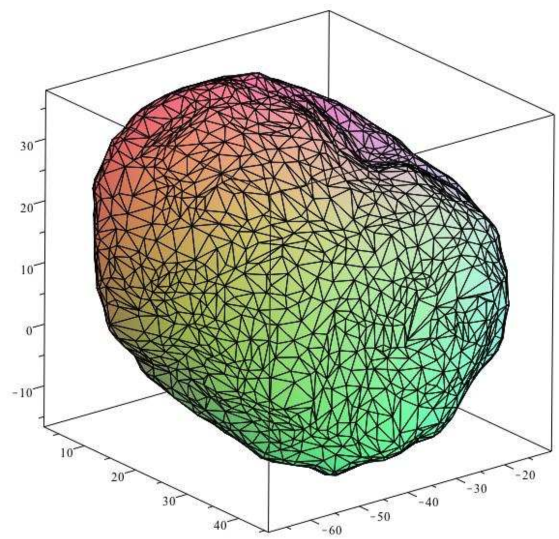

FIGURE 2. Triangulation of the surface of the left atrium surface, excluding the pulmonary veins and the appendage, of a human heart. The length units are milimeters.

Example 10. *The triangulation of the surface of the left atrium of a human heart shown in Figure 2 has* $N = 4000$ *triangles and* $V = 2002$ *vertex. From the numerical data we compute the approximations described below obtaining* $\|r\| \simeq 84.88$, $\|D_1 r\| \simeq 52.62$ *and* $\|D_2 r\| \simeq 156.86$. *Then, taking* $\varepsilon = 0.01$, *we have* $L_r(0.01) = \{(\ell, m) \in \mathbb{Z}^2, |m| \le \ell \le 23\}$ *because* $L_r^-(0.01) \simeq -582.73$, $L_r^+(0.01) \simeq 583.51$ *and* $\lambda_\ell = \ell(\ell+1)$. *After computing Fourier coefficients up to degree* $\ell = 5$, *i.e. taking* $I = \{(\ell, m) \in \mathbb{Z}^2, |m| \le \ell \le 5\}$, *we apply Corollary 6 obtaining an improved set of frequencies* $L_r(0.01, I) = I \cup \{(\ell, m) \in \mathbb{Z}^2, |m| \le \ell, 10 \le \ell \le 16\}$. *In Table 1 we list the* L^2*-norm of the degree* ℓ *homogeneous part* $r_\ell = \sum\limits_{m=-\ell}^{\ell} r_{\ell m} \psi_{\ell m}$ *of* r *for* $0 \le \ell \le 17$. *In Figure 3 we represent graphically these norms in function of* ℓ.

ℓ	0	1	2	3	4	5	6	7	8
$\Vert r_\ell \Vert$	83.4	1.46	14.8	3.13	2.39	1.59	1.74	1.51	1.09

ℓ	9	10	11	12	13	14	15	16	17
$\Vert r_\ell \Vert$	1.02	0.57	1.09	0.93	0.88	0.88	1.15	1.41	1.56

TABLE 1. L^2-norm of the degree ℓ homogeneous part r_ℓ of r for $0 \leq \ell \leq 17$.

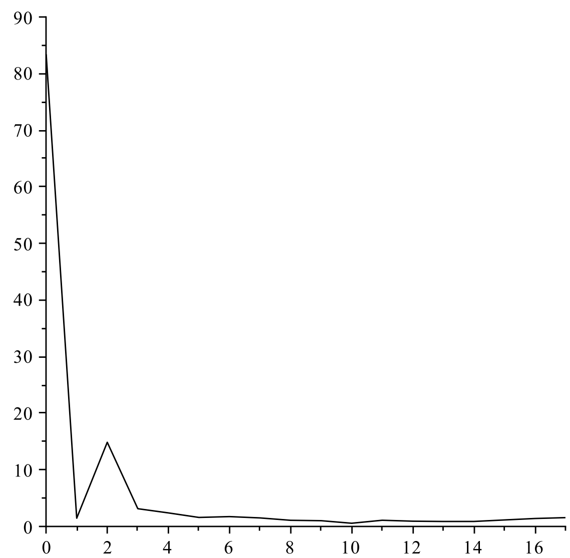

FIGURE 3. Graphic representation of the set $\{(\ell, \Vert r_\ell \Vert), 0 \leq \ell \leq 17\}$.

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