AVERAGING AND FIXED POINTS IN BANACH SPACES

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We use various averaging techniques to obtain results in different aspects of functional analysis and Banach space theory, particularly in fixed point theory.

Specifically, in the second chapter, we discuss the class of so-called *mean nonexpansive maps*, introduced in 2007 by Goebel and Japón Pineda, and we prove that mean isometries must be isometries in the usual sense. We further generalize this class of mappings to what we call the *affine combination maps*, give many examples, and study some preliminary properties of this class.

In the third chapter, we extend Browder's and Opial's famous Demiclosedness Principles to the class of mean nonexpansive mappings in the setting of uniformly convex spaces and spaces satisfying Opial's property. Using this new demiclosedness principle, we prove that the iterates of a mean nonexpansive map converge weakly to a fixed point in the presence of asymptotic regularity at a point.

In the fourth chapter, we investigate the geometry and fixed point properties of some equivalent renormings of the classical Banach space c_0. In doing so, we prove that all norms on ℓ^∞ which have a certain form must fail to contain asymptotically isometric copies of c_0.

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PREFACE

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1.0 INTRODUCTION

We begin with some preliminary notions essential to discussing fixed point theory.

Definition 1.0.1. Let C be a set and $f : C \to C$ a function. We say f has a *fixed point* in C if there exists $x_0 \in C$ such that $f(x_0) = x_0$.

Definition 1.0.2. Let (M, d) be a metric space and suppose $f : M \to M$ is a function.

1. We say f is *Lipschitz* (or k*-Lipschitz*) if there exists $k \geq 0$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in M$. We refer to the infimal Lipschitz constant of f as $k(f)$.

2. If f is 1-Lipschitz, we say it is *nonexpansive.* If f is k-Lipschitz for some $k \in [0, 1)$, we say f is a *strict contraction.*
3. If $(X, \|\cdot\|)$ is a normed vector space, we say that $(X, \|\cdot\|)$ has the *fixed point property for nonexpansive maps* (fpp(ne)) if, for every nonempty closed, bounded, convex (c.b.c.) $C \subset X$, and for every nonexpansive $T : C \to C$, it follows that T must have a fixed point in C.

The questions of fixed point theory are typically of a similar flavor: *given a subset C of a topological space and some collection of functions mapping C back into C satisfying certain properties, can we guarantee that every such function has a fixed point?* This setting is usually too general to say anything positive, so "topological space" will usually be replaced by "Banach space" (that is, a complete normed vector space), and the subsets in question are usually c.b.c. or compact, either in the topology induced by the norm or in some other useful topology on the space. In 1910, Brouwer proved his famous fixed point theorem [36, Ch. 8] about the simplest class of Banach spaces, $(\mathbb{R}^n, \|\cdot\|)$:

Theorem 1.0.1 (Brouwer)**.** *Suppose that $K \subset \mathbb{R}^n$ is (nonempty) norm-compact and convex and that $f : K \to K$ is continuous. Then f has a fixed point in K.*

Note that, in $(\mathbb{R}^n, \|\cdot\|)$, we have that a set K is compact if and only if K is closed and bounded. Thus, we may rephrase Brouwer's result to say "$(\mathbb{R}^n, \|\cdot\|)$ has the fixed point property for continuous maps." Schauder later extended this result [47] to the infinite-dimensional setting. We state a version of Schauder's theorem which elucidates the analogy.

Theorem 1.0.2 (Schauder)**.** *Suppose X is a locally convex topological vector space (e.g. any Banach space) and suppose that $K \subset X$ is nonempty, compact and convex. Then every continuous $T : K \to K$ has a fixed point.*

Answering the fixed point question for c.b.c., *noncompact* subsets of a given space is where the theory begins to come into its own. Banach's famous contraction mapping theorem has completeness as the essential assumption of the underlying space, but makes no mention of compactness. Indeed, Banach's theorem does not require boundedness, convexity, or even a linear structure.

Theorem 1.0.3 (Banach Contraction Mapping Theorem (BCMT))**.** *Suppose (M, d) is a complete metric space and $T : M \to M$ is a strict contraction; that is, there exists a $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in M.$$

Then T has a unique fixed point $x_0 \in M$. Moreover, $d(T^n z, x_0) \to_n 0$ for all $z \in M$, where $T^n := T \underbrace{\circ \cdots \circ}_{n} T$.

When the condition on the function in BCMT is relaxed to nonexpansiveness, additional assumptions must be placed on the domain of the function in order to say anything positive. Inspired by Brouwer and Schauder, we consider closed, bounded, convex subsets of an infinite dimensional Banach space (which crucially are not necessarily norm-compact). In this setting, the fixed point problem becomes significantly harder. To illustrate this, we present two classical examples of fixed-point-free nonexpansive mappings defined on closed, bounded, convex subsets of the Banach space of real sequences which converge to 0, $(c_0, \|\cdot\|_\infty)$, and absolutely-summable sequences, $(\ell^1, \|\cdot\|_1)$, endowed with their usual norms as defined below.

Example 1.0.1. Define

$$c_0 := \left\{(x_1, x_2, \ldots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} x_k = 0\right\}$$

and define the norm $\|(x_n)_n\|_\infty := \sup_n |x_n|$. We define our closed, bounded, convex (and not norm-compact) set

$$C := \{x \in c_0 : 0 \le x_k \le 1 \text{ for all } k \in \mathbb{N}\}.$$

We define a right-shift map on C,

$$T : C \to C : (x_1, x_2, \ldots) \mapsto (1, x_1, x_2, \ldots).$$

Then

$$\|Tx - Ty\|_\infty = \|(0, x_1 - y_1, x_2 - y_2, \ldots)\|_\infty = \sup_n |x_n - y_n| = \|x - y\|_\infty$$

and we have that T is an isometry. Finally, observe that $Tx = x$ if and only if $x = (1, 1, 1, \ldots)$. Not only is the sequence $(1, 1, \ldots)$ not in C, it is not even in c_0! Therefore, T is fixed-point-free on C, and we can say that $(c_0, \|\cdot\|_\infty)$ fails fpp(ne).

Example 1.0.2. Define

$$\ell^1 := \{(x_1, x_2, \ldots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |x_k| < \infty\},$$

and define the norm $\|x\|_1 := \sum_{k=1}^{\infty} |x_k|$. Now define the closed, bounded, convex (but not norm-compact) subset

$$C := \left\{x = (x_n)_n \in \ell^1 : x_n \ge 0 \text{ for all } n, \text{ and } \sum_{n=1}^{\infty} x_n = 1\right\}.$$

Let $T : C \to C$ be a right-shift map similar to the one defined above:

$$Tx := (0, x_1, x_2, \ldots).$$

Then for any $x, y \in C$,

$$\|Tx - Ty\|_1 = \|(0, x_1 - y_1, x_2 - y_2, \ldots)\|_1 = \sum_{n=1}^{\infty} |x_n - y_n| = \|x - y\|_1,$$

and we see that T is an isometry (and hence nonexpansive). Moreover, $Tx = x$ if and only if $x = 0$. But $0 \notin C$, so T is fixed-point-free on C. From this, we can say that $(\ell^1, \|\cdot\|_1)$ *fails fpp(ne)*.

In order to obtain positive results, we will ask an easier preliminary question: *does a nonexpansive map T on a c.b.c. C satisfy* $\inf_C \|Tx - x\| = 0$*?* The answer is *yes*, and the proof is a clever application of BCMT.

Theorem 1.0.4. *Let $(X, \|\cdot\|)$ be a Banach space, $C \subset X$ c.b.c., and $T : C \to C$ nonexpansive. Then* $\inf_C \|Tx - x\| = 0$.

Proof. Fix $z \in C$ and $\varepsilon \in (0,1)$. For $x \in C$, define $T_\varepsilon x := (1-\varepsilon)Tx + \varepsilon z$. Then

$$\|T_\varepsilon x - T_\varepsilon y\| = (1-\varepsilon)\|Tx - Ty\| \leq (1-\varepsilon)\|x - y\|.$$

That is, T_ε is a strict contraction for all $\varepsilon \in (0,1)$. Since X is Banach and C is closed, C is a complete metric space (with the metric induced by $\|\cdot\|$) and, by BCMT, T_ε has a unique fixed point $x_\varepsilon \in C$ for all $\varepsilon \in (0,1)$. Now

$$\|Tx_\varepsilon - x_\varepsilon\| = \|Tx_\varepsilon - T_\varepsilon x_\varepsilon\| = \varepsilon\|Tx_\varepsilon - z\| \to 0 \text{ as } \varepsilon \to 0$$

since C is bounded. Hence, $\inf_C \|Tx - x\| = 0$. □

Note that the above theorem guarantees the existence of an *approximate fixed point sequence* (afps) for any nonexpansive self-mapping of a closed, bounded, convex subset of a Banach space. That is, for any nonexpansive $T : C \to C$, there exists $(x_n)_n$ in C for which

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0,$$

provided that C is closed, bounded, and convex.

After Schauder, it took 35 years until the next major breakthroughs in understanding the "fixed point question," when, in 1965, several major positive results were proven independently by Browder [7, 8], Göhde [30], and Kirk [32]. First, Browder proved that Hilbert spaces (a Hilbert space is a vector space equipped with an inner product whose induced norm is complete) have fpp(ne) [7].

Theorem 1.0.5 (Browder's Theorem). *All Hilbert spaces have fpp(ne).*

His proof relied on the relationship between nonexpansive maps and the so-called *monotone* maps. Specifically, if $(H, (\cdot, \cdot))$ is Hilbert and $T : H \to H$, we say T is *monotone* if

$$(Tx - Ty, x - y) \geq 0$$

for all $x, y \in H$. Browder observed that for any nonexpansive map $U : H \to H$, the associated mapping $T := I - U$ (where I is the identity mapping on H) is monotone, and the fixed point property for Hilbert spaces follows from an extension theorem of Kirszbraun [48] in the Hilbert space setting and several lemmas which Browder proves about monotone mappings.

Also in 1965, this result was independently improved by Göhde [30] from Hilbert spaces to the class of *uniformly convex* Banach spaces, defined below.

Definition 1.0.3. A Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ for which

$$\begin{cases} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x - y\| \geq \varepsilon \end{cases} \implies \frac{1}{2}\|x + y\| \leq 1 - \delta.$$

This extends Browder's Theorem for Hilbert spaces because all Hilbert spaces are uniformly convex.

Theorem 1.0.6 (Göhde's Theorem). *If $(X, \|\cdot\|)$ is uniformly convex, then X has fpp(ne).*

Göhde's proof relied on the following keen observation about uniformly convex spaces: if $(X, \|\cdot\|)$ is uniformly convex and $C \subset X$ is bounded and convex, then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $u, v \in C$ and $z = tu + (1 - t)v$ for some $t \in [0, 1]$, then

$$\begin{cases} \|x - u\| \leq \|z - u\| + \delta \\ \|x - v\| \leq \|z - v\| + \delta \end{cases} \implies \|x - z\| \leq \varepsilon.$$

Again in 1965, Browder [8] also proved, independently from Göhde, that uniformly convex spaces have fpp(ne). He did so in a way that was distinct both from his own methods in the Hilbert space setting and from Göhde's methods. In short, he proved that uniformly convex spaces have fpp(ne) by using an argument about the diameters of certain sets in uniformly convex sets. Recall that we define the radius and diameter of a set C in a Banach space as follows:

Definition 1.0.4. Suppose $(X, \|\cdot\|)$ is a Banach space and $C \subset X$ is nonempty.

1. Fix $x \in C$. Define $r(x, C) := \sup\{\|x - z\| : z \in C\}$. We define the *radius* and *diameter of* C, resp., to be

$$\text{rad}(C) := \inf_{x \in C} r(x, C) \quad \text{and}$$
$$\text{diam}(C) := \sup_{x \in C} r(x, C)$$

   Note that we always have $\text{rad}(C) \leq \text{diam}(C)$.

2. If $\text{diam}(C) > 0$, we say that C has *normal structure* if $\text{rad}(C) < \text{diam}(C)$. If every such c.b.c. $C \subset X$ has normal structure, we say that the space $(X, \|\cdot\|)$ has normal structure.

Finally, Kirk [32] proved the strongest result of 1965 and did so independently from both Browder and Göhde. Before stating Kirk's Theorem, let's recall one final definition.

Recall that for any normed linear space $(X, \|\cdot\|)$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we define the *dual space* (sometimes called the *continuous dual space*) to X, denoted by X^*, as

$$X^* := \{f : X \to \mathbb{K} : f \text{ is continuous and linear}\}.$$

We define the *weak topology* on X to be the coarsest topology on X such that each $f \in X^*$ is continuous. We say that a Banach space $(X, \|\cdot\|)$ has the *weak fixed point property for nonexpansive maps* (w-fpp(ne)) if, for every nonempty, weakly compact, convex set $K \subset X$, every nonexpansive map $T : K \to K$ has a fixed point. Now we state two versions of Kirk's Theorem.

Theorem 1.0.7 (Kirk's Theorem, Version 1)**.** *Suppose X is a Banach space having weak normal structure (i.e. every weakly compact, convex subset of X with positive diameter has normal structure). Then X has w-fpp(ne).*

Let $X^{**} := (X^*)^*$ denote the second dual of X. Recall that a Banach space X is said to be *reflexive* if the linear map $j : X \to X^{**}$ is an isometric isomorphism, where j is defined for all $x \in X$ and for all $\varphi \in X^*$ by $j(x)(\varphi) := \varphi(x)$. That is, reflexive spaces are those which correspond exactly with their second dual space. It is a fact that, in a reflexive space, a set C is c.b.c. implies that C is weakly compact. With this in mind, we can state Kirk's theorem in a "strong" way.

Theorem 1.0.8 (Kirk's Theorem, Version 2)**.** *Suppose $(X, \|\cdot\|)$ is reflexive and has normal structure. Then X has fpp(ne).*

Note that Kirk's Theorem is a genuine extension of both Browder's and Göhde's results since all uniformly convex spaces are reflexive and have normal structure. There are two key elements to Kirk's proof, both of which quite naturally generalize the techniques developed by Browder in the uniformly convex setting. The first element of Kirk's proof is that any nonexpansive mapping $T : C \to C$, with C weakly compact, must have a *minimal invariant set*; that is, by Zorn's Lemma, there must exist a set $K \subseteq C$ for which $T(K) \subseteq K$ and if $K' \subseteq K$ also satisfies $T(K') \subseteq K'$, it must follow that $K' = K$. The second key element of his proof is that, in the presence of normal structure (or weak normal structure, as the case may be), any minimal invariant set must be a singleton; that is, T must have at least one fixed point.

In 1967, Opial [43] revealed the essential pieces of the proof of Browder's Theorem for Hilbert spaces and used them to prove a new and interesting generalization of Theorem 1.0.5 in a way distinct from both Göhde's and Kirk's methods. Recall that a sequence $(x_n)_n$ in X converges weakly to $x \in X$ if $\lim_n \varphi(x_n) = \varphi(x)$ for all $\varphi \in X^*$. We will use "$\rightharpoonup$" to denote weak convergence and "$\to$" to denote strong (norm) convergence throughout. Before stating Opial's result, we need two notions.

Definition 1.0.5 (Opial's Property)**.** Suppose $(X, \|\cdot\|)$ is a Banach space. We say X has *Opial's property* if, whenever $x_n \rightharpoonup x$, we have

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|$$

for all $y \neq x$.

Note that all Hilbert spaces have Opial's property, as do ℓ^p for all $p \in (1, \infty)$. However, L^p fails to have Opial's property for $p \neq 2$.

Definition 1.0.6 (Demiclosedness)**.** Suppose $(X, \|\cdot\|)$ is a Banach space, $C \subseteq X$, and $F : C \to X$ a function. For any $y \in X$, we say F is *demiclosed at* y if, whenever $x_n \rightharpoonup x$ in C and $Fx_n \to y$, it follows that $Fx = y$. If F is demiclosed for all $y \in X$, we call F *demiclosed*.

Note that if a function T is such that $I - T$ is demiclosed at 0 and T has a weakly convergent approximate fixed point sequence, then T will have a fixed point. A detailed proof of this fact can be found in Lemma 3.3.1.

Theorem 1.0.9 (Opial's Demiclosedness Principle)**.** *Suppose $(X, \|\cdot\|)$ is a Banach space with Opial's property. Then for any closed and convex $C \subseteq X$, any nonexpansive map $T : C \to X$ is such that $I - T$ is demiclosed at 0. In particular, $(X, \|\cdot\|)$ has w-fpp(ne).*

Using Opial's ideas and a subtle refinement of the techniques developed by Göhde, Browder [9] proved his famous Demiclosedness Principle in 1968.

Theorem 1.0.10 (Browder's Demiclosedness Principle)**.** *If $(X, \|\cdot\|)$ is uniformly convex, $C \subseteq X$ is closed, bounded, and convex, and $T : C \to X$ is nonexpansive, then $I - T$ is demiclosed. In particular, X has fpp(ne).*

It has been of interest to study classes of mappings broader than the nonexpansive maps. For instance, Goebel and Kirk [26] defined the class of *asymptotically nonexpansive mappings*; that is, we say a function $T : C \to C$ is asymptotically nonexpansive if, for all $n \in \mathbb{N}$ and $x, y \in X$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

where $k_n \searrow 1$. If a mapping is nonexpansive, it is easy to see that it is asymptotically nonexpansive for any choice of $(k_n)_n$ decreasing to 1. In 1991, Xu [51] extended Browder's Demiclosedness Principle to the class of asymptotically nonexpansive mappings, and Garcia-Falset, Sims, and Smyth [22] extended Opial's Demiclosedness Principle to asymptotically nonexpansive mappings in 1996. We provide generalizations of both Opial's and Browder's Demiclosedness Principles, which are distinct from the results of Xu, Garcia-Falset, Sims, and Smyth, in Chapter 3. In particular, we prove the Demiclosedness Principle for *mean* *nonexpansive mappings*, which were introduced in 2007 by Goebel and Japón Pineda [25].

We say a function $T : C \to C$ is *mean nonexpansive* if there is a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1, \alpha_n > 0$ and $\alpha_1 + \cdots + \alpha_n = 1$ for which

$$\sum_{k=1}^{n} \alpha_k \left\| T^k x - T^k y \right\| \leq \|x - y\|$$

for all $x, y \in C$. Again, notice that any nonexpansive mapping will be mean nonexpansive for any choice of α. Also note that we provide and study a further generalization of this class, which we call the class of *affine combination mappings*, in Chapter 2. Just like Xu, Garcia-Falset, Sims, and Smyth, we use our Demiclosedness Principle for mean nonexpansive mappings to prove fixed point theorems in the presence of an approximate fixed point sequence, which must be assumed to exist in the case of both asymptotically nonexpansive and mean nonexpansive mappings.

So far, we have only seen positive fixed point property results in two distinct contexts: closed, bounded, convex subsets of reflexive spaces, or weakly compact, convex subsets of possibly nonreflexive spaces. All Hilbert spaces are reflexive, uniformly convex spaces are reflexive, and we have a theorem about reflexive spaces having normal structure. Further, we have two results of Maurey (see [1, Ch. 3, §V] and [41]) regarding reflexivity and the fixed point property. First,

Theorem 1.0.11 (Maurey's L^1 Theorem)**.** *All reflexive subspaces of* $(L^1[0,1], \|\cdot\|_1)$ *have fpp(ne).*

Recall that a Banach space $(X, \|\cdot\|)$ is called *superreflexive* if and only if there exists an equivalent norm $\|\cdot\|_0$ on X which is uniformly convex, an equivalence which is due to Enflo [19]. Maurey's second result that we will state is of a slightly different flavor than the rest of the theorems listed here because it deals only with the fixed points of isometries.

Theorem 1.0.12 (Maurey's Superreflexive Theorem)**.** *Superreflexive Banach spaces have the fixed point property for isometries.*

Both of Maurey's theorems were proven using "nonstandard methods" regarding the ultrapowers of Banach spaces. For a thorough survey of nonstandard techniques in fixed point theory, see [1]. In Chapter 2, we will investigate whether or not Maurey's Superreflexive

Theorem extends to the class of *mean isometries*. Indeed, we will answer this question in the affirmative for the class of mean isometries with multi-index of length 2, and also that such mean isometries defined on closed, bounded, convex subsets of Banach spaces must also have approximate fixed point sequences.

In Examples 1.0.1 and 1.0.2, we saw that the lack of reflexivity led to the failure of fpp(ne). On the other hand, we do have the w-fpp(ne) results for possibly nonreflexive spaces having normal structure or Opial's property. Several questions naturally arise from these observations: *Do there exist weakly compact, convex sets which fail to have fpp(ne)?* and *must a Banach space be reflexive in order to have fpp(ne)?*

The answers to these questions are "yes" and "no," respectively.

In 1981, Alspach [2] answered the first question by providing an example of a weakly compact, convex set which fails to have the fixed point property for nonexpansive maps. His example, now called "Alspach's Map," is an isometry on a weakly compact, convex subset of $L^1[0,1]$ which is fixed-point-free. That is, he proved the following theorem.

Theorem 1.0.13 (Alspach)**.** *$(L^1[0,1], \|\cdot\|_1)$ fails to have the w-fpp for isometries. Hence, $(L^1[0,1], \|\cdot\|_1)$ fails to have w-fpp(ne).*

In 2008, Lin [40] answered the second question in the negative by providing an example of a nonreflexive space which has fpp(ne). His example is a renorming of $(\ell^1, \|\cdot\|_1)$. For all $x \in \ell^1$, define a new norm

$$\|x\| := \sup_{n\in\mathbb{N}} \left(\frac{\gamma_n}{1+\gamma_n} \sum_{k=n}^{\infty} |x_k| \right)$$

where $\gamma_n := 8^n$ for all n. Note that $\|\cdot\|$ is Lipschitz-equivalent to $\|\cdot\|_1$, with

$$\frac{8}{9} \|x\|_1 \leq \|x\| \leq \|x\|_1 ,$$

and thus $(\ell^1, \|\cdot\|)$ is nonreflexive. Lin proved the following theorem.

Theorem 1.0.14 (Lin)**.** *$(\ell^1, \|\cdot\|)$ has fpp(ne).*

Lin's result shows that the notions of reflexivity and fpp(ne) cannot be equivalent, but it is still possible that reflexivity implies the fixed point property. A partial positive result in this vein was given by Domínguez Benavides [4] in 2009, and, interestingly, his result also pertains to altering the original norm.

Theorem 1.0.15 (Domínguez Benavides)**.** *If* $(X, \|\cdot\|)$ *is reflexive, then there exists a norm* $\|\cdot\|_0$ *on* X *that is Lipschitz-equivalent to* $\|\cdot\|$ *such that* $(X, \|\cdot\|_0)$ *has fpp(ne).*

With both Lin's and Domínguez Benavides' results in mind, we study the following question in Chapter 4: *can* c_0 *be renormed to have fpp(ne)?*

2.0 NONEXPANSIVENESS AND AVERAGING

Beginning with Banach's Contraction Mapping Theorem, studying the iterates of functions has been of significant interest. In particular, fixed point theorems have been proven for uniformly Lipschitzian mappings [27] and asymptotically nonexpansive mappings [22, 26, 51], and both notions involve conditions being placed on the behavior of iterates. In the case of a uniformly Lipschitzian mapping T on a metric space (M, d), one would have some $k \geq 0$ for which

$$d(T^n x, T^n y) \leq k d(x, y) \quad \text{for all } n \text{ and all } x, y \in M,$$

and in the case of asymptotically nonexpansive mappings, one would have T satisfying

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \text{for all } n \text{ and all } x, y \in M,$$

where $(k_n)_n$ is a sequence of real numbers strictly decreasing to 1. With this in mind, Goebel and Japón Pineda [25] sought a fruitful generalization to the nonexpansive mappings which involved multiple iterates and which is distinct from the class of asymptotically nonexpansive mappings. A first attempt was to consider a weighted average of the first two iterates of a function $T : C \to C$ where C is a convex subset of a Banach space; i.e. given $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, let's consider a new function $T_\alpha : C \to C$, given by

$$T_\alpha := \alpha_1 T + \alpha_2 T^2.$$

If T_α is nonexpansive, can we garner any interesting information about T? The answer is generally "no," since this condition does not guarantee continuity or the existence of an approximate fixed point sequence, as we see in the next examples which appear in [46, Examples 3.5, 3.6].

Example 2.0.1. Let $T : [0,1] \to [0,1] : x \mapsto \chi_{[0,1/2]}(x)$. Then T is discontinuous, and $T^2 x = \chi_{(1/2,1]}(x)$ so T^2 is also discontinuous. Further,

$$\inf_{x\in[0,1]} |Tx - x| = \frac{1}{2},$$

so T does not have an approximate fixed point sequence. However, for $\alpha = (1/2, 1/2)$, we find that

$$T_\alpha x = \frac{1}{2}Tx + \frac{1}{2}T^2 x = \frac{1}{2}$$

for all $x \in [0,1]$. That is, $|T_\alpha x - T_\alpha y| = 0$ for all $x, y \in [0,1]$, so T_α is nonexpansive.

This is a rather extreme example since we do not even have continuity of T, so one could imagine considering only those functions T which were continuous and had T_α nonexpansive. This is still insufficient, as we see in the next example. Indeed, even if the original function T is uniformly Lipschitzian, nonexpansiveness of T_α still does not guarantee the existence of an approximate fixed point sequence for T.

Example 2.0.2. Benyamini and Sternfeld [5] showed in 1983 that, if $(X, \|\cdot\|)$ is infinite dimensional, then there always exists a Lipschitz retraction of the unit ball of X,

$$B_X := \{x \in X : \|x\| \le 1\},$$

onto the unit sphere of X,

$$S_X := \{x \in X : \|x\| = 1\}.$$

That is, there exists some function $R : B_X \to S_X$ which is Lipschitz and satisfies $Rx = x$ for all $x \in S_X$. Furthermore, Piasecki [46, Example 3.6] notes that the Lipschitz constant of such a retraction must be quite large: $k(R) \ge 3$, where $k(R)$ denotes the Lipschitz constant of R. Now define $T : B_X \to S_X \subset B_X$ to be $T := -R$. Then

$$T^2 x = -R(-Rx) = -(-Rx) = Rx \quad \text{since } Rx \in S_X \implies -Rx \in S_X,$$

and similarly,

$$T^n x = (-1)^n Rx$$

for all $x \in B_X$. Thus, for all $x, y \in B_X$ and any $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| = \|(-1)^n Rx - (-1)^n Ry\| = \|Rx - Ry\| \leq k(R)\,\|x - y\|$$

and T is uniformly Lipschitzian. However, for $\alpha = (1/2, 1/2)$,

$$T_\alpha x = \frac{1}{2}Tx + \frac{1}{2}T^2 x = \frac{1}{2}(-Rx) + \frac{1}{2}Rx = 0,$$

and again we see that T_α is a constant (and hence nonexpansive). However, for any $x \in B_X$,

$$\left\|T^2 x - Tx\right\| = \|Rx - (-Rx)\| = 2\,\|Rx\| = 2$$

since $Rx \in S_X$. On the other hand, this yields

$$2 = \left\|T^2 x - Tx\right\| \leq k(T)\,\|Tx - x\| \leq k(R)\,\|Tx - x\|\,,$$

which tell us that $0 < 2/k(R) \leq \|Tx - x\|$, and T cannot have an approximate fixed point sequence.

As a second attempt, consider a function T on a metric space (M, d) for which

$$\max\{d(Tx, Ty), d(T^2 x, T^2 y)\} \leq d(x, y)$$

for all $x, y \in M$. However, it is easy to see that any such function is already nonexpansive. Next, one could consider a function for which

$$\min\{d(Tx, Ty), d(T^2 x, T^2 y)\} \leq d(x, y)$$

for all $x, y \in M$. All nonexpansive mappings satisfy this condition, but such functions need not be continuous and do not necessarily have approximate fixed point sequences, as shown in the next example given by Goebel and Japón Pineda.

Example 2.0.3. Let $\varphi : [1/4, 1/2) \to [3/4, 1]$ be one-to-one and onto. Define $T : [0,1] \to [0,1]$ by

$$Tx := \begin{cases} x + \frac{1}{2} & x \in \left[0, \frac{1}{4}\right) \\ \varphi(x) & x \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ x - \frac{1}{2} & x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \varphi^{-1}(x) & x \in \left[\frac{3}{4}, 1\right] \end{cases}.$$

T is discontinuous, and $|x - Tx| \geq 1/4$ for all $x \in [0,1]$, so T cannot have an approximate fixed point sequence. A simple calculation shows that $T^2x = x$ for all $x \in [0,1]$, so T^2 is nonexpansive and hence

$$\min\{|Tx - Ty|, |T^2x - T^2y|\} \leq |x - y|$$

for all $x, y \in [0,1]$.

Lying between the numbers $\min\{d(Tx,Ty), d(T^2x,T^2y)\}$ and $\max\{d(Tx,Ty), d(T^2x,T^2y)\}$ are all of the weighted averages of $d(Tx,Ty)$ and $d(T^2x,T^2y)$; that is, all numbers of the form

$$td(Tx,Ty) + (1-t)d(T^2x,T^2y)$$

for some $t \in [0,1]$. With this in mind, Goebel and Japón Pineda defined and studied the class of *mean nonexpansive mappings.*

2.1 MEAN LIPSCHITZ CONDITION

Definition 2.1.1. Let (M,d) be a metric space. A function $T : C \to C$ is called *mean k-Lipschitzian* (or *α-k-Lipschitzian*) if there exists a $k > 0$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1, \alpha_n > 0$, each $\alpha_j \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$ such that, for all $x, y \in C$, we have

$$\sum_{j=1}^{n} \alpha_j d(T^j x, T^j y) \leq kd(x,y).$$

When $k = 1$, we say T is *α-nonexpansive* (or *mean nonexpansive*).

First, observe that if $T : C \to C$ is k-Lipschitzian for some $k \geq 0$, then

1. $k(T^n) \leq k^n$ for all $n \in \mathbb{N}$, and
2. for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with each $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$, we have that T is α-$\widetilde{k}$-Lipschitz, where $\widetilde{k} := \alpha_1 k + \alpha_2 k^2 + \cdots \alpha_n k^n$.

Thus, all Lipschitzian maps are mean-Lipschitzian. In particular, all nonexpansive maps are mean-nonexpansive for any choice of α. We will primarily be concerned with mean nonexpansive mappings defined on some closed, bounded, convex subset of a Banach space. To see that this is indeed a nontrivial definition, we have an example due to Goebel and Sims [28] of a function which is mean nonexpansive but none of its iterates are nonexpansive in the usual sense.

Example 2.1.1 (Goebel and Sims)**.** Let's define a map

$$T : B_{\ell^1} \to B_{\ell^1} : (x_1, x_2, \ldots) \mapsto \left(\tau(x_2), \frac{2}{3}x_3, x_4, \ldots\right),$$

where $\tau : [-1, 1] \to [-1, 1]$ is given by

$$\tau(t) := \begin{cases} 2t + 1 & -1 \leq t \leq -\frac{1}{2} \\ 0 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 2t - 1 & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Note the following facts about τ and T:

1. $|\tau(t)| \leq |t|$ for all $t \in [-1, 1]$,
2. $|\tau(t) - \tau(s)| \leq 2|t - s|$ for all $s, t \in [-1, 1]$,
3. $\|Tx - Ty\|_1 \leq 2\|x - y\|_1$,
4. $\|T^j x - T^j y\|_1 \leq \frac{4}{3}\|x - y\|_1$ for all $x, y \in B_{\ell^1}$, for all $j \geq 2$, and
5. each estimate above is sharp.

We claim that T is $(1/2, 1/2)$-nonexpansive. First we calculate a formula for T^2,

$$T^2x = T(Tx) = \left(\tau\left(\frac{2}{3}x_3\right), \frac{2}{3}x_4, x_5, \ldots\right).$$

Now consider, for any $x, y \in B_{\ell^1}$,

$$\begin{aligned}
\frac{1}{2}\|Tx - Ty\|_1 + \frac{1}{2}\left\|T^2x - T^2y\right\|_1 &= \frac{1}{2}\left(|\tau(x_2) - \tau(y_2)| + \frac{2}{3}|x_3 - y_3| + \sum_{n=4}^{\infty}|x_n - y_n|\right) \\
&\quad + \frac{1}{2}\left(\left|\tau\left(\frac{2}{3}x_3\right) - \tau\left(\frac{2}{3}y_3\right)\right| + \frac{2}{3}|x_4 - y_4| + \sum_{n=5}^{\infty}|x_n - y_n|\right) \\
&\leq \frac{1}{2}\left(2|x_2 - y_2| + \frac{2}{3}|x_3 - y_3| + \sum_{n=4}^{\infty}|x_n - y_n|\right) \\
&\quad + \frac{1}{2}\left(\frac{4}{3}|x_3 - y_3| + \frac{2}{3}|x_4 - y_4| + \sum_{n=5}^{\infty}|x_n - y_n|\right) \\
&= |x_2 - y_2| + |x_3 - y_3| + \left(\frac{1}{3} + \frac{1}{2}\right)|x_4 - y_4| + \sum_{n=5}^{\infty}|x_n - y_n| \\
&\leq \|x - y\|_1.
\end{aligned}$$

Thus, T is a mean nonexpansive map for which no iterate is nonexpansive.

Note that, for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, if we define

$$T_\alpha := \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_n T^n,$$

then by the triangle inequality, we have that T is α-nonexpansive implies T_α is nonexpansive. We have the following results due to Goebel and Japón Pineda, which we state in the simple case of $\alpha = (\alpha_1, \alpha_2)$ before stating the analogous results for multi-indices of arbitrary length.

Theorem 2.1.1 (Goebel and Japón Pineda)**.** *If $T : C \to C$ is (α_1, α_2)-nonexpansive, then $\inf_C \|Tx - x\| = 0$, provided that $\alpha_1 \geq \frac{1}{2}$. That is, T has an approximate fixed point sequence.*

If it happens that the underlying space is already known to have fpp(ne), we can say more:

Theorem 2.1.2 (Goebel and Japón Pineda)**.** *Suppose that X has fpp(ne). Then X has the fixed point property for (α_1, α_2)-nonexpansive maps, provided $\alpha_1 \geq 1/2$.*

We will present the proofs of Theorems 2.1.1 and 2.1.2, which rely entirely on the nonexpansiveness of T_α, in Chapter 3 (see the proofs of Theorems 3.0.1 and 3.0.2). Goebel and Japón-Pineda proved generalizations of Theorems 2.1.1 and 2.1.2 for the case when T is $(\alpha_1, \ldots, \alpha_n)$-nonexpansive for arbitrary $n \in \mathbb{N}$. The proofs, along with slight extensions in specific cases (e.g. multi-indices $(\alpha_1, \alpha_2, \alpha_3)$ with $1/2 \leq \alpha_1 < 1/\sqrt{2}$ and $\alpha_2 \geq \alpha_3$), can be found in [46, pp. 35-37].

Theorem 2.1.3 (Goebel and Japón Pineda)**.** *Suppose $T : C \to C$ is α-nonexpansive for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 \geq 2^{\frac{1}{1-n}}$. Then*

1. $\inf_C \|Tx - x\| = 0$, *and*
2. *If X has fpp(ne), then T has a fixed point in C. That is, X has fpp(α-ne).*

These results lead to two open questions: *can we obtain any results, positive or negative, for $\alpha_1 < 1/2$?* and *can one classify the set of all multi-indices for which a given function is mean nonexpansive?* We provide partial answers to these questions in the following sections and in Chapter 3.

Seeking intuition for these questions, we want tangible examples of mean nonexpansive maps with $\alpha_1 < 1/2$. The following is an example of a family of maps $T_{A,\beta} : B_{\ell^1} \to B_{\ell^1}$ that are (α_1, α_2)-n.e. for $\alpha_1 < 1/2$, but which are not nonexpansive.

Example 2.1.2 (Modifications to the map of Goebel and Sims from Example 2.1.1)**.** Fix $A \in \mathbb{R}$ and $\beta \in (0, 1)$ with

$$2 < A < A_+ \qquad \text{and} \qquad \max\left\{\frac{1}{A}, \frac{A-1}{A+1}\right\} < \beta \leq \frac{A}{A^2 - A + 1},$$

where A_+ is the unique real number satisfying $A_+^3 - 3A_+^2 + A_+ - 1 = 0$; i.e.

$$A_+ := 1 + (1/3)(27 - 3\sqrt{57})^{1/3} + (1/3)^{2/3}(9 + \sqrt{57})^{1/3} \approx 2.7693.$$

Some notes:

1. The inequalities listed above are nontrivial: $\max\{1/A, (A-1)/(A+1)\} < A/(A^2 - A + 1)$ for $A \in (2, A_+)$ as above.

2. Depending on the value of A, we have the following:

$$\max\left\{\frac{1}{A}, \frac{A-1}{A+1}\right\} = \begin{cases} \frac{1}{A} & 2 < A \le 1+\sqrt{2} \\ \frac{A-1}{A+1} & 1+\sqrt{2} \le A < A_+ \end{cases}.$$

Define $\tau_A : [-1,1] \to [-1,1]$ as follows:

$$\tau_A(t) := \begin{cases} At + (A-1) & -1 \le t \le -\frac{A-1}{A+1} \\ -t & -\frac{A-1}{A+1} \le t \le \frac{A-1}{A+1} \\ At - (A-1) & \frac{A-1}{A+1} \le t \le 1 \end{cases}.$$

Finally, define $T = T_{A,\beta} : \ell^1 \to \ell^1$ by $Tx := T_{A,\beta}(x) := (\tau_A(x_2), \beta x_3, x_4, \ldots)$.

Notes on T:

1. $|\tau_A(t)| \le |t|$ for all $t \in [-1,1]$, so we have that $T(B_{\ell^1}) \subset B_{\ell^1}$.
2. $|\tau_A(s) - \tau_A(t)| \le A|s-t|$ for all s, t and

$$\left|\tau_A(1) - \tau_A\left(\frac{A-1}{A+1}\right)\right| = A\left|1 - \frac{A-1}{A+1}\right|,$$

so we have that $k(T) = A > 2$.

3. $|\tau_A(\beta s) - \tau_A(\beta t)| \le A\beta|s-t|$ for all s, t and $\tau_A(\beta \cdot 1) - \tau_A(\beta \cdot u)| = A\beta|1-u|$ where

$$u := \frac{1}{2}\left(1 + \frac{1}{\beta}\frac{A-1}{A+1}\right),$$

so we have that $k(T^2) = A\beta > 1$.

4. Further, we have $k(T^n) = A\beta$ for all $n \ge 2$.
5. $Tx = x \iff x = 0$.

Claim 2.1.1. *Fix A, β as above and let $T = T_{A,\beta} : B_{\ell^1} \to B_{\ell^1}$ be defined as above. Then T is (α_1, α_2)-nonexpansive if*

$$(0 <)\ \frac{A\beta - 1}{A\beta - \beta} \le \alpha_1 \le \frac{1}{A}\ \left(< \frac{1}{2}\right).$$

Proof of Claim. Fix α_1 as in the statement of the claim and let $\alpha_2 := 1 - \alpha_1$. Fix $x, y \in B_{\ell^1}$. Then

$$\begin{aligned}
\alpha_1 \|Tx - Ty\| + \alpha_2 \left\|T^2x - T^2y\right\| &= \alpha_1 \left(|\tau_A(x_2) - \tau_A(y_2)| + \beta|x_3 - y_3| + \sum_{n=4}^{\infty} |x_n - y_n| \right) \\
&\quad + (1-\alpha_1) \left(|\tau_A(\beta x_3) - \tau_A(\beta y_3)| + \beta|x_4 - y_4| + \sum_{n=5}^{\infty} |x_n - y_n| \right) \\
&\le A\alpha_1|x_2 - y_2| + (\alpha_1\beta + (1-\alpha_1)A\beta)|x_3 - y_3| \\
&\quad + (\alpha_1 + (1-\alpha_1)\beta)|x_4 - y_4| + \sum_{n=5}^{\infty} |x_n - y_n| \\
&\le \|x - y\|
\end{aligned}$$

since we have

1. $A\alpha_1 \le 1 \iff \alpha_1 \le A^{-1}$,
2. $\alpha_1\beta + (1-\alpha_1)A\beta = A\beta + \alpha_1(\beta - A\beta) \le 1 \iff \frac{A\beta - 1}{A\beta - \beta} \le \alpha_1$, and
3. $\alpha_1 + (1-\alpha_1)\beta \le 1$ whenever $\alpha_1 \in (0,1)$.

□

For a concrete example of such a map, choose $A = 5/2$ and $\beta = 1/2$. Then

$$T_{A,\beta}x = T_{\frac{5}{2},\frac{1}{2}}x = \left(\tau_{\frac{5}{2}}(x_2), \frac{1}{2}x_3, x_4, \dots \right),$$

where

$$\tau_{\frac{5}{2}}(t) := \begin{cases} \frac{5}{2}t + \frac{3}{2} & -1 \le t \le -\frac{3}{7} \\ -t & -\frac{3}{7} \le t \le \frac{3}{7} \\ \frac{5}{2}t - \frac{3}{2} & \frac{3}{7} \le t \le 1 \end{cases}.$$

Then $T_{A,\beta}$ is (α_1, α_2)-nonexpansive for any α_1 with

$$\frac{1}{3} \le \alpha_1 \le \frac{2}{5}.$$

One of the more interesting and fruitful observations about mean nonexpansive maps is the fact that a mean nonexpansive map is actually *nonexpansive* with respect to an equivalent metric. Before we define the metric, first note that T is $\alpha = (\alpha_1, \ldots, \alpha_n)$-nonexpansive (with $\alpha_1 \neq 0$) implies that $\alpha_1 \|Tx - Ty\| \leq \|x - y\|$, and we therefore have that

$$\|Tx - Ty\| \leq \alpha_1^{-1} \|x - y\|$$

for all $x, y \in C$. That is, T is α_1^{-1}-Lipschitz. Now we define the metric in the case when $\alpha = (\alpha_1, \alpha_2)$.

Theorem 2.1.4 (Goebel and Japón Pineda)**.** *Suppose* (M, d) *is a metric space and* $T : M \to M$ *is* (α_1, α_2)*-nonexpansive. Then*

1. *For all* $x, y \in M$*, let*

$$\rho(x, y) := d(x, y) + \alpha_2 d(Tx, Ty).$$

Then ρ *is a metric on* M *that is Lipschitz-equivalent to* d*.*

2. T *is a nonexpansive map on* (M, ρ)*.*

Proof. 1. ρ is a metric on M, and for all $x, y \in M$ we have

$$d(x, y) \leq \rho(x, y) \leq \left(1 + \frac{\alpha_2}{\alpha_1}\right) d(x, y) = \alpha_1^{-1} d(x, y).$$

That is, $d \sim \rho$.

2. Fix $x, y \in M$. Then

$$\begin{aligned}
\rho(Tx, Ty) &= d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \\
&= \alpha_2 d(Tx, Ty) + \alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \\
&\leq d(x, y) + \alpha_2 d(Tx, Ty) \\
&= \rho(x, y).
\end{aligned}$$

Hence, $\rho(Tx, Ty) \leq \rho(x, y)$ and T is nonexpansive on the metric space (M, ρ).

□

Note that if T is defined on a subset C of a Banach space, then

$$\rho(x, y) = \|x - y\| + \alpha_2 \|Tx - Ty\|$$

is a metric (*not* generally a norm) which is equivalent to $d(x, y) := \|x - y\|$. Again, a general form of the above theorem is given for $(\alpha_1, \ldots, \alpha_n)$-nonexpansive maps. We omit the proof.

Theorem 2.1.5 (Goebel and Japón Pineda)**.** *If (M, d) is a metric space and $T : M \to M$ is $(\alpha_1, \ldots, \alpha_n)$-nonexpansive, then*

$$\rho(x, y) := \sum_{j=1}^{n} \left(\sum_{k=j}^{n} \alpha_k \right) d(T^{j-1}x, T^{j-1}y)$$

is a metric that is Lipschitz-equivalent to d and T is nonexpansive with respect to ρ.

We have an immediate consequence of this theorem.

Corollary 2.1.1. *Any α-nonexpansive map $T : M \to M$ is uniformly Lipschitzian with $d(T^j x, T^j y) \leq \alpha_1^{-1} d(x, y)$ for all $j \in \mathbb{N}$.*

Proof. Fix $x, y \in C$ and $j \in \mathbb{N}$. Then

$$d(T^j x, T^j y) \leq \rho(T^j x, T^j y) \leq \rho(x, y) \leq \frac{1}{\alpha_1} d(x, y).$$

□

This estimate is not sharp for $j > 1$; i.e. for $\alpha = (\alpha_1, \alpha_2)$ and $j \geq 2$, every α-nonexpansive map T is such that $k(T^j) < \alpha_1^{-1}$. In [46, ch. 4], Piasecki calculated sharp bounds for the Lipschitz constants of iterates of mean nonexpansive maps, as we see in the next section.

2.1.1 Lipschitz constants for iterates of mean nonexpansive maps

Theorem 2.1.6 (Piasecki)**.** *Suppose T is (α_1, α_2)-nonexpansive for $\alpha_2 \neq 1$. Then*

1. $k(T^n) \leq \dfrac{1+\alpha_2}{1-(-\alpha_2)^{n+1}}$,
2. *Moreover, this estimate is sharp. That is, for any (α_1, α_2), there exists a Banach space X and an α-nonexpansive map $T : X \to X$ with $k(T^n) = (1+\alpha_2)/(1-(-\alpha_2)^{n+1})$ for all $n \in \mathbb{N}$.*

Proof of 1. Refer to [46, pp. 41-43].

Proof of 2. Let $(X, \|\cdot\|) = (\ell^1, \|\cdot\|_1)$. Fix $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$. For our α-nonexpansive map T, we take a variation on the left-shift. For all $x = (x_1, x_2, \ldots) \in \ell^1$, define

$$Tx := \left(\frac{1-(-\alpha_2)^1}{1-(-\alpha_2)^2} x_2, \frac{1-(-\alpha_2)^2}{1-(-\alpha_2)^3} x_3, \ldots, \frac{1-(-\alpha_2)^k}{1-(-\alpha_2)^{k+1}} x_{k+1}, \ldots \right).$$

T is linear, and it is easy to check that $\alpha_1 \|Tx\|_1 + \alpha_2 \|T^2 x\|_1 \leq \|x\|_1$ for all $x \in \ell^1$. By part 1, we know that

$$k(T^n) \leq \frac{1+\alpha_2}{1-(-\alpha_2)^{n+1}}.$$

All that is left to do is to show that this estimate is sharp. Taking $e_n := (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots)$, we have

$$\|T^n e_{n+1}\|_1 = \left\| \left(\frac{1-(-\alpha_2)^1}{1-(-\alpha_2)^{n+1}}, 0, 0, \ldots \right) \right\|_1 = \frac{1+\alpha_2}{1-(-\alpha_2)^{n+1}}$$

as desired. □

Piasecki calculated sharp bounds in a similar, but naturally more complicated, fashion for $(\alpha_1, \ldots, \alpha_n)$-nonexpansive and for $(\alpha_1, \ldots, \alpha_n)$-$k$-Lipschitzian maps. We omit the general treatment, which can be found in [46, pp. 46-69].

2.1.2 Classifying multi-indices

In this section, we give a partial answer to the question of Goebel and Japón Pineda regarding the classification of all multi-indices for which a given function is mean nonexpansive.

Let us recall Example 2.1.1 due to Goebel and Sims. The map, defined on the closed unit ball of ℓ^1 was found to be $(1/2, 1/2)$-nonexpansive with $k(T) = 2$ and $k(T^j) = \frac{4}{3}$ for all $j > 1$. Thus, $2 \leq 1/\alpha_1 \iff \alpha_1 \leq 1/2$, and

$$\frac{4}{3} = k(T^2) \leq 1/(\alpha_2^2 - \alpha_2 + 1) \iff \alpha_2 = \frac{1}{2}.$$

Thus the only (α_1, α_2) for which T is (α_1, α_2)-nonexpansive is $(1/2, 1/2)$. In contrast with the Goebel and Sims example, we also saw in example 2.1.2 examples of (α_1, α_2)-nonexpansive maps that remained mean nonexpansive over a range of multi-indices. In the case when a given map is known to be mean nonexpansive for multiple multi-indices of the same length, we have the following theorem.

Theorem 2.1.7 (Interpolation of multi-indices)**.** *Suppose (M, d) is a metric space and $T : M \to M$ is mean nonexpansive for each multi-index $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$, $j = 1, \ldots, J$. Then T is μ-nonexpansive for any $\mu \in co\{\alpha^{(j)} : j = 1, \ldots, J\}$.*

Proof. Fix $t_1, \ldots, t_J \in [0, 1]$ with $t_1 + \cdots + t_J = 1$. For each j and for all $x, y \in C$,

$$\begin{aligned} t_j \sum_{m=1}^{n} \alpha_m^{(j)} d(T^m x, T^m y) \leq t_j d(x, y) &\implies \sum_{j=1}^{J} t_j \sum_{m=1}^{n} \alpha_m^{(j)} d(T^m x, T^m y) \leq d(x, y) \\ &\iff \sum_{m=1}^{n} \left(\sum_{j=1}^{J} t_j \alpha_m^{(j)} \right) d(T^m x, T^m y) \leq d(x, y) \\ &\iff \sum_{m=1}^{n} \mu_m d(T^m x, T^m y) \leq d(x, y), \end{aligned}$$

where $(\mu_m)_{m=1}^n := \left(\sum_{j=1}^J t_j \alpha_m^{(j)} \right)_{m=1}^n \in \text{co}\{\alpha^{(j)} : j = 1, \ldots, J\}$. □

Generalizing the above question, can we characterize all multi-indices (not necessarily of the same length) for which a given map is mean nonexpansive?

We have an elementary result.

Theorem 2.1.8. *Let (M, d) be a metric space. The following are equivalent.*

1. $T : M \to M$ *is* (α_1, α_2)*-nonexpansive.*
2. T *is* $(\gamma_1(\beta), \gamma_2(\beta), \gamma_3(\beta))$*-nonexpansive, where*

$$\gamma_1(\beta) := \beta\alpha_1, \quad \gamma_2(\beta) := (1-\beta)\alpha_1^2 + \alpha_2, \quad \textit{and } \gamma_3(\beta) := (1-\beta)\alpha_1\alpha_2$$

for all $\beta \in [0, 1]$.

Proof. First, note that

$$\gamma_1(\beta) + \gamma_2(\beta) + \gamma_3(\beta) = \beta\alpha_1 + (1-\beta)\alpha_1^2 + \alpha_2 + (1-\beta)\alpha_1\alpha_2 = 1.$$

Next, we know that for all $x, y \in C$, $\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \le d(x, y)$. Thus

$$\begin{aligned}
&\gamma_1(\beta)d(Tx, Ty) + \gamma_2(\beta)d(T^2x, T^2y) + \gamma_3(\beta)d(T^3x, T^3y) \\
&\quad = \beta\alpha_1 d(Tx, Ty) + ((1-\beta)\alpha_1^2 + \alpha_2)d(T^2x, T^2y) + (1-\beta)\alpha_1\alpha_2 d(T^3x, T^3y) \\
&\quad = \beta\alpha_1 d(Tx, Ty) + (1-\beta)\alpha_1(\alpha_1 d(T^2x, T^2y) + \alpha_2 d(T^3x, T^3y)) + \alpha_2 d(T^2x, T^2y) \\
&\quad \le \beta\alpha_1 d(Tx, Ty) + (1-\beta)\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \\
&\quad = \alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \\
&\quad \le d(x, y)
\end{aligned}$$

Taking $\beta = 1$ yields the converse. □

This theorem generalizes to multi-indices of arbitrary length.

Theorem 2.1.9. *Let* (M, d) *be a metric space and* $T : M \to M$*. The following are equivalent.*

1. T *is* $(\alpha_1, \dots, \alpha_n)$*-nonexpansive.*
2. T *is* $(\gamma_1(t), \dots, \gamma_{n+1}(t))$*-nonexpansive for all* $t \in [0, 1]$*, where*

$$\begin{aligned}
\gamma_k(t) &:= \alpha_k + t\alpha_1\alpha_{k-1} \quad \textit{for } k = 1, \dots, n, \textit{ and} \\
\gamma_{n+1}(t) &:= t\alpha_1\alpha_n,
\end{aligned}$$

with $\alpha_0 := -1$

Proof. This proof is entirely similar to the proof presented above. Note that each $\gamma_k(t) \geq 0$ and

$$\begin{aligned}
&\gamma_1(t) + \cdots + \gamma_{n+1}(t) \\
&\qquad = (\alpha_1 + t\alpha_1\alpha_0) + (\alpha_2 + t\alpha_1^2) + \cdots + (\alpha_n + t\alpha_1\alpha_{n-1}) + t\alpha_1\alpha_n \\
&\qquad = (\alpha_1 + \cdots + \alpha_n) + t\alpha_1(\alpha_0 + \alpha_1 + \cdots + \alpha_n) \\
&\qquad = 1.
\end{aligned}$$

Finally, T is $\gamma(t)$-nonexpansive:

$$\begin{aligned}
&\gamma_1(t)d(Tx,Ty) + \cdots + \gamma_{n+1}(t)d(T^{n+1}x, T^{n+1}y) \\
&\qquad = (\alpha_1 d(Tx,Ty) + \cdots + \alpha_n d(T^n x, T^n y)) \\
&\qquad\quad + t\alpha_1(\alpha_0 d(Tx,Ty) + \alpha_1 d(T^2x, T^2y) + \cdots + \alpha_n d(T^{n+1}x, T^{n+1}y)) \\
&\qquad \leq d(x,y) + t\alpha_1(-d(Tx,Ty) + d(Tx,Ty)) \\
&\qquad = d(x,y).
\end{aligned}$$

□

Mean Lipschitzian maps are not necessarily uniformly Lipschitzian, so we have a question: *Can we come up with a notion of mean-uniformly Lipschitzian maps?*

One idea is to try the following: consider a map T for which there exists a k and a sequence of pairs $\left(\alpha_1^{(n)}, \alpha_2^{(n)}\right)$ (with $\alpha_1^{(n)}, \alpha_2^{(n)} > 0$ and summing to 1) such that for all n and for all x, y,

$$\alpha_1^{(n)} d(T^n x, T^n y) + \alpha_2^{(n)} d(T^{n+1}x, T^{n+1}y) \leq k d(x,y).$$

Note preliminarily that, if this notion is to have any merit, we should have that all (α_1, α_2)-nonexpansive mappings are also $(\alpha_1^{(n)}, \alpha_2^{(n)})$-1-uniformly Lipschitzian for some sequence of pairs $(\alpha_1^{(n)}, \alpha_2^{(n)})$. This is indeed the case.

Claim 2.1.2. *If $T : M \to M$ is (α_1, α_2)-nonexpansive, then T is $(\alpha_1^{(n)}, \alpha_2^{(n)})$-1-uniformly Lipschitzian, with*

$$\alpha_1^{(n+1)} := \alpha_1\alpha_1^{(n)} + \alpha_2^{(n)} \quad \text{and} \quad \alpha_2^{(n+1)} := \alpha_2\alpha_1^{(n)} \quad \text{for all } n \in \mathbb{N},$$

where $\alpha_1^{(1)} := \alpha_1$ and $\alpha_2^{(1)} := \alpha_2$.

Proof of the claim. We proceed by induction. First, it is clear that $\alpha_j^{(n)} > 0$ for all $n \in \mathbb{N}$ and $j = 1, 2$. Next, let us check that $\alpha_1^{(n)} + \alpha_2^{(n)} = 1$ for all n. When $n = 1$, $\alpha_1^{(1)} + \alpha_2^{(1)} = \alpha_1 + \alpha_2 = 1$. If it happens that $\alpha_1^{(n)} + \alpha_2^{(n)} = 1$ for some $n \in \mathbb{N}$, then

$$\alpha_1^{(n+1)} + \alpha_2^{(n+1)} = \alpha_1\alpha_1^{(n)} + \alpha_2^{(n)} + \alpha_2\alpha_1^{(n)} = \alpha_1^{(n)} + \alpha_2^{(n)} = 1.$$

Now, we must check that $\alpha_1^{(n)} d(T^n x, T^n y) + \alpha_2^{(n)} d(T^{n+1}x, T^{n+1}y) \leq d(x, y)$ for all n. When $n = 1$, it follows from the fact that T is (α_1, α_2)-nonexpansive. For any n and $x, y \in M$, let $D_n := d(T^n x, T^n y)$ for simplicity, and note that the (α_1, α_2)-nonexpansiveness of T gives us that $\alpha_1 D_{n+1} + \alpha_2 D_{n+2} \leq D_n$ for all n. Supposing that the inequality $\alpha_1^{(n)} D_n + \alpha_2^{(n)} D_{n+1} \leq D_0 := d(x, y)$ is satisfied for some $n \in \mathbb{N}$, we then have

$$\begin{aligned}
\alpha_1^{(n+1)} D_{n+1} + \alpha_2^{(n+1)} D_{n+2} &= \left(\alpha_1\alpha_1^{(n)} + \alpha_2^{(n)}\right) D_{n+1} + \alpha_2\alpha_1^{(n)} D_{n+2} \\
&= \alpha_1^{(n)} \left(\alpha_1 D_{n+1} + \alpha_2 D_{n+2}\right) + \alpha_2^{(n)} D_{n+1} \\
&\leq \alpha_1^{(n)} D_n + \alpha_2^{(n)} D_{n+1} \\
&\leq D_0,
\end{aligned}$$

as desired. ∎

All (α_1, α_2)-nonexpansive maps are therefore $(\alpha^{(n)})_n$-1-uniformly Lipschitzian, where $\alpha^{(n)} = (\alpha_1^{(n)}, \alpha_2^{(n)})$ is given above. Similarly, if T is already k-uniformly Lipschitzian, then T is $(\alpha^{(n)})$-k-uniformly Lipschitzian for any choice of $(\alpha^{(n)})_n$. If we can fully formalize this notion, find nontrivial examples, and begin to study it in earnest, then it may be possible to obtain theorems like Theorem 2.1.1 and Theorem 2.1.2 which would extend the results of Goebel and Kirk in uniformly convex spaces [27].

2.2 MEAN ISOMETRIES

Recall Maurey's Superreflexive Theorem (Theorem 1.0.12), which states that superreflexive Banach spaces have the fixed point property for isometries. With the notion of a mean nonexpansive mapping in mind, one could ask if Maurey's Superreflexive Theorem extends to the class of *mean isometries*. In the natural way, we say a function $T : (M, d) \to (M, d)$ a *mean isometry* if, for all $x, y \in M$,

$$\sum_{j=1}^{n} \alpha_j d(T^j x, T^j y) = d(x, y)$$

for some multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ for which $\alpha_1, \alpha_n > 0$, $\alpha_j \geq 0$ for all j with $\alpha_1 + \cdots + \alpha_n = 1$. Notice that all isometries are mean isometries for any choice of α. Indeed, as we will see, the converse is also true.

2.2.1 Mean isometries when $\alpha = (\alpha_1, \alpha_2)$

Theorem 2.2.1. *Let (M, d) be a metric space. Then $T : M \to M$ is an isometry if and only if it is an (α_1, α_2)-isometry.*

We will have two immediate and trivial corollaries.

Corollary 2.2.1. *Superreflexive Banach spaces have the fixed point property for (α_1, α_2)-isometries.*

Corollary 2.2.2. *If X is a Banach space, $C \subset X$ is closed, bounded, and convex, and $T : C \to C$ is a mean isometry, then T has an approximate fixed point sequence (regardless of the size of α_1).*

The proof of Theorem 2.2.1 relies on the following lemma, stated without proof.

Lemma 2.2.1. *Suppose that a recurrence relation is given by $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $(n \geq 3)$ for constants c_1, c_2, with $a_1, a_2 \in \mathbb{R}$ given. Suppose further that the equation $t^2 - c_1 t - c_2 = 0$ has two distinct real roots, t_1 and t_2. Then for all n, $a_n = b_1 (t_1)^n + b_2 (t_2)^n$, where $b_1, b_2 \in \mathbb{R}$.*

Proof of Theorem 2.2.1. Let (M, d) be a metric space. First, if $T : M \to M$ is an isometry, then T is a mean isometry with respect to any (α_1, α_2).

Conversely, suppose for a contradiction that $T : M \to M$ is not an isometry, but is a mean isometry; that is, T satisfies $\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) = d(x, y)$ for all $x, y \in M$, but there exist $x, y \in M$ such that $d(Tx, Ty) \neq d(x, y)$. For all $n \in \mathbb{N}$, define

$$\mu_n = \mu_n(x, y) := d(T^n x, T^n y).$$

From the mean isometry condition, we know that $d(Tx, Ty) \neq d(x, y) \implies d(T^n x, T^n y) \neq d(x, y)$ for all n. We know further that the sequence μ_n satisfies

$$\alpha_1 \mu_n + \alpha_2 \mu_{n+1} = \mu_{n-1} \quad \iff \quad \mu_{n+1} = -\frac{\alpha_1}{\alpha_2}\mu_n + \frac{1}{\alpha_2}\mu_{n-1}.$$

Consider the equation $t^2 + \frac{\alpha_1}{\alpha_2}t - \frac{1}{\alpha_2} = 0$. Solving, we find the solutions t_1 and t_2 to be

$$t_1 = -\frac{1}{\alpha_2} \quad \text{and} \quad t_2 = 1.$$

From Lemma 2.2.1, we know that $\mu_n = b_1 \left(-\frac{1}{\alpha_2}\right)^n + b_2$ for all n. Solving for b_1 and b_2 yields

$$b_1 = \frac{\alpha_2}{1+\alpha_2}(\mu_0 - \mu_1) \quad \text{and} \quad b_2 = \frac{1}{1+\alpha_2}(\mu_0 + \alpha_2\mu_1).$$

The important thing to note is that $d(Tx, Ty) \neq d(x, y) \implies \mu_0 - \mu_1 \neq 0 \iff b_1 \neq 0$. Since $\alpha_2 \in (0, 1)$, we have $\alpha_2^{-1} > 1$, and therefore $\mu_n = b_1 \left(-\frac{1}{\alpha_2}\right)^n + b_2 < 0$ for sufficiently large n. But $\mu_n = d(T^n x, T^n y) \geq 0$ for all n. Contradiction. Thus it must be that $d(Tx, Ty) = d(x, y)$ for all $x, y \in M$. □

Remark 2.2.1. The techniques in the above proof work even in the case when $\alpha_2 \in (-1, 0)$. We will explore this notion in depth in Sections 2.3 and 2.4.

We close this section with a conjecture regarding mean isometries for multi-indices of arbitrary length.

Conjecture 2.2.1. *If $T : M \to M$ is a mean isometry for some $\alpha = (\alpha_1, \ldots, \alpha_n)$, then T is an isometry.*

2.3 AFFINE COMBINATION (A.C.) MAPS

Upon inspection of the proof of Theorem 2.2.1, we see that the assumption that $\alpha_2 \in (0,1)$ may, in the right context, be relaxed to $\alpha_2 \in (-1,1)$. First, if $\alpha_2 = 0$, then $\alpha_1 = 1$ and T is already an isometry. If $\alpha_2 \in (-1,0)$, following the same argument as above, we would have

$$d(T^n x, T^n y) = b_1 \left(-\frac{1}{\alpha_2}\right)^n + b_2 \to_n \infty.$$

We have proved the following theorem.

Theorem 2.3.1. *Let (M,d) be a bounded metric space. The following are equivalent.*

1. *T is an isometry.*
2. *There exist α_1, α_2 with $\alpha_2 \in (-1,1)$ and $\alpha_1 + \alpha_2 = 1$ for which*

$$\alpha_1 d(Tx,Ty) + \alpha_2 d(T^2x, T^2y) = d(x,y)$$

for all $x, y \in M$.

Remark 2.3.1. The assumption of boundedness is essential in the above theorem, as Example 2.3.5 demonstrates.

Let us examine this new class of functions in more detail, first by stating the definition.

Definition 2.3.1. Let (M,d) be a metric space and $T : M \to M$. We call T *affine combination Lipschitz* (or *a.c. Lipschitz*) if, for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\alpha_1, \alpha_n \neq 0$ with $\alpha_1 + \dots + \alpha_n = 1$, we have

$$\sum_{j=1}^{n} \alpha_j d(T^j x, T^j y) \leq k d(x,y)$$

for all $x, y \in M$ and for some $k > 0$.

If $k = 1$, we say T is a.c. nonexpansive, and if $k < 1$, we say T is an a.c. contraction.

In other words, Theorem 2.3.1 may be rephrased to say T is an isometry on a bounded metric space if and only if T is an a.c. isometry for a length 2 multi-index with $\alpha_2 \in (-1,1)$. Also, it is clear that the class of a.c. nonexpansive mappings includes the class of mean nonexpansive mappings. Indeed, the class of a.c. nonexpansive mappings is a nontrivial extension of the class of mean nonexpansive mappings, as the following examples demonstrate.

2.3.1 Examples of a.c. nonexpansive mappings

Example 2.3.1. Let $X = \mathbb{R}^2$, equipped with any ℓ^p norm $\|\cdot\|_p$, $p \geq 1$. Let $C = B_{\mathbb{R}^2} = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_p \leq 1\}$ and let

$$T := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then T is linear and for any $(x,y) \in C$,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

so $\|T(x,y)\|_p \leq \|(x,y)\|_p$, and we have that $T(C) \subseteq C$. Also note that $T^2 = 0$, so for any $k \in [0,1)$ and any $\alpha_1 < 0$ and $\alpha_2 = 1 - \alpha_1$,

$$\begin{aligned} \alpha_1 \|T(x,y) - T(u,v)\|_p + \alpha_2 \left\|T^2(x,y) - T^2(u,v)\right\|_p &= \alpha_1 \|T(x,y) - T(u,v)\|_p \\ &\leq 0 \\ &\leq k \|(x,y) - (u,v)\|_p . \end{aligned}$$

Hence T is an a.c. contraction for any $\alpha_1 < 0$ and any $k \geq 0$.

Note that a function may be an a.c. contraction without even being continuous, as we see in the next example.

Example 2.3.2. Let $f : [-1,1] \to [-1,1]$ be given by

$$f(x) := \begin{cases} -1 & -1 < x < 0 \\ 1 & \text{otherwise} \end{cases}.$$

Then $f^2(x) = 1$ for all x, and $|f^2(x) - f^2(y)| = 0$ for all x, y, just as in the above example. Thus, f is an a.c. contraction for all $k \in [0,1)$ and for any $\alpha_1 < 0$.

Also note that a discontinuous function with non-constant second iterate may be a.c. nonexpansive:

Example 2.3.3. Let $f : [0,1] \to [0,1]$ be given by

$$f(x) := \begin{cases} \frac{x}{5} & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{6} & x \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Then

1. for all $x, y \in [0, \frac{1}{2}]$, $|f(x) - f(y)| = \frac{1}{5}|x - y|$,
2. for $x, y \in (\frac{1}{2}, 1]$, $|f(x) - f(y)| = \frac{1}{6}|x - y|$,
3. for any $x \in [0,1]$, $f(x) \in [0, \frac{1}{2})$, so

$$f^2(x) = \begin{cases} \frac{x}{25} & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{30} & x \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Now let $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, so

$$|f(x) - f(y)| = \frac{1}{30}|6x - 5y|, \text{ and}$$

$$|f^2(x) - f^2(y)| = \frac{1}{150}|6x - 5y|.$$

Let $\alpha_1 := -\frac{1}{4}$ and $\alpha_2 := \frac{5}{4}$. Then

$$\begin{aligned} \alpha_1|f(x) - f(y)| + \alpha_2|f^2(x) - f^2(y)| &= -\frac{1}{4} \cdot \frac{1}{30}|6x - 5y| + \frac{5}{4} \cdot \frac{1}{150}|6x - 5y| \\ &= 0 \\ &\leq |x - y|. \end{aligned}$$

Thus, f is $(-1/4, 5/4)$-nonexpansive.

If we have equality, as in the above, (that is, $\alpha_1 \|Tx - Ty\| + \alpha_2 \|T^2x - T^2y\| = \|x - y\|$) we say T is an a.c. isometry.

Example 2.3.4. Consider $\mathbb{R}^2$ endowed with the 1-norm, $\|(x,y)\|_1 := |x| + |y|$, and let $A : B_{\mathbb{R}^2} \to B_{\mathbb{R}^2}$ be given by

$$A := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then

$$\|A(x,y)\|_1 = \left\|(x, \frac{1}{2}y)\right\|_1 = |x| + \frac{1}{2}|y|, \text{ and}$$

$$\begin{aligned} 3\|A(x,y)\|_1 - 2\left\|A^2(x,y)\right\|_1 &= 3|x| + \frac{3}{2}|y| - 2|x| - \frac{1}{2}|y| \\ &= |x| + |y| \\ &= \|(x,y)\|_1 . \end{aligned}$$

That is, A is a $(3,-2)$-isometry without being an isometry in the usual sense.

Note also that by changing the above example slightly, we see that a.c. isometries with $\alpha_2 < 0$ may be norm-expanding or even strict contractions in the usual sense.

Example 2.3.5. Fix $\alpha_2 < 0$ and let $\alpha_1 := 1 - \alpha_2$. Define

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto -\frac{1}{\alpha_2}x.$$

Then for any $x, y \in \mathbb{R}$,

$$\alpha_1|f(x) - f(y)| + \alpha_2|f^2(x) - f^2(y)| = -\frac{\alpha_1}{\alpha_2}|x-y| + \frac{1}{\alpha_2}|x-y| = |x-y|,$$

and we have that f is an a.c. isometry. Further more, if $\alpha_2 \in (-1,0)$, then $|f(x) - f(y)| > |x-y|$ for all x, y, and if $\alpha_2 < -1$, then $|f(x) - f(y)| = -\alpha_2^{-1}|x-y|$ for all x, y and is hence a strict contraction in the usual sense.

Finally, we have examples of fixed-point-free a.c. nonexpansive mappings defined on closed, bounded, convex sets in $(\ell^1, \|\cdot\|_1)$. Given a bounded sequence $\lambda = (\lambda_n)_n$ of real numbers, define the closed, bounded, convex set $C_\lambda \subset \ell^1$ via

$$C_\lambda := \left\{ x = \sum_{n=1}^{\infty} t_n f_n \;:\; t_n \geq 0,\; \sum_{n=1}^{\infty} t_n = 1,\; f_n = \lambda_n e_n \right\}$$

and define the map $T : C_\lambda \to C_\lambda$ via

$$T\left(\sum_{n=1}^{\infty} t_n f_n\right) := \sum_{n=1}^{\infty} t_n f_{n+1}.$$

Note that showing T is a.c.-nonexpansive amounts to finding $\alpha_1 \in \mathbb{R}$ for which $\alpha_1(\lambda_{n+1} - \lambda_{n+2}) \leq \lambda_n - \lambda_{n+2}$ for all n. This is because $\alpha_2 = 1 - \alpha_1$ and

$$\alpha_1 \|Tx - Ty\|_1 + \alpha_2 \left\|T^2 x - T^2 y\right\|_1 \leq \|x - y\|_1 \iff \alpha_1 \lambda_{n+1} + \alpha_2 \lambda_{n+2} \leq \lambda_n.$$

Example 2.3.6 (A norm-expanding map that is a.c. nonexpansive for some $\alpha_2 < 0$)**.** Let $\lambda_n := 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Then $\lambda_n < \lambda_{n+1}$ for all n, and for $x = \sum t_n f_n$, $y = \sum s_n f_n \in C_\lambda$, we have

$$\|Tx - Ty\|_1 = \sum_{n=1}^{\infty} \lambda_{n+1} |t_n - s_n| > \sum_{n=1}^{\infty} \lambda_n |t_n - s_n| = \|x - y\|_1$$

provided that $x \neq y$, and we have that T is expansive on C_λ with respect to $\|\cdot\|_1$.

For $\alpha_1, \alpha_2 \in \mathbb{R}$, we know that

$$\alpha_1 \|Tx - Ty\|_1 + \alpha_2 \|Tx - Ty\|_1 \leq \|x - y\|_1 \iff \alpha_1 \lambda_{n+1} + \alpha_2 \lambda_{n+2} \leq \lambda_n$$

for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \alpha_1(\lambda_{n+1} - \lambda_{n+2}) \leq \lambda_n - \lambda_{n+2} &\iff \alpha_1 \left(\frac{1}{2^{n+2}} - \frac{1}{2^{n+1}}\right) \leq \frac{1}{2^{n+2}} - \frac{1}{2^n} \\ &\iff \alpha_1 \left(-\frac{1}{2^{n+2}}\right) \leq -\frac{3}{2^{n+2}} \\ &\iff \alpha_1 \geq 3 \end{aligned}$$

We know three things about T:

1. T is a $(3, -2)$-isometry,
2. T is (α_1, α_2)-nonexpansive for all $\alpha_1 > 3$ (or, equivalently, for all $\alpha_2 < -2$), and

3. $(\ell^1, \|\cdot\|_1)$ fails to have the fpp for a.c. nonexpansive maps and a.c. isometries for $\alpha_2 < 0$.

Example 2.3.7 (An a.c. isometry for some $\alpha_1 < 0$ which is not an isometry)**.** Let $\lambda_{2k} := 1 - 2^{-2k}$ and $\lambda_{2k-1} := 1 + 2^{-(2k-1)}$ for all k.

Claim 2.3.1. *T is a $(-1, 2)$-isometry.*

Proof. We consider two separate cases.

1. First,

$$\begin{aligned}
-\lambda_{2k+1} + 2\lambda_{2k+2} &= -1 - \frac{1}{2^{2k+1}} + 2 - \frac{2}{2^{2k+2}} \\
&= 1 - \frac{2}{2^{2k+1}} \\
&= 1 - \frac{1}{2^{2k}} \\
&= \lambda_{2k}.
\end{aligned}$$

2. Similarly,

$$\begin{aligned}
-\lambda_{2k+2} + 2\lambda_{2k+3} &= -1 + \frac{1}{2^{2k+2}} + 2 + \frac{2}{2^{2k+3}} \\
&= 1 + \frac{2}{2^{2k+2}} \\
&= 1 + \frac{1}{2^{2k+1}} \\
&= \lambda_{2k+1}.
\end{aligned}$$

Thus, $-\lambda_{n+1} + 2\lambda_{n+2} = \lambda_n$ for all $n \in \mathbb{N}$, and T is a $(-1, 2)$-isometry. Finally, T is not an isometry with respect to $\|\cdot\|_1$ since $\|Tf_1 - Tf_2\|_1 = \lambda_2 + \lambda_3 \neq \lambda_1 + \lambda_2 = \|f_1 - f_2\|_1$.

Now we can also say that $(\ell^1, \|\cdot\|_1)$ fails to have the fpp for a.c. isometries with $\alpha_1 < 0$. □

2.4 FIXED POINT RESULTS FOR A.C. NONEXPANSIVE MAPPINGS WITH $\alpha_1 < 0$

Notice that, if T is a.c. nonexpansive for $\alpha_1 < 0$, then

$$\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \le d(x, y) \iff d(T^2x, T^2y) \le \beta_1 d(x, y) + \beta_2 d(Tx, Ty),$$

where $\beta_1 = \alpha_2^{-1}$ and $\beta_2 = -\alpha_1\alpha_2^{-1}$, and that $\beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$.

Similarly, T is a.c. nonexpansive for $\alpha_2 < 0$ if and only if

$$d(Tx, Ty) \le \beta_1 d(x, y) + \beta_2 d(T^2x, T^2y),$$

where $\beta_1 = \alpha_1^{-1}, \beta_2 = -\alpha_2\alpha_1^{-1} > 0$ and $\beta_1 + \beta_2 = 1$.

Definition 2.4.1. We will say $T : M \to M$ is a *strong a.c. contraction with* $\alpha_1 < 0$ if, for some $k \in [0, 1)$, $\beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$, and for all $x, y \in M$ we have

$$d(T^2x, T^2y) \le k(\beta_1 d(x, y) + \beta_2 d(Tx, Ty)).$$

2.4.1 A contraction mapping theorem

Theorem 2.4.1. *Suppose that* (M, d) *is a complete metric space and that* $T : M \to M$ *is continuous and a strong a.c. contraction with* $\alpha_1 < 0$*. Then* T *has a unique fixed point* $x_0 \in M$*, and* $\lim_{n\to\infty} d(T^n x, x_0) = 0$ *for all* $x \in M$*.*

Proof. Fix $x_0 \in M$ and let $x_n := T^n x_0$ for all $n \in \mathbb{N}$. Let $\gamma_n := d(x_{n+1}, x_n)$. Then

$$\begin{aligned}\gamma_n = d(x_{n+1}, x_n) &= d(T^2x_{n-1}, T^2x_{n-2})\\ &\le k(\beta_1 d(x_{n-1}, x_{n-2}) + \beta_2 d(Tx_{n-1}, Tx_{n-2}))\\ &= k(\beta_1\gamma_{n-2} + \beta_2\gamma_{n-1})\end{aligned}$$

Let $\delta_0 := \gamma_0$, $\delta_1 := \gamma_1$, and $\delta_n := k(\beta_1\delta_{n-2} + \beta_2\delta_{n-1})$. Note that $\gamma_n \le \delta_n$ for all n. If $\delta_0 = 0$, then $\delta_n = 0$ for all n, so assume $\delta_0 > 0$, which gives us that $\delta_n > 0$ for all n. We want to use lemma 2.2.1, so consider the equation $t^2 - k\beta_2 t - k\beta_1 = 0$. Solving yields

$$t_{-,+} = \frac{1}{2}\left(k\beta_2 \pm \sqrt{k^2\beta_2^2 + 4k\beta_1}\right)$$

and $\delta_n = b_1(t_-)^n + b_2(t_+)^n$. It is easy to check that $k < 1 \implies |t_-| < |t_+| < 1$.

Now for any $m < n$,

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) = \sum_{k=m}^{n-1} \gamma_k \leq \sum_{k=m}^{n-1} \delta_k = b_1 \sum_{k=m}^{n-1} (t_-)^k + b_2 \sum_{k=m}^{n-1} (t_+)^k$$

which is small for sufficiently large m. That is, $(x_n)_n$ is Cauchy. Since (M, d) is complete, $(x_n)_n$ is convergent to some $z \in M$. Since T is continuous, $z = \lim_n x_n = \lim_n T(x_{n-1}) = T(\lim_n x_{n-1}) = T(z)$ and z is a fixed point of T. To see that z is unique, suppose z' is another fixed point of T. Then

$$\begin{aligned} 0 \leq d(z', z) = d(T^2 z', T^2 z) &\leq k(\beta_1 d(z', z) + \beta_2 d(Tz', Tz)) \\ &= k(\beta_1 d(z', z) + \beta_2 d(z', z)) \\ &= k d(z', z) \\ &< d(z', z), \end{aligned}$$

which is a contradiction. Thus, z is unique. □

Remark 2.4.1. The technique in the proof above works in the slightly broader setting of continuous a.c. contractions with $\alpha_1 < 0$; i.e. continuous functions satisfying

$$\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2 x, T^2 y) \leq k d(x, y)$$

for some $k \in [0, 1)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 < 0$, for which $\alpha_1 + \alpha_2 = 1$. In this case, the solutions to the characteristic equation are

$$t_{-,+} = \frac{1}{2\alpha_2}\left(-\alpha_1 \pm \sqrt{\alpha_1^2 + 4k\alpha_2}\right).$$

Whenever one proves an extension of Banach's Contraction Mapping Theorem, one must ask whether it is genuinely an extension. That is, one must ask if it is possible that a.c. contractions with $\alpha_1 < 0$ are strict contractions with respect to another complete metric. A theorem of Bessaga [6], which is quite general, tells us that any a.c. contraction T with $\alpha_1 < 0$ is indeed a strict contraction with respect to a family of different metrics.

Theorem 2.4.2 (Bessaga)**.** *Suppose X is an abstract set and $f : X \to X$ is such that f^n has a unique fixed point for all $n \in \mathbb{N}$. Let $\lambda \in (0,1)$. Then there exists a complete metric ρ_λ on X such that f is a strict contraction on (X, ρ_λ) with contraction constant λ.*

Bessaga's proof relies on the axiom of choice; in fact, he remarks that the theorem above is equivalent to a special case of the axiom of choice. We give constructive results for a.c. contractions.

2.4.2 Equivalent metrics

Suppose that (M, d) is a complete metric space and $T : M \to M$ is an (α_1, α_2)-contraction; that is, suppose T satisfies

$$\alpha_1 d(Tx, Ty) + \alpha_2 d(T^2x, T^2y) \leq kd(x, y)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 + \alpha_2 = 1$, $k \in [0, 1)$, and $x, y \in M$. By the proof of Theorem 2.4.1 and Remark 2.4.1, we know that T has a unique fixed point if $\alpha_1 < 0$ with T continuous, and that $T^n x$ converges to the fixed point for any $x \in M$ with the rate of convergence being on the order of $\frac{1}{2\alpha_2}(-\alpha_1 + \sqrt{\alpha_1^2 + 4k\alpha_2})$.

Choose k_0 with

$$\frac{1}{2\alpha_2}\left(-\alpha_1 + \sqrt{\alpha_1^2 + 4k\alpha_2}\right) \leq k_0 < 1$$

and define a new metric ρ on M:

$$\rho(x, y) := c_0 d(x, y) + c_1 d(Tx, Ty) + d(T^2x, T^2y),$$

where $0 \leq c_0 \leq k_0^2 + \frac{\alpha_1}{\alpha_2}k_0 - \frac{k}{\alpha_2}$ and $c_1 = k_0 + \frac{\alpha_1}{\alpha_2}$.

Note that $c_0 = 0$ if $k_0 = \frac{1}{2\alpha_2}\left(-\alpha_1 + \sqrt{\alpha_1^2 + 4k\alpha_2}\right)$ and that all of the above inequalities are nontrivial. Note also that we have $c_0 d(x, y) \leq \rho(x, y)$ for all $x, y \in M$, and if T is γ-Lipschitz on (M, d), then $c_0 d(x, y) \leq \rho(x, y) \leq (c_0 + \gamma c_1 + \gamma^2 c_2) d(x, y)$ for all $x, y \in M$. Similarly, if we have only continuity of T on (M, d), then we know that ρ is complete if d is complete.

Theorem 2.4.3. *T is a contraction with respect to ρ.*

Proof.

$$\begin{aligned}
\rho(Tx,Ty) &= c_0 d(Tx,Ty) + c_1 d(T^2x,T^2y) + d(T^3x,T^3y) \\
&\le c_0 d(Tx,Ty) + c_1 d(T^2x,T^2y) + \frac{k}{\alpha_2} d(Tx,Ty) - \frac{\alpha_1}{\alpha_2} d(T^2x,T^2y) \\
&= \left(c_0 + \frac{k}{\alpha_2}\right) d(Tx,Ty) + \left(c_1 - \frac{\alpha_1}{\alpha_2}\right) d(T^2x,T^2y) \\
&\le \left(k_0^2 + \frac{\alpha_1}{\alpha_2} k_0\right) d(Tx,Ty) + k_0 d(T^2x,T^2y) \\
&= k_0 \left(\left(k_0 + \frac{\alpha_1}{\alpha_2}\right) d(Tx,Ty) + d(T^2x,T^2y)\right) \\
&\le k_0 \rho(x,y)
\end{aligned}$$

□

Note that if we take $k_0 > \frac{1}{2\alpha_2}\left(-\alpha_1 + \sqrt{\alpha_1^2 + 4k\alpha_2}\right)$, then the method of Theorem 2.4.1 (and Remark 2.4.1) gives a faster convergence rate than that of the equivalent metric. On the other hand, if we take $k_0 = \frac{1}{2\alpha_2}\left(-\alpha_1 + \sqrt{\alpha_1^2 + 4k\alpha_2}\right)$, then $c_0 = 0$ and $\rho(x,y) = (k_0 + \alpha_1/\alpha_2) d(Tx,Ty) + d(T^2x,T^2y)$, which does not necessarily define a metric.

If we relax the condition on T to a.c. nonexpansiveness, then we can find a metric with respect to which a.c. nonexpansive maps are nonexpansive in the usual sense.

Theorem 2.4.4. *Suppose (M,d) is a metric space and $T : M \to M$ is a.c. nonexpansive with $\alpha_1 < 0$. For all $x,y \in M$, let $\rho(x,y) := \beta_1 d(x,y) + d(Tx,Ty)$, where $\beta_1 := \alpha_2^{-1}$. Then T is nonexpansive on (M,ρ).*

Proof. Let $\beta_2 = 1 - \beta_1 = -\alpha_1\alpha_2^{-1}$. For any $x,y \in M$, we have

$$\begin{aligned}
\rho(Tx,Ty) &= \beta_1 d(Tx,Ty) + d(T^2x,T^2y) \\
&\le \beta_1 d(Tx,Ty) + \beta_1 d(x,y) + \beta_2 d(Tx,Ty) \\
&= \beta_1 d(x,y) + d(Tx,Ty) \\
&= \rho(x,y).
\end{aligned}$$

□

2.4.3 Lipschitz constants of iterates

Finally, in the case when $\alpha_1 < 0$ and T is assumed to be Lipschitz, we deduce that T is in fact uniformly Lipschitzian with bounds for $k(T^j)$ for all j.

Theorem 2.4.5. *If (M, d) is a metric space and $T : M \to M$ is γ-Lipschitz and a.c. nonexpansive with $\alpha_1 < 0$, then T is uniformly Lipschitzian. Furthermore,*

$$k(T^n) \leq \frac{\gamma + \beta_1 + (-\beta_1)^n(1-\gamma)}{1+\beta_1},$$

where $\beta_1 = \alpha_2^{-1}$ and $\beta_2 = -\alpha_1\alpha_2^{-1}$ are positive and sum to 1.

Proof. For any $x, y \in M$, let $\mu_n = \mu_n(x, y) := d(T^n x, T^n y)$. We have the following claim.

Claim 2.4.1. *$\mu_n \leq c_n\mu_0$, where $c_0 = 1$, $c_1 = \gamma$, and $c_{n+1} = \beta_1 c_{n-1} + \beta_2 c_n$ for all $n \in \mathbb{N}$.*

Proof of the claim. We proceed by (strong) induction on n. We easily see that $\mu_0 \leq c_0\mu_0$ and $\mu_1 \leq c_1\mu_0$. Also,

$$\mu_2 \leq \beta_1\mu_0 + \beta_2\mu_1 \leq (\beta_1 + \beta_2\gamma)\mu_0 = c_2\mu_0,$$

where $c_2 = \beta_1 + \beta_2\gamma = \beta_1 c_0 + \beta_2 c_1$. Suppose that, for some $n_0 \in \mathbb{N}$, we have $\mu_k \leq c_k\mu_0$ for $k = 1, \ldots, n_0$. Then

$$\mu_{n+1} \leq \beta_1\mu_{n-1} + \beta_2\mu_n \leq (\beta_1 c_{n-1} + \beta_2 c_n)\mu_0 = c_{n+1}\mu_0$$

as desired. This completes the proof of the claim. □

From the claim and Lemma 2.2.1 we know that $c_n = a + (-\beta_1)^n b$ for some $a, b \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Solving for a and b we find that

$$a = \frac{\beta_1 + \gamma}{\beta_1 + 1} \quad \text{and} \quad b = \frac{1-\gamma}{\beta_1 + 1}.$$

Thus,

$$k(T^n) \leq c_n = a + (-\beta_1)^n b = \frac{\gamma + \beta_1 + (-\beta_1)^n(1-\gamma)}{1+\beta_1}.$$

□

Uniformly Lipschitzian maps were first studied in fixed point theory by Goebel and Kirk [27], and their results were improved by Lifshitz [39]. Before stating his result, let's recall a few pertinent notions. For $\emptyset \neq C \subset M$ and $x \in C$, let

$$r_x(C) := \sup\{d(x,y) : y \in C\}, \text{ and}$$

$$r(C) := \inf_{x \in C} r_x(C).$$

Lifshitz defined the *character of the metric space* M, which we state in the case when M is a Banach space $(X, \|\cdot\|)$ for simplicity:

$$\kappa(X) := \sup\{c > 0 : r\left(B(0,1) \cap B(x,c)\right) < 1, \|x\| \leq 1\},$$

where $B(z,r) := \{w \in X : \|z - w\| \leq r\}$. It is known that, for any Banach space X, $\kappa(X) \in [1,2]$ (by definition), and that in particular, $\kappa(H) = \sqrt{2}$ for any Hilbert space H. Lifshitz proved a fixed point theorem using this notion.

Theorem 2.4.6 (Lifshitz)**.** *If (M,d) is bounded and complete and $T : M \to M$ is uniformly Lipschitzian with $\sup_n k(T^n) < \kappa(M)$, then T has a fixed point.*

García and Piasecki [24, Theorem 3.2] note that, if C is a closed, bounded, convex subset of a Banach space X and $T : C \to C$ is such that

$$k_\infty(T) := \limsup_n k(T^n) < \kappa(X)$$

then T must have a fixed point.

From Theorem 2.4.5, we obtain a fixed point theorem for a.c. nonexpansive maps with $\alpha_1 < 0$.

Theorem 2.4.7. *If $C \subset X$ is closed, bounded, and convex, $T : C \to C$ is γ-Lipschitz, a.c. nonexpansive with $\alpha_1 < 0$, and*

$$\frac{\gamma + \beta_1}{1 + \beta_1} < \kappa(X),$$

where $\beta_1 = \alpha_2^{-1}$ and $\beta_2 = -\alpha_1\alpha_2^{-1}$ as usual, then T has a fixed point.

Proof. $k_\infty(T) = \limsup_n k(T^n) \leq (\gamma + \beta_1)/(1 + \beta_1)$ since $\alpha_1 < 0 \implies \alpha_2 > 1 \implies \beta_1 \in (0,1)$, so $(-\beta_1)^n \to 0$ as $n \to \infty$. □

2.5 MISCELLANEOUS QUESTIONS

Since we have a generalized notion of nonexpansiveness and an analogue of Banach's Contraction Mapping Theorem, it is natural to ask whether the proof of Theorem 1.0.4 which guarantees the existence of approximate fixed point sequences for nonexpansive mappings defined on closed, bounded, convex subsets of a Banach space can be adapted to the new context. The formal statement of the question is as follows: *Can we use Theorem 2.4.1 to obtain approximate fixed point sequences for a.c. nonexpansive mappings with* $\alpha_1 < 0$ *which are defined on closed, bounded, convex subsets of a Banach space?*

We showed in Theorem 2.3.1 that (α_1, α_2)-isometries on bounded metric spaces were isometries in the usual sense for $\alpha_2 \in (-1, 1)$, and it seems natural to ask whether this theorem extends to the case of arbitrary length multi-indices. So we have a question: *Must a.c. isometries for multi-indices of arbitrary length be, in the appropriate context, isometries in the usual sense?*

3.0 THE DEMICLOSEDNESS PRINCIPLE

We remark that the theorems in Sections 3.1 - 3.3 have recently appeared in [20]. Goebel and Japón Pineda suggested, but did not study, the class of (α, p)-*nonexpansive maps*. A function $T : (M, d) \to (M, d)$ is called (α, p)-*nonexpansive* if, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k \geq 0$ for all k, $\alpha_1, \alpha_n > 0$, and for some $p \in [1, \infty)$,

$$\sum_{k=1}^{n} \alpha_k d(T^k x, T^k y)^p \leq d(x, y)^p, \text{ for all } x, y \in M.$$

For simplicity, we will generally discuss the case when $n = 2$ and when M is a subset of a Banach space. That is, for $(X, \|\cdot\|)$ a Banach space, $C \subseteq X$, and $T : C \to C$, we say T is $((\alpha_1, \alpha_2), p)$-*nonexpansive* if for some $p \in [1, \infty)$, we have

$$\alpha_1 \|Tx - Ty\|^p + \alpha_2 \left\|T^2 x - T^2 y\right\|^p \leq \|x - y\|^p, \text{ for all } x, y \in C.$$

When $p = 1$, we have the original notion of (α_1, α_2)-nonexpansiveness.

As one can imagine, (α, p)-nonexpansive maps can be quite natural to study in L^p spaces. The following is an example of a $((1/2, 1/2), 2)$-nonexpansive map defined on $(\ell^2, \|\cdot\|_2)$ for which none of its iterates are nonexpansive. The map below is based on Example 2.1.1.

Example 3.0.1. Let $(\ell^2, \|\cdot\|_2)$ be the Hilbert space of square-summable sequences endowed with its usual norm. Let $\tau : [-1, 1] \to [-1, 1]$ be given by

$$\tau(t) := \begin{cases} \sqrt{2}\,t + (\sqrt{2} - 1) & -1 \leq t \leq -t_0 \\ 0 & -t_0 \leq t \leq t_0 \\ \sqrt{2}\,t - (\sqrt{2} - 1) & t_0 \leq t \leq 1 \end{cases}$$

where $t_0 := (\sqrt{2}-1)/\sqrt{2}$.

Note the following facts about τ:

1. τ is Lipschitz with $k(\tau) = \sqrt{2}$, and
2. $|\tau(t)| \leq |t|$ for all $t \in [-1, 1]$.

Let B_{ℓ^2} denote the closed unit ball of $(\ell^2, \|\cdot\|_2)$ and for any $x \in \ell^2$, define T by

$$T(x_1, x_2, \ldots) := \left(\tau(x_2), \sqrt{\frac{2}{3}}\, x_3, x_4, x_5, \ldots\right)$$

and

$$T^2(x_1, x_2, \ldots) = \left(\tau\left(\sqrt{\frac{2}{3}}\, x_3\right), \sqrt{\frac{2}{3}}\, x_4, x_5, \ldots\right).$$

Observe that $|\tau(t)| \leq |t|$ implies that $T(B_{\ell^2}) \subseteq B_{\ell^2}$, and $k(T) = \sqrt{2} > 1$ and $k(T^j) = \frac{2}{\sqrt{3}} > 1$ for all $j \geq 2$. Now, for any $x, y \in B_{\ell^2}$ we find

$$\begin{aligned}
&\frac{1}{2}\|Tx - Ty\|_2^2 + \frac{1}{2}\left\|T^2x - T^2y\right\|_2^2 \\
&\qquad = \frac{1}{2}\left(|\tau(x_2) - \tau(y_2)|^2 + \frac{2}{3}|x_3 - y_3|^2 + \sum_{j=4}^{\infty}|x_j - y_j|^2\right) \\
&\qquad + \frac{1}{2}\left(\left|\tau\left(\sqrt{\frac{2}{3}}\, x_3\right) - \tau\left(\sqrt{\frac{2}{3}}\, y_3\right)\right|^2 + \frac{2}{3}|x_4 - y_4|^2 + \sum_{j=5}^{\infty}|x_j - y_j|^2\right) \\
&\qquad \leq \frac{1}{2}\left(2|x_2 - y_2|^2 + \frac{4}{3}|x_3 - y_3|^2 + \frac{5}{3}|x_4 - y_4|^2 + 2\sum_{j=5}^{\infty}|x_j - y_j|^2\right) \\
&\qquad \leq \|x - y\|_2^2.
\end{aligned}$$

Hence, $T : B_{\ell^2} \to B_{\ell^2}$ is a $((1/2, 1/2), 2)$-nonexpansive map for which each iterate T^j is not nonexpansive.

It is easy to check that all (α, p)-nonexpansive maps with $p > 1$ are α-nonexpansive. Indeed, if $T : (M, d) \to (M, d)$ is (α, p)-nonexpansive, then for any $x, y \in M$ we use the Hölder inequality to see that

$$\begin{aligned}\sum_{k=1}^{n} \alpha_k d(T^k x, T^k y) &= \sum_{k=1}^{n} \alpha_k^{\frac{p-1}{p}} \alpha_k^{\frac{1}{p}} d(T^k x, T^k y) \\ &\leq \left(\sum_{k=1}^{n} \alpha_k\right)^{\frac{p-1}{p}} \left(\sum_{k=1}^{n} \alpha_k d(T^k x, T^k y)^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{n} \alpha_k d(T^k x, T^k y)^p\right)^{\frac{1}{p}} \\ &\leq d(x, y).\end{aligned}$$

Hence, T is α-nonexpansive.

The converse, however, does not hold; that is, there is an α-nonexpansive map which is not (α, p)-nonexpansive for any $p > 1$, as we see in the following example due to Piasecki [46, Ch. 5].

Example 3.0.2. Let $T : \ell^1 \to \ell^1$ be the linear mapping given by

$$Tx := \left(2x_2, \frac{2}{3}x_3, x_4, x_5, \dots\right),$$

and we find that

$$T^2 x = \left(\frac{4}{3}x_3, \frac{2}{3}x_4, x_5, \dots\right).$$

Now

$$\frac{1}{2}\left(\|Tx\|_1 + \left\|T^2 x\right\|_1\right) = |x_2| + |x_3| + \frac{5}{6}|x_4| + \sum_{n=5}^{\infty} |x_n| \leq \|x\|_1$$

for any $x \in \ell^1$, and we have that T is $(1/2, 1/2)$-nonexpansive. However, note that

$$Te_3 := T(0, 0, 1, 0, 0, \dots) = \left(0, \frac{2}{3}, 0, 0, \dots\right),$$

$$T^2 e_3 = \left(\frac{4}{3}, 0, 0, 0, \dots\right),$$

and finally, for any $\alpha_1, \alpha_2 > 0$ with $\alpha_1 \leq 1/2$ and $\alpha_1 + \alpha_2 = 1$ and for any $p > 1$, by the (strict) convexity of the real-valued function $u \mapsto u^p$, we see that

$$\begin{aligned}\alpha_1 \|Te_3\|_1^p + \alpha_2 \left\|T^2 e_3\right\|_1^p &= \alpha_1 \left(\frac{2}{3}\right)^p + \alpha_2 \left(\frac{4}{3}\right)^p \\ &> \left(\alpha_1 \cdot \frac{2}{3} + \alpha_2 \cdot \frac{4}{3}\right)^p \\ &\geq 1 \\ &= \|e_3\|_1^p .\end{aligned}$$

Hence, T is not $((\alpha_1, \alpha_2), p)$-nonexpansive. In particular, T is not $((1/2, 1/2), p)$-nonexpansive for any $p > 1$.

Goebel and Japón Pineda gave preliminary fixed point results for the class of mean nonexpansive maps. Specifically, they proved two important theorems, which we state for the case when $n = 2$ for simplicity. The proofs are elementary and are given for completeness. Furthermore, we present the proof to emphasize the contrast between the methods of Goebel and Japón Pineda and the methods which are presented in this thesis. In particular, their proofs rely heavily on the nonexpansiveness of the map $T_\alpha := \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_n T^n$ (which is seen to be nonexpansive by the triangle inequality whenever T is α-nonexpansive).

Theorem 3.0.1 (Goebel and Japón Pineda)**.** *If $(X, \|\cdot\|)$ is a Banach space, C is a closed, bounded, convex subset of X and $T : C \to C$ is (α_1, α_2)-nonexpansive, then T has an approximate fixed point sequence, provided that $\alpha_1 \geq \frac{1}{2}$.*

Proof. Fix $\varepsilon > 0$. Since T_α is a nonexpansive self-map on C, we know that $\inf_C \|T_\alpha x - x\| = 0$ and thus there exists $x_\varepsilon \in C$ for which $\|T_\alpha x_\varepsilon - x_\varepsilon\| \leq \alpha_2 \varepsilon$. Since T is (α_1, α_2)-nonexpansive, we have

$$\begin{aligned}\alpha_1 \left\|T^2 x_\varepsilon - T x_\varepsilon\right\| + \alpha_2 \left\|T^3 x_\varepsilon - T^2 x_\varepsilon\right\| &\leq \|T x_\varepsilon - x_\varepsilon\| \\ &= \|T x_\varepsilon - T_\alpha x_\varepsilon + T_\alpha x_\varepsilon - x_\varepsilon\| \\ &\leq \|T x_\varepsilon - T_\alpha x_\varepsilon\| + \|T_\alpha x_\varepsilon - x_\varepsilon\| \\ &\leq \left\|(1 - \alpha_1) T x_\varepsilon - \alpha_2 T^2 x_\varepsilon\right\| + \alpha_2 \varepsilon \\ &= \alpha_2 \left\|T x_\varepsilon - T^2 x_\varepsilon\right\| + \alpha_2 \varepsilon.\end{aligned}$$

Thus, $(\alpha_1 - \alpha_2)\left\|Tx_\varepsilon - T^2x_\varepsilon\right\| + \alpha_2\left\|T^3x_\varepsilon - T^2x_\varepsilon\right\| \le \alpha_2\varepsilon \iff (2\alpha_1 - 1)\left\|Tx_\varepsilon - T^2x_\varepsilon\right\| + \alpha_2\left\|T^3x_\varepsilon - T^2x_\varepsilon\right\| \le \alpha_2\varepsilon$. Since $\alpha_1 \ge \frac{1}{2}$, we know $2\alpha_1 - 1 \ge 0$, so $\|Tz_\varepsilon - z_\varepsilon\| \le \varepsilon$, where $z_\varepsilon := T^2x_\varepsilon \in C$. □

Taking $\varepsilon = 0$ in the above proof yields the following result.

Theorem 3.0.2 (Goebel and Japón Pineda)**.** *If $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps, then any (α_1, α_2)-nonexpansive map $T : C \to C$ has a fixed point, provided that $\alpha_1 \ge \frac{1}{2}$.*

Piasecki [46, Theorems 8.3 and 8.4] generalizes these results for (α, p)-nonexpansive mappings as follows.

Theorem 3.0.3 (Piasecki)**.** *If $T : C \to C$ is (α, p)-nonexpansive for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $p \ge 1$, then T has an approximate fixed point sequence provided that*

$$(1 - \alpha_1)\left(1 - \alpha_1^{\frac{n-1}{p}}\right) \le \alpha_1^{\frac{n-1}{p}}\left(1 - \alpha_1^{\frac{1}{p}}\right).$$

Furthermore, if X has the fixed point property for nonexpansive maps, then T has a fixed point.

Goebel and Japón Pineda asked what, if anything, can be said for mean nonexpansive maps with $\alpha_1 < \frac{1}{2}$ (the more general version of the theorems for $\alpha = (\alpha_1, \ldots, \alpha_n)$ requires that $\alpha_1 \ge 2^{(1-n)^{-1}}$). In light of Theorem 3.0.3 above, one may ask even more generally what can be said for (α, p)-nonexpansive maps for which

$$(1 - \alpha_1)\left(1 - \alpha_1^{\frac{n-1}{p}}\right) > \alpha_1^{\frac{n-1}{p}}\left(1 - \alpha_1^{\frac{1}{p}}\right).$$

We give a partial answer to this question in Theorem 3.3.1, which has no restriction on the value of α_1, but does rely on the existence of an approximate fixed point sequence.

Similar to Goebel and Japón Pineda, García and Piasecki [24] have proven fixed point theorems that place restrictions on α in spaces X with characteristic of convexity $\varepsilon_0(X) < 1$, which in turn gives Lifshitz constant $\kappa(X) > 1$ [17]. The main fixed point theorem in their paper, given below, relies on the fact that mean nonexpansive mappings are uniformly Lipschitzian with precise estimates for the Lipschitz constant of T^n for all n.

Theorem 3.0.4 (García and Piasecki (Thm. 3.4)). *Let X be a Banach space with $\varepsilon_0(X) < 1$ and let $C \subset X$ be nonempty, closed, bounded, and convex. If $T : C \to C$ is (α, p)-nonexpansive for $p \geq 1$ with*

$$\left(\sum_{j=1}^{n}\left(\sum_{i=j}^{n}\alpha_i\right)\right)^{\frac{1}{p}} < \kappa(X),$$

then T has a fixed point in C.

This yields the following interesting corollary.

Corollary 3.0.1 (García and Piasecki (Cor. 3.7)). *If H is a Hilbert space, $C \subset H$ is closed, bounded, and convex, and $T : C \to C$ is $((\alpha_1, \alpha_2), 2)$-nonexpansive, then T has a fixed point.*

Recall that a mapping $F : C \subseteq X \to X$ is called *demiclosed at y* (Definition 1.0.6) if, whenever x_n converges weakly to x in C and $Fx_n \to y$ strongly in X, it follows that $Fx = y$. Browder's famous Demiclosedness Principle [9] states that if $(X, \|\cdot\|)$ is uniformly convex, $C \subseteq X$ is closed, bounded, and convex, and $U : C \to X$ is nonexpansive, then $I - U$ is demiclosed, where I is the identity operator. Note that U need not map C into C, an observation which is essential for our study.

Suppose that $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$. Recall that we say C has the *Opial property* (Definition 1.0.5) if, for any sequence $(x_n)_n$ in C converging weakly to some $x \in X$ and for any $y \in X$ with $y \neq x$, we have $\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|$. When $C = X$ we say the space $(X, \|\cdot\|)$ has the Opial property. It is well known that all Hilbert spaces and the sequence spaces $(\ell^p, \|\cdot\|_p)$ have the Opial property for all $p \in (1, \infty)$, yet $(L^p, \|\cdot\|_p)$ fails to have the Opial property when $p \neq 2$. In particular, uniform convexity does not imply Opial's property, and spaces having Opial's property need not be isomorphic to a uniformly convex space [18].

3.1 DEMICLOSEDNESS AND UNIFORM CONVEXITY

We will begin by formally stating the main theorem of this section.

Theorem 3.1.1. *Suppose $(X, \|\cdot\|)$ is uniformly convex, $C \subseteq X$ is nonempty, closed, bounded, and convex, and $T : C \to C$ is $((\alpha_1, \alpha_2), p)$-nonexpansive for some $p \in (1, \infty)$. Then $I - T$ is demiclosed at 0.*

In order to prove this theorem, we need a lemma and classical results of Clarkson [13, Theorem 1] and Browder [9, Theorem 3] (c.f. Theorem 1.0.10). First, the result of Clarkson:

Theorem 3.1.2 (Clarkson)**.** *If $(X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)$ are uniformly convex, then for $p > 1$, $(X_1 \oplus \cdots \oplus X_n, \|\cdot\|_p)$ is uniformly convex, where $\|(x_1, \ldots, x_n)\|_p := (\|x_1\|_1^p + \cdots + \|x_n\|_n^p)^{\frac{1}{p}}$.*

Corollary 3.1.1. *If $(X, \|\cdot\|)$ is uniformly convex and $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, then for $p > 1$, $(X^2, \|\cdot\|_{\alpha,p})$ is uniformly convex, where $\|(x, y)\|_{\alpha,p} := (\alpha_1 \|x\|^p + \alpha_2 \|y\|^p)^{\frac{1}{p}}$.*

Now we state a version of Browder's Demiclosedness Principle for uniformly convex spaces:

Theorem 3.1.3 (Browder's Demiclosedness Principle)**.** *Suppose $(X, \|\cdot\|)$ is uniformly convex, $K \subset X$ is closed, bounded, and convex, and $U : K \to X$ is nonexpansive. Then*

1. *$I - U$ is demiclosed, and*
2. *$(I - U)(K)$ is closed in X.*

Finally, we have a straightforward lemma regarding weak convergence.

Lemma 3.1.1. *$(X^2, \|\cdot\|_{\alpha,p})^* = (X^* \oplus X^*, \|\cdot\|_{\alpha,p}^*)$, where $\|\cdot\|_{\alpha,p}$ is the norm defined above and $\|\cdot\|_{\alpha,p}^*$, the dual norm to $\|\cdot\|_{\alpha,p}$, is defined as usual. In particular, if the sequences $(y_n)_n, (z_n)_n$ converge weakly to y, z, respectively, in X, then $(y_n, z_n)_n$ converges weakly to (y, z) in X^2.*

Proof. First, given any two $\varphi, \psi \in X^*$, the linear map $\Phi(x, y) := \varphi(x) + \psi(y)$ is $\|\cdot\|_{\alpha,p}^*$-bounded, so $X^* \oplus X^* \subseteq (X^2)^*$. Conversely, given any $\Phi \in (X^2)^*$, define $\varphi(x) := \Phi(x, 0)$ and $\psi(y) := \Phi(0, y)$. Then both φ, ψ are linear and bounded, and thus elements of X^*. Hence, $X^* \oplus X^* = (X^2)^*$.

Second, for any sequences $y_n \rightharpoonup y$ and $z_n \rightharpoonup z$ in X and for any $\Phi \in (X^2)^*$, we have that $\Phi(y_n, z_n) = \varphi(y_n) + \psi(z_n) \to \varphi(y) + \psi(z) = \Phi(y, z)$. □

We can now prove Theorem 3.1.1. The techniques below are inspired by the proof, due to Kirk, Martinez Yañez, and Sik Shin [35, Theorem 4.1], that asymptotically nonexpansive mappings have approximate fixed point sequences in Banach spaces which have the so-called "super fixed point property for nonexpansive maps;" i.e. every ultrapower of the space X has fpp(ne). Sparing most of the details, the key observation made by the authors was that, if T is asymptotically nonexpansive on X, then the associated function T', defined on the ultrapower of X, given by

$$T'(x_1, x_2, x_3, \ldots) := (Tx_1, T^2x_2, T^3x_3, \ldots)$$

is nonexpansive.

Proof of Theorem 3.1.1. Suppose that $(X, \|\cdot\|)$ is uniformly convex, $C \subset X$ is closed, bounded, and convex, and $T : C \to C$ is $((\alpha_1, \alpha_2), p)$-nonexpansive for some $p \in (1, \infty)$.

Let $(X^2, \|\cdot\|_{\alpha,p})$ be as defined above. Define the function $\widetilde{T} : C^2 \to C^2$ by

$$\widetilde{T}(x, y) := (Tx, T^2y).$$

Generally speaking, we cannot say very much about $\left\|\widetilde{T}(x, y) - \widetilde{T}(u, v)\right\|_{\alpha,p}$, since

$$\left\|\widetilde{T}(x, y) - \widetilde{T}(u, v)\right\|_{\alpha,p} = \left(\alpha_1 \|Tx - Tu\|^p + \alpha_2 \left\|T^2y - T^2v\right\|^p\right)^{\frac{1}{p}}.$$

However, consider the set

$$D := \{(x, x) : x \in C\} \subseteq C^2,$$

and observe that for any $(x, x), (y, y) \in D$, we now have

$$\begin{aligned}
\left\|\widetilde{T}(x, x) - \widetilde{T}(y, y)\right\|_{\alpha,p} &= \left\|(Tx - Ty, T^2x - T^2y)\right\|_{\alpha,p} \\
&= \left(\alpha_1 \|Tx - Ty\|^p + \alpha_2 \left\|T^2x - T^2y\right\|^p\right)^{\frac{1}{p}} \\
&\le (\|x - y\|^p)^{\frac{1}{p}} \\
&= (\alpha_1 \|x - y\|^p + \alpha_2 \|x - y\|^p)^{\frac{1}{p}} \\
&= \|(x, x) - (y, y)\|_{\alpha,p}.
\end{aligned}$$

Thus, $\widetilde{T}\big|_D : D \to C^2$ is nonexpansive and, by Browder's Demiclosedness Principle, $I - \widetilde{T}\big|_D$ is demiclosed. To see that $I - T$ is also demiclosed at 0, let $(x_n)_n$ be any sequence converging weakly to $x \in C$ for which $(I - T)x_n \to 0$ (that is, $(x_n)_n$ is an approximate fixed point sequence for T). Then $(I - T^2)x_n \to 0$ as well, since

$$\left\|x_n - T^2 x_n\right\| \leq \|x_n - Tx_n\| + \left\|Tx_n - T^2 x_n\right\| \leq \left(1 + \alpha_1^{-\frac{1}{p}}\right) \|x_n - Tx_n\|.$$

Thus, $x_n \rightharpoonup x$ implies $(x_n, x_n) \rightharpoonup (x, x)$ and $\|x_n - Tx_n\|, \|x_n - T^2 x_n\| \to 0$ implies $(I - \widetilde{T})(x_n, x_n) \to (0, 0)$, so the demiclosedness of $I - \widetilde{T}\big|_D$ tells us that $(I - \widetilde{T})(x, x) = (0, 0)$. That is, $(Tx, T^2 x) = (x, x)$ and $Tx = x$. Therefore, $I - T$ is demiclosed at 0. □

This result easily generalizes to (α, p)-nonexpansive maps for arbitrary length $\alpha = (\alpha_1, \ldots, \alpha_n)$, as shown in the following theorem.

Theorem 3.1.4. *Suppose $(X, \|\cdot\|)$ is uniformly convex, $C \subseteq X$ is nonempty, closed, bounded, and convex, and $T : C \to C$ is (α, p)-nonexpansive for some $p \in (1, \infty)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then $I - T$ is demiclosed at 0.*

Proof. If $\{1 < j < n : \alpha_j = 0\} \neq \emptyset$, then write $\{1 < j < n : \alpha_j = 0\} = \{j_1, j_2, \ldots, j_m\}$ and $\{1 \leq k \leq n : \alpha_j > 0\} = \{k_1, k_2, \ldots, k_{n-m}\}$, where $k_1 = 1$ and $k_{n-m} = n$. Then the space $(X^\nu, \|\cdot\|_{\alpha,p})$ is uniformly convex, where $\nu := n - m$ and

$$\|(x_1, \ldots, x_\nu)\|_{\alpha,p} := \left(\sum_{r=1}^{\nu} \alpha_{k_r} \|x_r\|^p\right)^{\frac{1}{p}}.$$

Again, the map $\widetilde{T} : C^\nu \to C^\nu$, defined by

$$\begin{aligned}\widetilde{T}(x_1, \ldots, x_\nu) &:= (T^{k_1} x_1, T^{k_2} x_2, \ldots, T^{k_{\nu-1}} x_{\nu-1}, T^{k_\nu} x_\nu) \\ &= (Tx_1, T^{k_2} x_2, \ldots, T^{k_{\nu-1}} x_{\nu-1}, T^n x_\nu)\end{aligned}$$

is nonexpansive on $D := \{(x, x, \ldots, x) : x \in C\}$, which is a closed, bounded, and convex set in X^ν. Thus, $I - \widetilde{T}\big|_D$ is demiclosed, and an analogous argument to the one presented above gives us that $I - T$ is demiclosed at 0 as well. □

Remark 3.1.1. For any function f with domain A, let $F(f)$ denote the fixed point set of f: $F(f) := \{x \in A : fx = x\}$. Observe that for $T : C \to C$ and $\widetilde{T}\big|_D : D \to C^\nu$ defined as above, we have

$$F\left(\widetilde{T}\big|_D\right) = \{(x, \ldots, x) \in D : (Tx, \ldots, T^n x) = (x, \ldots, x)\} = \{(x, \ldots, x) \in D : Tx = x\},$$

which may be easily identified with the set $\{x \in C : Tx = x\} = F(T)$. Also note that $F(T) = (I - \widetilde{T})^{-1}\{0\} \cap D$. Note that the above proof tells us that $(I - \widetilde{T})(D)$ is closed in $(X^\nu, \|\cdot\|_{\alpha,p})$, and this gives a dichotomous scenario regarding the existence of fixed points for an (α, p)-nonexpansive map T: either $0 \in (I - \widetilde{T})(D)$ or $0 \notin (I - \widetilde{T})(D)$. In other words, T either has a fixed point, or T admits no approximate fixed point sequences. For more about fixed point results, see Section 3.3.

Remark 3.1.2. It should also be noted that Klin-eam and Suantai [37, Theorem 3.4] proved a version of the demiclosedness principle for $(\alpha_1, \ldots, \alpha_n)$-nonexpansive mappings in uniformly convex spaces, but with the restriction that α_1 is sufficiently large, specifically $\alpha_1 > \sqrt{2}^{\,1-n}$. Their method of proof resembles those of Goebel and Japón Pineda and primarily utilizes the triangle inequality.

3.2 DEMICLOSEDNESS AND OPIAL'S PROPERTY

Now we will establish the demiclosedness principle for (α_1, α_2)-nonexpansive maps whose domains satisfy Opial's property. That is, the definition of Opial's property can be applied to subsets of Banach spaces rather than to the entire space. We say that $C \subseteq X$ has the *Opial property* if, whenever $(x_n)_n$ is a sequence in C converging weakly to some $x \in X$, then

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|$$

for all $y \neq x$ in X.

We first make an easy observation for length 2 multi-indices which extends easily to multi-indices of arbitrary length.

Lemma 3.2.1. *If $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$ is closed and convex with the Opial property, then $(D, \|\cdot\|_\alpha) \subset (C^2, \|\cdot\|_\alpha)$ has the Opial property, where $\|\cdot\|_\alpha := \|\cdot\|_{\alpha,1}$.*

Proof. Let $(x_n, x_n)_n$ be a weakly convergent sequence in D. Then $(x_n, x_n) \rightharpoonup (x, x)$ for some $x \in C$ since C closed and convex implies C is weakly closed. Let $(u, v) \in X^2$ be such that $(u, v) \neq (x, x)$. Without loss of generality, suppose that $u \neq x$. Then

$$\liminf_n \|x_n - x\| \leq \liminf_n \|x_n - v\|,$$

and since C is Opial we have that

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - u\|.$$

Thus,

$$\begin{aligned}\liminf_n \|(x_n, x_n) - (x, x)\|_\alpha &= \liminf_n (\alpha_1 \|x_n - x\| + \alpha_2 \|x_n - x\|) \\ &< \liminf_n (\alpha_1 \|x_n - u\| + \alpha_2 \|x_n - v\|) \\ &= \liminf_n \|(x_n, x_n) - (u, v)\|_\alpha .\end{aligned}$$

Hence, D is Opial. □

The following theorem is straightforward, and we present the proof for completeness.

Theorem 3.2.1. *Suppose $C \subseteq X$ is closed and convex and C has the Opial property. If $U : C \to X$ is nonexpansive, then $I - U$ is demiclosed at* 0.

Proof. Suppose $(x_n)_n$ is a sequence in C weakly convergent to some $x \in X$. Since C is closed and convex, it is weakly closed and $x \in C$. Suppose further that $(I - U)x_n \to 0$. Since U is nonexpansive,

$$\|x_n - (I - U)x_n - Ux\| = \|Ux_n - Ux\| \leq \|x_n - x\|$$

and

$$\begin{aligned}\liminf_n \|x_n - (I - U)x_n - Ux\| &\geq \liminf_n (\|x_n - Ux\| - \|(I - U)x_n\|) \\ &= \liminf_n \|x_n - Ux\| .\end{aligned}$$

Thus,

$$\liminf_n \|x_n - Ux\| \leq \liminf_n \|x_n - x\|.$$

Since C is Opial, this is only possible if $x = Ux$, and we have $I - U$ is demiclosed at 0. □

Notice that, in contrast with Browder's Demiclosedness Principle, we don't need the assumption of boundedness in Theorem 3.2.1, but the most that we can conclude is demiclosedness of $I - T$ at 0 (rather than demiclosedness of $I - T$ at every point). Now we have our theorem, the proof of which is nearly identical to the proof of Theorem 3.1.1.

Theorem 3.2.2. *If $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$ is closed and convex with the Opial property, then any α-nonexpansive map $T : C \to C$ is such that $I - T$ is demiclosed at 0.*

Recall that a mapping $J : X \to X^*$ (where $(X^*, \|\cdot\|^*)$ is the dual space of X with dual norm $\|\cdot\|^*$ defined as usual) is called *a duality mapping of X* if, for all $x \in X$, J satisfies both

1. $(Jx)(x) = \|Jx\|^* \|x\|$, and
2. $\|Jx\|^* = \mu(\|x\|)$, where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, strictly increasing function with $\mu(0) = 0$.

Before drawing an application to uniformly convex spaces, we state a lemma of Opial [43, Lemma 3]:

Lemma 3.2.2 (Opial)**.** *If a Banach space X has a weakly continuous duality mapping and $(x_n)_n$ converges weakly to x_0, then for any $x \in X$,*

$$\liminf_n \|x_n - x_0\| \leq \liminf_n \|x_n - x\|.$$

If X is also uniformly convex, then X has the Opial property.

Corollary 3.2.1. *If $(X, \|\cdot\|)$ is uniformly convex with a weakly continuous duality map and $C \subset X$ is closed, bounded, and convex, then any α-nonexpansive map $T : C \to C$ is demiclosed at 0. In particular, this result holds when $(X, \|\cdot\|) = (\ell^p, \|\cdot\|_p)$ for $p \in (1, \infty)$.*

Proof. Since X is uniformly convex with a weakly continuous duality map, X has the Opial property by Lemma 3.2.2. □

Remark 3.2.1. Our results are genuinely distinct from those of Xu for asymptotically nonexpansive maps [51] since the class of mean nonexpansive maps and asymptotically nonexpansive maps do not coincide. For instance, if T is asymptotically nonexpansive, then $\limsup_n k(T^n) = \lim_n k(T^n) = 1$, and yet Piasecki [46, Ch. 4] constructed an example of an (α_1, α_2)-nonexpansive mapping T for which $\limsup_n k(T^n) = \lim_n k(T^n) = 1 + \alpha_2$.

Furthermore, our results are not included in those of Goebel and Kirk for uniformly Lipschitzian maps [27]. As we have seen, every mean nonexpansive map is uniformly Lipschitzian [23, 46], but $\sup_n k(T^n) \leq \alpha_1^{-1}$ (and the inequality is sharp), where α_1 can be arbitrarily small.

Remark 3.2.2. Finally, as noted in the introduction, spaces having uniform normal structure are of interest in fixed point theory, but our techniques fail in such spaces. In particular, the demiclosedness principle does not necessarily hold in spaces having uniform normal structure, with the space $(\ell^2 \oplus \mathbb{R}, \|\cdot\|)$ (where $\|(x,t)\| := \max\{\|x\|_2, |t|\}$) serving as an example [34].

3.3 FIXED POINT RESULTS

Now we present a standard argument to show that any function T whose domain is a weakly compact subset of a Banach space with $I - T$ demiclosed at 0 has a fixed point, provided that T has an approximate fixed point sequence.

Lemma 3.3.1. *Suppose $C \subset X$ is weakly compact and $T : C \to X$ is such that T has an approximate fixed point sequence and $I - T$ is demiclosed at 0. Then T has a fixed point.*

Proof. Let the approximate fixed point sequence for T be denoted by $(z_n)_n$. Since $(z_n)_n$ is a sequence in C, which is weakly compact, there is a weakly convergent subsequence $(z_{n_k})_k$ with $z_{n_k} \rightharpoonup z$ for some $z \in C$. For simplicity, denote z_{n_k} by z_k. Also note that $(I-T)z_k \to 0$. Thus, by the demiclosedness of $I - T$ at 0, we have $Tz = z$. □

Now we have a partial extension of Theorem 3.0.2 that has no requirement on the size of α_1, and thus we have a partial answer to the fixed point question for mean nonexpansive maps, which we summarize in the following theorem.

Theorem 3.3.1. *Suppose* $(X, \|\cdot\|)$ *is a Banach space,* $C \subset X$ *is closed, bounded, and convex, and* $T : C \to C$ *is* (α, p)*-nonexpansive for some* $\alpha = (\alpha_1, \dots, \alpha_n)$ *and* $p \in [1, \infty)$*. Suppose further that* T *has an approximate fixed point sequence. Then* T *has a fixed point if*

1. $(X, \|\cdot\|)$ *is uniformly convex and* $1 < p$*, or*
2. $(C, \|\cdot\|)$ *is weakly compact with the Opial property.*

Proof. In either case, $I - T$ is demiclosed at 0 and a fixed point of T is a weak-subsequential limit point of its approximate fixed point sequence. □

Since it is known that uniformly convex spaces and weakly compact sets with the Opial property have the fixed point property for nonexpansive maps, the above theorem yields a special case of Theorem 3.0.3 as a corollary.

Corollary 3.3.1. *Suppose* $(X, \|\cdot\|)$ *is a Banach space,* $C \subset X$ *is closed, bounded, and convex, and* $T : C \to C$ *is* (α, p)*-nonexpansive for some* $\alpha = (\alpha_1, \dots, \alpha_n)$ *with*

$$(1 - \alpha_1)\left(1 - \alpha_1^{\frac{n-1}{p}}\right) \leq \alpha_1^{\frac{n-1}{p}}\left(1 - \alpha_1^{\frac{1}{p}}\right)$$

and $p \in [1, \infty)$*. Then* T *has a fixed point if either*

1. X *is uniformly convex and* $p > 1$*, or*
2. C *is weakly compact with the Opial property.*

Proof. Since $(1 - \alpha_1)\left(1 - \alpha_1^{\frac{n-1}{p}}\right) \leq \alpha_1^{\frac{n-1}{p}}\left(1 - \alpha_1^{\frac{1}{p}}\right)$, T has an approximate fixed point sequence by Theorem 3.0.3. □

Finally, recall that a Banach space $(X, \|\cdot\|)$ is called *strictly convex* if $x, y \in X$ with $\|x\|, \|y\| \leq 1$, then

$$\|x - y\| > 0 \quad \Longrightarrow \quad \frac{1}{2}\|x + y\| < 1.$$

It is well-known that, if $(X, \|\cdot\|)$ is strictly convex, $C \subset X$ is closed and convex, and $T : C \to C$ is nonexpansive, then $F(T)$ is closed and convex (see, for instance, [29, p. 34]). There is nothing in the proof which requires $T(C) \subset C$, however, so the same result holds for nonexpansive non-selfmaps $T : C \to X$.

García and Piasecki proved [24, Theorem 4.2] that any α-nonexpansive self-map of a closed, convex subset of a strictly convex space must have a closed and convex fixed point set. Our techniques do not yield this result in its entirety since $\|\cdot\|_{\alpha,1}$ is not generally strictly convex. However, the techniques in this paper do yield the following theorem, which is a special case of the García and Piasecki result, essentially for free:

Theorem 3.3.2. *If $(X, \|\cdot\|)$ is strictly convex, $C \subset X$ is closed, bounded, convex, and $T : C \to C$ is (α, p)-nonexpansive for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $p > 1$, then $F(T)$ is closed and convex.*

Proof. Without loss of generality, $\alpha_j > 0$ for all $j = 1, \ldots, n$. Using the notation established previously, $(X^n, \|\cdot\|_{\alpha,p})$ is strictly convex and $\widetilde{T}\big|_D : D \to C^n$ is nonexpansive, which implies that its fixed point set is closed and convex in $(X^n, \|\cdot\|_{\alpha,p})$. It follows that $F(T)$ is also closed and convex in $(X, \|\cdot\|)$. □

Remark 3.3.1. In light of the above theorems, is now important to determine precisely when a mean nonexpansive map admits an approximate fixed point sequence. Some partial results toward this end are known. For example, it is shown in [25] that if T is $(\alpha_1, \alpha_2, \alpha_3)$-nonexpansive with $1/2 \leq \alpha_1$ and $\alpha_2 \geq (1 - \alpha_1)/2$, then T has an approximate fixed point sequence. Piasecki [46, Theorem 8.23] then extends this result to $\alpha_2 \geq 1/2 - \alpha_1^2$.

While all known examples of mean nonexpansive maps admit approximate fixed point sequences (regardless of the size of α_1), it is possible that α_1 must be sufficiently large in order to *guarantee* the existence of an approximate fixed point sequence, which would be interesting in its own right. If this is the case, then Theorems 3.1.1, 3.1.4, and 3.2.2 would still be of interest for convergence and approximation purposes, but Theorem 3.3.1 would reduce to the special case of Theorem 3.0.3 stated in the above corollary.

Remark 3.3.2. It is worth noting that fixed point theory for nonexpansive non-self maps in Banach spaces usually requires some kind of boundary condition, most notably *inwardness* of the map. Recall that a function $F : K \to X$ is called *inward at x* if $Fx \in I_K(x) := \{x + \lambda(y - x) : y \in K, \lambda \geq 0\}$. However, it is easy to check that $\widetilde{T}$ is inward at $(x, \ldots, x) \in D$ if and only if $Tx = T^2x$, and hence no new fixed point information can be garnered from inwardness of $\widetilde{T}$.

3.4 WEAK CONVERGENCE RESULTS

In this section, we prove generalizations to the classical theorems of Browder and Petryshyn [10] and Opial [43] from 1966 and 1967, respectively, regarding weak convergence of iterates of a nonexpansive map to a fixed point in the presence of asymptotic regularity and Opial's property.

Recall that, for any subset C of a Banach space X and any $x \in C$, a mapping $T : C \to C$ is called *asymptotically regular at* x if

$$\lim_{n\to\infty} \left\| T^n x - T^{n+1} x \right\| = 0.$$

If T is asymptotically regular for all $x \in C$, we say T is *asymptotically regular*. Let us state the main theorem of this section regarding (α_1, α_2)-nonexpansive mappings in Opial spaces. The proofs of the following theorem and lemmas will follow.

Theorem 3.4.1. *Suppose $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$ is weakly compact, convex, and has the Opial property. Suppose further that $T : C \to C$ is (α_1, α_2)-nonexpansive and asymptotically regular at some point $x \in C$. Then $(T^n x)_n$ converges weakly to a fixed point of T.*

To ensure that this theorem is a genuine extension of the classical theorems for nonexpansive maps, note that Example 3.0.1 is a mean nonexpansive map defined on $(\ell^2, \|\cdot\|_2)$ for which none of its iterates are nonexpansive, and this map is asymptotically regular. Moreover, since ℓ^2 is Hilbert, it is Opial.

Before proving the theorem, let's state some preliminary definitions and results. For any $x \in C$, let

$$\omega_w(x) := \{y \in C : y \text{ is a weak subsequential limit of } (T^n x)_n\}$$

and note that if C is weakly compact, then $\omega_w(x) \neq \emptyset$. Further note that if $I - T$ is demiclosed at 0 and asymptotically regular at x, then $\emptyset \neq \omega_w(x) \subseteq F(T)$. We have a lemma.

Lemma 3.4.1. *Suppose C is weakly compact and convex with the Opial property, and suppose that $T : C \to C$ is (α_1, α_2)-nonexpansive and asymptotically regular at some $x \in C$. Then for all $y \in \omega_w(x)$, $\lim_n \|T^n x - y\|$ exists.*

Our theorem will be proved if we can show that $\omega_w(x)$ is a singleton. This follows from the fact that C is Opial and the knowledge that $(\|T^n x - y\|)_n$ converges for all $y \in \omega_w(x)$, as summarized in the following lemma.

Lemma 3.4.2. *If $C \subseteq X$ is Opial, $T : C \to C$ is a function, and for some $x \in C$, $\lim_n \|T^n x - y\|$ exists for all $y \in \omega_w(x)$, then $\omega_w(x)$ is empty or consists of a single point.*

3.4.1 Proofs

Proof of Lemma 3.4.1. C closed and convex with the Opial property implies that $I - T$ is demiclosed at 0. That is, whenever $(z_n)_n$ is a sequence in C converging weakly to some z (which is necessarily in C since closed and convex implies weakly closed) for which $\|(I - T)z_n\| \to_n 0$, it follows that $(I - T)z = 0$.

By the asymptotic regularity of T at x, we have that $(T^n x)_n$ is an approximate fixed point sequence for T.

Since $y \in \omega_w(x)$ and $I - T$ is demiclosed at 0, we have that y is a fixed point of T and we see that

$$\begin{aligned}\alpha_1 \|Tx - y\| + \alpha_2 \left\|T^2 x - y\right\| &= \alpha_1 \|Tx - Ty\| + \alpha_2 \left\|T^2 x - T^2 y\right\| \\ &\leq \|x - y\| .\end{aligned}$$

Hence, at least one of $\|Tx - y\|$ or $\|T^2 x - y\|$ must be less than or equal to $\|x - y\|$. Let $k_1 \in \{1, 2\}$ be such that $\left\|T^{k_1} x - y\right\| \leq \|x - y\|$.

Next, we know that

$$\begin{aligned}\alpha_1 \left\|T^{k_1+1} x - y\right\| + \alpha_2 \left\|T^{k_1+2} x - y\right\| &= \alpha_1 \left\|T^{k_1+1} x - T^{k_1+1} y\right\| + \alpha_2 \left\|T^{k_1+2} x - T^{k_1+2} y\right\| \\ &\leq \left\|T^{k_1} x - T^{k_1} y\right\| \\ &= \left\|T^{k_1} x - y\right\|\end{aligned}$$

and so one of $\left\|T^{k_1+1} x - y\right\|$ or $\left\|T^{k_1+2} x - y\right\|$ must be less than or equal to $\left\|T^{k_1} x - y\right\|$. As above, let $k_2 \in \{k_1 + 1, k_1 + 2\}$ be such that $\left\|T^{k_2} x - y\right\| \leq \left\|T^{k_1} x - y\right\|$.

Inductively, build a sequence $(k_n)_n$ which satisfies

1. $k_n + 1 \leq k_{n+1} \leq k_n + 2$, and
2. $\left\|T^{k_{n+1}}x - y\right\| \leq \left\|T^{k_n}x - y\right\|$

for all $n \in \mathbb{N}$. Now $(\left\|T^{k_n}x - y\right\|)_n$ is a non-increasing sequence in $\mathbb{R}^+$, and is thus convergent to some $q \in \mathbb{R}^+$.

Consider the set $M := \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$. We have two cases. First, if M is a finite set, then the claim is proved. Second, if M is infinite, write $M = \{m_n : n \in \mathbb{N}\}$, where $(m_n)_n$ is strictly increasing. Note that, by property (1) of the sequence $(k_n)_n$ above, we must have that for all $n \in \mathbb{N}$, there exists a $j_n \in \mathbb{N}$ for which

$$m_n = k_{j_n} + 1.$$

Also, $(j_n)_n$ is strictly increasing. Asymptotic regularity of T at x and the fact that $\lim_n \left\|T^{k_n}x - y\right\| = q$ gives us that for any $\varepsilon > 0$, there is n large enough such that

1. $\|T^{m_n}x - T^{m_n-1}x\| < \varepsilon/2$, and
2. $\left|\ \left\|T^{k_{j_n}}x - y\right\| - q\ \right| < \varepsilon/2$.

Thus,

$$\begin{aligned}
\|T^{m_n}x - y\| - q &\leq \left\|T^{m_n}x - T^{m_n-1}x\right\| + \left\|T^{m_n-1}x - y\right\| - q \\
&= \left\|T^{m_n}x - T^{m_n-1}x\right\| + \left\|T^{k_{j_n}}x - y\right\| - q \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Entirely similarly, we have that

$$\begin{aligned}
\|T^{m_n}x - y\| - q &\geq -\left\|T^{m_n}x - T^{m_n-1}x\right\| + \left\|T^{m_n-1}x - y\right\| - q \\
&= -\left\|T^{m_n}x - T^{m_n-1}x\right\| + \left\|T^{k_{j_n}}x - y\right\| - q \\
&> -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon.
\end{aligned}$$

Hence, $\left|\ \|T^{m_n}x - y\| - q\ \right| < \varepsilon$ for n large enough. Since $\{m_n : n \in \mathbb{N}\} \cup \{k_n : n \in \mathbb{N}\} = \mathbb{N}$, we have finally that $\lim_n \|T^n x - y\|$ exists for any $y \in \omega_w(x)$. □

Remark 3.4.1. The above argument presented in the proof above actually works for any $y \in F(T)$, but in particular for $y \in \omega_w(x)$. This will be of use to us in Theorem 3.4.3.

Proof of Lemma 3.4.2. Suppose for a contradiction that z and y are distinct elements of $\omega_w(x)$. Then there exist $(n_k)_k$ and $(m_k)_k$ for which $T^{n_k}x \rightharpoonup_k z$ and $T^{m_k}x \rightharpoonup_k y$. Thus, using the fact that C is Opial, we have

$$\begin{aligned}
\lim_n \|T^n x - y\| &= \lim_k \|T^{m_k} x - y\| \\
&< \lim_k \|T^{m_k} x - z\| \\
&= \lim_n \|T^n x - z\| \\
&= \lim_k \|T^{n_k} x - z\| \\
&< \lim_k \|T^{n_k} x - y\| \\
&= \lim_n \|T^n x - y\|,
\end{aligned}$$

which is a contradiction. Thus, $\omega_w(x)$ is a singleton. □

Proof of Theorem 3.4.1. As stated above, let

$$\omega_w(x) := \{y \in C : y \text{ is a weak subsequential limit of } (T^n x)_n\}$$

and note that $\omega_w(x) \neq \emptyset$ since C is weakly compact, as well as that the demiclosedness of $I - T$ at 0 gives us that $\omega_w(x) \subseteq F(T)$, where $F(T)$ is the set of fixed points of T. By Lemma 3.4.2, we know that $\omega_w(x)$ consists of a single point, say y. Thus, $T^n x \rightharpoonup_n y$, and the theorem is proved. □

3.4.2 Results for arbitrary α

We have the corresponding theorem for α of arbitrary length.

Theorem 3.4.2. *If $C \subseteq X$ is weakly compact, convex, and has the Opial property, $T : C \to C$ is α-nonexpansive and asymptotically regular at some point $x \in C$, then $T^n x$ converges weakly to a fixed point of T.*

The theorem will follow immediately from the analogous lemma concerning convergence of the sequence $(\|T^n x - y\|)_n$ for any $y \in \omega_w(x)$.

Lemma 3.4.3. *Suppose C is weakly compact and convex with the Opial property, and suppose that $T : C \to C$ is α-nonexpansive and asymptotically regular at some $x \in C$. Then for all $y \in \omega_w(x)$, $\lim_n \|T^n x - y\|$ exists.*

Proof of the lemma. Let $\alpha = (\alpha_1, \ldots, \alpha_{n_0})$. In the same way as above, we build a sequence $(k_n)_n$ for which

1. $k_n + 1 \le k_{n+1} \le k_n + n_0$, and
2. $\left\|T^{k_{n+1}}x - y\right\| \le \left\|T^{k_n}x - y\right\|$.

Again, as above, let $M = \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$. If M is finite, we are done. If M is infinite, then write the elements of M as $(m_n)_n$, strictly increasing. Note that for all $n \in \mathbb{N}$, there exist $j_n \in \mathbb{N}$ and $i_n \in \{1, \ldots, n_0 - 1\}$ for which

$$m_n = k_{j_n} + i_n.$$

Also, $(j_n)_n$ is strictly increasing. Now, for any $\varepsilon > 0$, we can find n large enough so that

$$\left\|T^{m_n - j+1}x - T^{m_n - j}x\right\| < \frac{\varepsilon}{n_0} \quad \text{for all } j = 1, \ldots, n_0 - 1, \text{ and}$$

$$\left| \left\|T^{k_{j_n}}x - y\right\| - q \right| < \frac{\varepsilon}{n_0}, \quad \text{where } q = \lim_{n \to \infty} \left\|T^{k_n}x - y\right\|.$$

Thus, for n large, we have

$$\begin{aligned}
\|T^{m_n}x - y\| - q &\le \left\|T^{m_n}x - T^{m_n-1}x\right\| + \cdots + \left\|T^{m_n-i_n+1}x - T^{m_n-i_n}x\right\| + \left\|T^{m_n-i_n}x - y\right\| - q \\
&= \left\|T^{m_n}x - T^{m_n-1}x\right\| + \cdots + \left\|T^{m_n-i_n+1}x - T^{m_n-i_n}x\right\| + \left\|T^{k_{j_n}}x - y\right\| - q \\
&< \underbrace{\frac{\varepsilon}{n_0} + \cdots + \frac{\varepsilon}{n_0}}_{i_n \text{ times}} + \frac{\varepsilon}{n_0} \\
&\le (n_0 - 1)\frac{\varepsilon}{n_0} + \frac{\varepsilon}{n_0} = \varepsilon.
\end{aligned}$$

A similar argument proves that $\left| \|T^{m_n}x - y\| - q \right| < \varepsilon$ for n large, and the lemma is proved.

□

3.4.3 Losing boundedness of C

Similar arguments show that, under appropriate circumstances, the assumption of boundedness of C may be dropped. First, it is easy to see that there is an equivalent sequential notion of uniform convexity. That is, X is uniformly convex if and only if for every $R > 0$ and for any sequences $(u_n)_n$ and $(v_n)_n$ in X,

$$\begin{cases} \|u_n\|, \|v_n\| \leq R \text{ for all n, and} \\ \frac{1}{2}\|u_n + v_n\| \to R \end{cases} \implies \lim_n \|u_n - v_n\| = 0.$$

Using this equivalent notion of uniform convexity, Lemma 3.2.2, and the fact that the fixed point sets of mean nonexpansive self-maps of closed, convex subsets of strictly convex spaces are closed and convex, we can prove a theorem:

Theorem 3.4.3. *Suppose $(X, \|\cdot\|)$ is uniformly convex with a weakly sequentially continuous duality map and $C \subseteq X$ is closed and convex. Assume further that $T : C \to C$ is α-nonexpansive, $F(T) \neq \emptyset$, and T is asymptotically regular at some $x \in C$. Then $(T^n x)_n$ converges weakly to some $z \in F(T)$.*

The proof follows largely from the work done above and the original proof for nonexpansive mappings due to Opial [43, Theorem 1], and we present it here for completeness.

Proof. By Opial's Lemma (Lemma 3.2.2), X is uniformly convex with a weakly continuous duality map implies that X is Opial. Thus, for every $y \in F(T)$, by the proof of Lemma 3.4.1 and Remark 3.4.1, we know that $\lim_n \|T^n x - y\|$ exists. In particular, this implies that $\{T^n x : n \in \mathbb{N}\}$ is bounded. Let $\varphi : F(T) \to [0, \infty)$ be given by $\varphi(y) := \lim_n \|T^n x - y\|$. For any $r \in [0, \infty)$, consider the set

$$\begin{aligned} F_r :&= \{y \in F(T) : \varphi(y) \leq r\} \\ &= \varphi^{-1}([0, r]) \cap F(T). \end{aligned}$$

We summarize the relevant facts about F_r.

Claim 3.4.1. *The sets F_r satisfy the following four properties:*

1. *F_r is nonempty for r sufficiently large,*

2. F_r is closed, bounded, and convex for all $r \geq 0$,

3. there is a minimal r_0 for which F_{r_0} is nonempty, and

4. F_{r_0} is a singleton.

Proof of Claim 3.4.1. (1) and (2) are easy to verify.

(3) follows from the fact that each F_r is weakly compact (since X is reflexive) and $\{F_r : r \geq 0\}$ forms a nested family. Thus, if each $F_r \neq \emptyset$ for $r > t$ for some $t \geq 0$, it follows that

$$F_t = \bigcap_{r>t} F_r \neq \emptyset.$$

(4) follows from uniform convexity. Suppose $u, v \in F_{r_0}$ with $u \neq v$, and let $z := \frac{1}{2}(u+v)$. Note that $z \in F_{r_0}$ since F_{r_0} is convex. Because r_0 is minimal for which $F_{r_0} \neq \emptyset$, it follows that $\varphi(u) = r_0 = \varphi(v)$. We want to show that $\varphi(z) < r_0$. Suppose for a contradiction that $\varphi(z) = r_0$. Then

$$\lim_n \frac{1}{2} \|(T^n x - u) + (T^n x - v)\| = \lim_n \|T^n x - z\| = r_0$$

and uniform convexity implies that

$$\lim_n \|(T^n x - u) - (T^n x - v)\| = \|u - v\| = 0,$$

but $\|u - v\| > 0$. This tells us that $\varphi(z) < r_0$, which contradicts the minimality of r_0. Hence, F_{r_0} must be a singleton. This completes the proof of the claim. □

Let $F_{r_0} = \{y_0\}$. We aim to show that $T^n x \rightharpoonup y_0$. For a contradiction, suppose this is not the case. Since $\{T^n x : n \in \mathbb{N}\}$ is bounded and X is reflexive, there is some subsequence $(T^{n_k} x)_k$ converging weakly to some $y \neq y_0$. By asymptotic regularity of T and demiclosedness of $I - T$ at 0, we know that $\|(I - T)T^{n_k} x\| \to 0$ yields $Ty = y$. That is, $y \in F(T)$. Thus,

$$\begin{aligned} r_0 = \varphi(y_0) &= \lim_n \|T^n x - y_0\| \\ &= \lim_k \|T^{n_k} x - y_0\| \\ &> \lim_k \|T^{n_k} x - y\| \\ &= \lim_n \|T^n x - y\| = \varphi(y), \end{aligned}$$

which contradicts the minimality of r_0. Finally, we have that $T^n x \rightharpoonup y_0$, and the proof is complete. $\square$

Remark 3.4.2. We note here, just as Opial did, that the same result will hold in any reflexive Opial space where $F(T)$ is convex and F_{r_0} is a singleton. For example, to guarantee that $F(T)$ is convex for a mean nonexpansive map, we need only assume strict convexity of X as opposed to uniform convexity.

3.5 MISCELLANEOUS QUESTIONS

3.5.1 T "almost" commutes with T_α

The observation essential to the proof of Theorem 3.1.1 gives a bit more than advertised. In particular, we know that, if C is a closed, bounded, convex subset of a uniformly convex Banach space X and $T : C \to C$ is (α, p)-nonexpansive for some $p > 1$, then $\widetilde{T} : D \to C^n : (x, x, \ldots, x) \mapsto (Tx, T^2x, \ldots, T^n x)$ is nonexpansive with respect to the uniformly convex norm $\|\cdot\|_{\alpha,p}$, where $D := \{(x, x, \ldots, x) : x \in C\}$. In 1981, Bruck [11] proved an interesting result about nonexpansive maps defined on closed, bounded, convex subsets of uniformly convex Banach spaces.

Theorem 3.5.1 (Bruck)**.** *If $(X, \|\cdot\|)$ is uniformly convex, $C \subset X$ is closed, bounded, and convex, and $F : C \to X$ is nonexpansive, then there exists a continuous, convex, and strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that, for any $u_1, \ldots, u_n \in C$ and $t_1, \ldots, t_n \geq 0$ with $t_1 + \cdots + t_n = 1$, we have*

$$g\left(\left\|F\left(\sum_{j=1}^{n} t_j u_j\right) - \sum_{j=1}^{n} t_j F u_j\right\|\right) \leq \max_{1\leq i,j\leq n} \left(\|u_i - u_j\| - \|Fu_i - Fu_j\|\right).$$

In fact, the function g may be chosen independently of the nonexpansive mapping F.

So, for our (α, p)-nonexpansive mapping $\widetilde{T}$, we know that there is some function g which is strictly increasing, continuous, convex, with $g(0) = 0$ for which

$$g\left(\left\|\widetilde{T}\left(\sum_{j=1}^{n} t_j(u_j, u_j)\right) - \sum_{j=1}^{n} t_j\widetilde{T}(u_j, u_j)\right\|_{\alpha,p}\right)$$
$$\leq \max_{1\leq i,j\leq n}\left(\|(u_i, u_i) - (u_j, u_j)\|_{\alpha,p} - \left\|\widetilde{T}(u_i, u_i) - \widetilde{T}(u_j, u_j)\right\|_{\alpha,p}\right)$$

for any $u_1, \ldots, u_n \in C$ and any $t_1, \ldots, t_n \geq 0$ for which $t_1 + \cdots + t_n = 1$.

For example, let's assume that T is $((\alpha_1, \alpha_2), p)$-nonexpansive, and let's take $t_i := \alpha_i$ and $u_i := T^i x$, $i = 1, 2$, where $x \in C$ is arbitrary. Then we would have

$$g\left(\left\|\widetilde{T}\left(\alpha_1(Tx, Tx) + \alpha_2(T^2x, T^2x)\right) - \left(\alpha_1\widetilde{T}(Tx, Tx) + \alpha_2\widetilde{T}(T^2x, T^2x)\right)\right\|_{\alpha,p}\right)$$
$$= g\left(\left\|\widetilde{T}(T_\alpha x, T_\alpha x) - (T_\alpha Tx, T_\alpha T^2x)\right\|_{\alpha,p}\right)$$
$$= g\left(\left\|\widetilde{T}\widetilde{T_\alpha}(x, x) - \widetilde{T_\alpha}\widetilde{T}(x, x)\right\|_{\alpha,p}\right)$$

on the left-hand side, and on the right-hand side,

$$\max_{1\leq i,j\leq 2}\left(\|(u_i, u_i) - (u_j, u_j)\|_{\alpha,p} - \left\|\widetilde{T}(u_i, u_i) - \widetilde{T}(u_j, u_j)\right\|_{\alpha,p}\right)$$
$$= \left\|(Tx, Tx) - (T^2x, T^2x)\right\|_{\alpha,p} - \left\|(T^2x, T^3x) - (T^3x, T^4x)\right\|_{\alpha,p}$$
$$= \left(\alpha_1\left\|Tx - T^2x\right\|^p + \alpha_2\left\|Tx - T^2x\right\|^p\right)^{\frac{1}{p}} - \left(\alpha_1\left\|T^2x - T^3x\right\|^p + \alpha_2\left\|T^3x - T^4x\right\|^p\right)^{\frac{1}{p}}$$
$$= \left\|Tx - T^2x\right\| - \left(\alpha_1\left\|T^2x - T^3x\right\|^p + \alpha_2\left\|T^3x - T^4x\right\|^p\right)^{\frac{1}{p}}$$
$$= \frac{1}{\alpha_2}\|Tx - T_\alpha x\| - \left(\alpha_1\left\|T^2x - T^3x\right\|^p + \alpha_2\left\|T^3x - T^4x\right\|^p\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{\alpha_2}\|Tx - T_\alpha x\|.$$

Before we continue, let's make a simple observation.

Lemma 3.5.1. *If $g : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous, and convex on $[0, \infty)$, then g^{-1} is strictly increasing and concave.*

Proof. Since g is strictly increasing, we know g^{-1} exists. To see that it is strictly increasing, note that, for all $u, v \in g([0,\infty)) \subseteq [0,\infty)$, $u = g^{-1}(x)$ and $v = g^{-1}(y)$ for some $x, y \in [0,\infty)$, and

$$u < v \iff g(u) < g(v), \text{ so } g^{-1}(x) < g^{-1}(y) \iff g\left(g^{-1}(x)\right) < g\left(g^{-1}(y)\right).$$

Finally, to see that g^{-1} is concave, fix $u, v \in g([0,\infty))$ and $s, t > 0$ with $s + t = 1$. Note also that continuity of g ensures that $su + tv \in g([0,\infty))$. Then, since g^{-1} is increasing,

$$g(sg^{-1}(u) + tg^{-1}(v)) \leq su + tv \iff sg^{-1}(u) + tg^{-1}(v) \leq g^{-1}(su + tv).$$

□

In particular, this lemma tells us that

$$\left\|\widetilde{T}\widetilde{T_\alpha}(x,x) - \widetilde{T_\alpha}\widetilde{T}(x,x)\right\|_{\alpha,p} \leq g^{-1}\left(\frac{1}{\alpha_2}\|Tx - T_\alpha x\|\right),$$

and, expanding the left-hand side,

$$\alpha_1 \|TT_\alpha x - T_\alpha Tx\|^p + \alpha_2 \left\|T^2T_\alpha x - T_\alpha T^2 x\right\|^p \leq \left(g^{-1}\left(\frac{1}{\alpha_2}\|Tx - T_\alpha x\|\right)\right)^p.$$

Finally, by making another elementary estimate,

$$\|TT_\alpha x - T_\alpha Tx\| \leq \left(\frac{1}{\alpha_1}\right)^{\frac{1}{p}} g^{-1}\left(\frac{1}{\alpha_2}\|Tx - T_\alpha x\|\right) = h\left(\|Tx - T_\alpha x\|\right), \tag{3.1}$$

where $h(u) := \alpha_1^{-p^{-1}} g^{-1}\left(\alpha_2^{-1}u\right)$ is strictly increasing, $h(0) = 0$, and concave by the lemma above.

Now, we have three generalized notions of commutativity due to Jungck [31] and Pant [44, 45].

Definition 3.5.1. Let $A, B : (M, d) \to (M, d)$ be functions.

1. (Jungck) We say A, B are *compatible* if, whenever $(x_n)_n$ is a sequence in M for which

$$\lim_n A(x_n) = \lim_n B(x_n) = x$$

for some $x \in M$, it follows that

$$\lim_n d(B(A(x_n)), A(B(x_n))) = 0.$$

2. (Pant) If there exists an $R > 0$ such that, for all $x \in M$,

$$d(ABx, BAx) \leq Rd(Ax, Bx),$$

then we say A and B are R *weakly commuting.*

3. (Pant) If for all $x \in M$, there exists an $R = R_x > 0$ for which

$$d(ABx, BAx) \leq Rd(Ax, Bx),$$

we say A and B are *pointwise* R *weakly commuting.*

4. We say a point $x \in M$ is a *coincidence point* of A, B if $Ax = Bx$. If it happens that $Ax = Bx = x$, we say x is a *common fixed point* of A, B, and if $d(Ax, Bx) < \varepsilon$ for some $\varepsilon > 0$, we say x is an (ε-)*approximate coincidence point.*

There are theorems about the existence of coincidence and common fixed points of pointwise R weakly commuting mappings [44, 45] as well as theorems about the existence of approximate coincidence points [14]. For example,

Theorem 3.5.2 (Pant)**.** *Let (M, d) be any metric space (not necessarily complete). Suppose $f, g : M \to M$ are noncompatible, pointwise R weakly commuting, and suppose they satisfy*

1. $\overline{f(M)} \subseteq g(M)$,
2. $d(fx, fy) \leq kd(gx, gy)$ *for some* $k \geq 0$, *and*
3. $d(fx, f^2x) \neq \max\{d(fx, gfx), d(f^2x, gfx)\}$

whenever the right-hand side is nonzero. Then f and g have a common fixed point.

Also,

Theorem 3.5.3 (Dey, Kumar Laha, Saha)**.** *Let (M, d) be any metric space (not necessarily complete). Suppose $f, g : M \to M$ are such that*

1. $f(M) \subseteq g(M)$, *and*
2. $d(fx, fy) \leq \beta\left(d(gx, gy)\right) \cdot d(gx, gy)$ *for all* $x, y \in M$, *where* $\beta : [0, \infty) \to [0, 1)$ *is a function for which* $\beta(t_n) \to 1 \implies t_n \to 0$.

Then for all $\varepsilon > 0$, f and g have an ε-approximate coincidence point. That is,

$$\inf_{x \in M} d(fx, gx) = 0.$$

Notice that in the case of T and T_α, we would know that T has an approximate fixed point sequence if T and T_α have an ε-approximate coincidence point for all $\varepsilon > 0$, since

$$\|Tx - T_\alpha x\| = \alpha_2 \left\|Tx - T^2x\right\|,$$

so if $\|Tx - T_\alpha x\| < \varepsilon$, we would have

$$\|z - Tz\| < \alpha_2^{-1}\varepsilon, \text{ where } z = Tx.$$

Note also, by Equation 3.1, we have that T and T_α are pointwise R weakly commuting, with

$$R_x := \frac{\|Tx - T_\alpha x\|}{h(\|Tx - T_\alpha x\|)} \quad \text{if } \|Tx - T_\alpha x\| \neq 0, \quad \text{and} \quad R_x > 0 \quad \text{if } \|Tx - T_\alpha x\| = 0.$$

We have a question: *Is there any way to use this notion of commutativity and the fact that T_α is known to be nonexpansive (and hence must have an approximate fixed point sequence) to obtain an ε-approximate coincidence point for T and T_α for all $\varepsilon > 0$?*

If the answer is affirmative, then we would obtain an approximate fixed point sequence for T, and demiclosedness of $I - T$ at 0 would tell us that uniformly convex spaces have the fixed point property for mean nonexpansive mappings, regardless of the size of α_1.

3.5.2 Approximate fixed point sequences for T^2

Intuitively speaking, the restriction that $\alpha_1 \geq 1/2$ in Theorems 3.0.1 and 3.0.2 indicates that T is "close" to being nonexpansive since the majority of the weight in the average is given to the $\|Tx - Ty\|$ term, and subsequently $\|T^2x - T^2y\|$ plays a less significant role in the inequality. When $\alpha_1 \geq 1/2$, Theorem 3.0.1 guarantees the existence of an approximate fixed point sequence for T. So we have a question: *If $\alpha_2 \geq 1/2$, can we ensure the existence of an approximate fixed point sequence for T^2 rather than T? More generally, if T is $(\alpha_1, \ldots, \alpha_n)$-nonexpansive and $j \in \{1, \ldots, n\}$ is such that $\alpha_j = \max\{\alpha_k : k = 1, \ldots, n\}$, can we ensure that T^j has an approximate fixed point sequence?*

Note also that the proofs of Theorems 3.1.1 and 3.2.2 can be altered to give us demiclosedness information about $I - T^2$ in the case of an $((\alpha_1, \alpha_2), p)$-nonexpansive mapping (where $p > 1$ or $p = 1$, depending on the context). In particular, observe that

$$\begin{aligned}(\alpha_1^2 + \alpha_2) \left\|T^2x - T^2y\right\|^p + \alpha_1\alpha_2 \left\|T^3x - T^3y\right\|^p \\ &= \alpha_1(\alpha_1 \left\|T^2x - T^2y\right\|^p + \alpha_2 \left\|T^3x - T^3y\right\|^p) + \alpha_2 \left\|T^2x - T^2y\right\|^p \\ &\leq \alpha_1 \left\|Tx - Ty\right\|^p + \alpha_2 \left\|T^2x - T^2y\right\|^p \\ &\leq \left\|x - y\right\|^p .\end{aligned}$$

and it is easy to check that $\alpha_1^2 + \alpha_2 + \alpha_1\alpha_2 = 1$. Thus, if we put a new norm on X^2, say

$$\|(x, y)\|'_{\alpha,p} := \left((\alpha_1^2 + \alpha_2) \|x\|^p + \alpha_1\alpha_2 \|y\|^p\right)^{\frac{1}{p}} ,$$

then the function $\widetilde{T}\,' : C^2 \to C^2$, given by

$$\widetilde{T}\,'(x, y) := (T^2x, T^3y)$$

would again be $\|\cdot\|'_{\alpha,p}$-nonexpansive when restricted to the set $D = \{(x, x) : x \in C\}$, and Browder's (or Opial's, as the case may be) Demiclosedness Principle would tell us that $I - \widetilde{T}\,'\big|_D$ is demiclosed at 0. Using the same argument as in the proofs of Theorems 3.1.1 and 3.2.2, we would have that $I - T^2$ is demiclosed at 0 as well. Hence, the existence of an approximate fixed point sequence for T^2 would guarantee the existence of a fixed point for T^2 while, a priori, it may be the case that neither exist for the original function T.

3.5.3 Asymptotic regularity of Krasnoselkii-type iterates

As noted in Section 3.4, asymptotic regularity at a point is quite a strong tool insofar as fixed point theory is concerned. While nonexpansive mappings generally fail to be asymptotically regular at even a single point, Krasnoselkii [38] first noted in 1955 that, if X is uniformly convex, $C \subset X$ is closed bounded, convex, and $T : C \to C$ is nonexpansive with $T(C)$ contained in a norm-compact subset of C, then the sequence $(F^n x)_n$ converges strongly to a fixed point of T, where

$$F := \frac{I+T}{2}.$$

This result was substantially generalized by Browder and Petryshyn [10] in 1966.

Theorem 3.5.4 (Browder and Petryshyn)**.** *If X is uniformly convex, $C \subset X$ is closed, bounded, and convex, and $T : C \to C$ is nonexpansive, then the associated function*

$$F := \lambda I + (1-\lambda)T$$

maps C into C, is nonexpansive, has the same fixed points as T, and is asymptotically regular at every $x \in C$.

A further generalization was then provided by Kirk in 1971 [33]:

Theorem 3.5.5 (Kirk)**.** *Suppose C is a closed, bounded, convex subset of a uniformly convex Banach space X and $T : C \to C$ is nonexpansive. Let*

$$F := \alpha_0 I + \alpha_1 T + \cdots + \alpha_k T^k,$$

where $\alpha_j \geq 0$ for all j, $\alpha_1 > 0$, and $\alpha_0 + \cdots + \alpha_k = 1$. Then $F(C) \subseteq C$, F has the same fixed points as T, is nonexpansive, and asymptotically regular on C.

We have a natural question: *Supposing we have an (α, p)-nonexpansive mapping, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $p > 1$, on a closed, bounded, convex subset of a uniformly convex space, are the analogues of Browder and Petryshyn's and Kirk's theorems still valid? Perhaps it will be impossible to obtain the fully general versions of these theorems (i.e. where the coefficients on T^j are allowed to vary), but could we obtain asymptotic regularity of the function $F := \alpha_1 I + \alpha_2 T + \cdots + \alpha_n T^{n-1}$ for the prescribed values of $\alpha_1, \ldots, \alpha_n$?*

3.5.4 An ergodic theorem

In 1931, von Neumann [42] proved his famous "mean ergodic theorem" for unitary operators on Hilbert space:

Theorem 3.5.6 (von Neumann's Ergodic Theorem)**.** *Suppose H is a Hilbert space and $U : H \to H$ is a linear operator such that $U^*U = I$. Then, for all $v \in H$,*

$$\lim_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k v - Pv \right\| = 0,$$

where $U^0 := I$, and P is the orthogonal projection onto $\{x \in H : Ux = x\}$.

Rephrasing this theorem in terms of fixed point theory, we can say that the sequence

$$v_n := \frac{v + Uv + U^2 v + \cdots + U^{n-1} v}{n}$$

converges in norm to a fixed point of U. Furthermore, the sequence $(v_n)_n$ forms an approximate fixed point sequence for U, with

$$\|v_n - Uv_n\| \leq \|v_n - U(Pv)\| + \|U(Pv) - Uv_n\| = 2\|v_n - Pv\| \to 0$$

since U is an isometry and since $U(Pv) = Pv$ by definition. With this slightly different viewpoint in mind (that is, the viewpoint of building approximate fixed point sequences rather than obtaining strong convergence to fixed points), Baillon [3] extended this theorem to *nonlinear* nonexpansive mappings on Hilbert space in 1975, and this result was further improved by Bruck [11] in 1981. We state a version of Bruck's more general result here.

Theorem 3.5.7 (Bruck's Ergodic Theorem)**.** *Suppose $(X, \|\cdot\|)$ is a uniformly convex Banach space, $C \subset X$ is closed, bounded, and convex, and $T : C \to C$ is (nonlinear and) nonexpansive. Then*

$$\lim_n \|S_n x - TS_n x\| = 0$$

uniformly on C, where

$$S_n := \frac{I + T + T^2 + \cdots + T^{n-1}}{n}.$$

From Bruck's Ergodic Theorem and Browder's Demiclosedness Principle, we have a simple corollary.

Corollary 3.5.1. *If X is uniformly convex, $C \subset X$ is closed, bounded, and convex, and $T : C \to C$ is nonexpansive, then for all $x \in C$, there is a subsequence $(S_{n_k}x)_k$ for which*

$$S_{n_k}x \rightharpoonup x_0,$$

where x_0 is a fixed point of T in C.

There is a natural question that arises from this: *Can we prove a version of Bruck's Ergodic Theorem for mean nonexpansive mappings? Would a stronger summability method (e.g. Abel or Borel) be required to guarantee convergence of $(T^n)_{n\geq 0}$?*

4.0 EQUIVALENT RENORMINGS OF C_0

Recall the definition of the Banach space of real-valued, convergent-to-zero sequences,

$$c_0 := \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}, \text{ and } \lim_n |x_n| = 0\},$$

with its usual norm given by $\|x\|_\infty := \sup_n |x_n|$.

As we saw in Example 1.0.1, $(c_0, \|\cdot\|_\infty)$ fails to have the fixed point property for nonexpansive maps. Despite this fact, Maurey [36, p. 194] was able to prove the following deep result about $(c_0, \|\cdot\|_\infty)$.

Theorem 4.0.1 (Maurey)**.** *$(c_0, \|\cdot\|_\infty)$ has w-fpp(ne). That is, for any weakly compact, convex set $C \subset c_0$, every nonexpansive mapping $T : C \to C$ has a fixed point.*

This result was improved by Dowling, Lennard, and Turett [16] when, in 2004, they proved that the converse of Maurey's theorem also holds.

Theorem 4.0.2 (Dowling, Lennard, Turett)**.** *A closed, bounded, convex subset of $(c_0, \|\cdot\|)$ has the fixed point property for nonexpansive maps if and only if it is weakly compact.*

It is also known that the space of real-valued, convergent sequences $(c, \|\cdot\|_\infty)$ has w-fpp(ne), but, interestingly, it was shown in 2015 by Lennard, Popescu, and the present author [21] that weak compactness is *not* equivalent to the fixed point property for nonexpansive maps in c. That is, we found an example of a closed, bounded, convex subset of c which is not weakly compact, but which is *hyperconvex*. By the theorems of Sine [49] and Soardi [50], such a set must have fpp(ne). From this study, we then found an equivalent renorming of c_0 and a closed, bounded, convex (but non-weakly compact) subset of c_0 which was hyperconvex and therefore has fpp(ne). The set in question is given by

$$K := \{x \in c_0 : \text{ each } x_k \geq 0, x_1 + x_2 \leq 1, \text{ and } x_2 \geq x_3 \geq x_4 \geq \cdots\},$$

and the equivalent norm on c_0 is given by

$$\|x\| := \sup_k |x_{k+1} + x_1|.$$

This was the first known example of such a set in an equivalent renorming of c_0. For more details, see [21, Theorem 5.1]

We have one more preliminary notion and result, both of which are due to Dowling, Lennard, and Turett [15].

Definition 4.0.1 (a.i. c_0)**.** Let $(X, \|\cdot\|)$ be a Banach space. We say X contains an *asymptotically isometric copy of* c_0 ("a.i. c_0" for short) if there exists a sequence $(x_n)_n$ in X and a sequence $(\varepsilon_n)_n$ in $(0, 1)$ decreasing to 0 such that for all $t \in c_0$, we have

$$\sup_n (1 - \varepsilon_n)|t_n| \leq \left\| \sum_{k=1}^{\infty} t_k x_k \right\| \leq \sup_n (1 + \varepsilon_n)|t_n|$$

Theorem 4.0.3 (Dowling, Lennard, Turett)**.** *If X contains an asymptotically isometric copy of c_0, then X fails fpp(ne).*

As the authors noted in [15], a consequence of this theorem is that *every* equivalent renorming of both $(\ell^\infty, \|\cdot\|_\infty)$ and $(c_0(\Gamma), \|\cdot\|)$ (for Γ uncountable) fails fpp(ne).

4.1 A NEW NORM ON C_0

In this chapter, we will be concerned with studying an entirely different class of renormings of c_0 from the one mentioned above. In particular, given two real, positive, summable sequences $(\lambda_n)_n$ and $(\kappa_n)_n$, let's define a new norm on ℓ^∞ by

$$\|x\| := \sum_{k=2}^{\infty} \lambda_k \sup_{1 \leq j \leq k-1} |x_j| \; + \; \sum_{k=1}^{\infty} \kappa_k \sup_{j \geq k} |x_j|.$$

First, note that $\|\cdot\|$ is Lipschitz-equivalent to $\|\cdot\|_\infty$, since

$$\begin{aligned}\kappa_1 \|x\|_\infty &\le \sum_{k=1}^{\infty} \kappa_k \sup_{j\ge k} |x_j| \\ &\le \|x\| \\ &\le \left(\sum_{k=2}^{\infty}\lambda_k + \sum_{k=1}^{\infty}\kappa_k\right)\|x\|_\infty\end{aligned}$$

for all $x \in \ell^\infty$ (or, in particular, for all $x \in c_0$).

As a preliminary step to determining whether or not $\|\cdot\|$ has the fixed point property, we check if the usual example of a fixed-point-free nonexpansive mapping on a closed, bounded, convex subset of c_0 is nonexpansive with respect to $\|\cdot\|$.

Example 4.1.1. Let $C := \{x \in c_0 : 0 \le x_k \le 1 \text{ for all } k\}$, and let $T : C \to C$ be given by

$$Tx = T(x_1, x_2, x_3, \ldots) := (1, x_1, x_2, x_3, \ldots).$$

Given two real numbers a, b, denote $\max\{a, b\}$ as $a \vee b$. Then, for any $x, y \in C$,

$$\begin{aligned}\|Tx - Ty\| &= \|(0, x_1 - y_1, x_2 - y_2, x_3 - y_3, \ldots)\| \\ &= \left(\lambda_2 \cdot 0 + \sum_{k=3}^{\infty}\lambda_k \sup_{2\le j\le k-1} |x_{j-1} - y_{j-1}|\right) \\ &\quad + \left(\kappa_1 \|x-y\|_\infty + \sum_{k=2}^{\infty}\kappa_k \sup_{j\ge k}|x_{j-1} - y_{j-1}|\right) \\ &= \sum_{k=3}^{\infty}\lambda_k \sup_{2\le j\le k-1}|x_{j-1}-y_{j-1}| + \kappa_1\|x-y\|_\infty + \sum_{k=2}^{\infty}\kappa_k \sup_{j\ge k}|x_{j-1}-y_{j-1}| \\ &= \lambda_3|x_1 - y_1| + \lambda_4(|x_1-y_1| \vee |x_2 - y_2|) + \cdots \\ &\quad + (\kappa_1 + \kappa_2)\|x-y\|_\infty + \kappa_3 \sup_{k\ge 2}|x_k - y_k| + \kappa_4 \sup_{k\ge 3}|x_k - y_k| + \cdots ,\end{aligned}$$

and

$$\begin{aligned}\|x - y\| &= \sum_{k=2}^{\infty}\lambda_k \sup_{1\le j\le k-1}|x_j - y_j| + \sum_{k=1}^{\infty}\kappa_k \sup_{j\ge k}|x_j - y_j| \\ &= \lambda_2|x_1 - y_1| + \lambda_3(|x_1 - y_1| \vee |x_2 - y_2|) + \cdots \\ &\quad + \kappa_1\|x-y\|_\infty + \kappa_2 \sup_{k\ge 2}|x_k - y_k| + \kappa_3\sup_{k\ge 3}|x_k - y_k| + \cdots .\end{aligned}$$

In order to make sure that this function is not nonexpansive on C, we need only use specific values for x and y and determine which sequences $(\lambda_n)_n$ and $(\kappa_n)_n$ will force T to expand the distance between x and y. In particular, we know that the summing basis elements σ_k are all contained in C, where

$$\begin{aligned}\sigma_k &:= e_1 + e_2 + \cdots + e_k \\ &= (1,0,0,\ldots) + (0,1,0,\ldots) + \cdots + (\underbrace{0,0,\ldots,0}_{k-1},1,0,\ldots) \\ &= (\underbrace{1,1,\ldots,1}_{k},0,0,\ldots).\end{aligned}$$

So let's begin calculating using the σ_k's. First note that $T\sigma_k = \sigma_{k+1}$. Then

$$\begin{aligned}\|T\sigma_k - T\sigma_{k-1}\| - \|\sigma_k - \sigma_{k-1}\| &= \|\sigma_{k+1} - \sigma_k\| - \|\sigma_k - \sigma_{k-1}\| \\ &= \|(\underbrace{0,\ldots,0}_{k},1,0,\ldots)\| - \|\underbrace{0,\ldots,0}_{k-1},1,0,\ldots)\| \\ &= \kappa_1 + \kappa_2 + \cdots + \kappa_{k+1} + \lambda_{k+2} + \lambda_{k+3} + \cdots \\ &\quad - (\kappa_1 + \kappa_2 + \cdots + \kappa_k + \lambda_{k+1} + \lambda_{k+2} + \cdots) \\ &= \kappa_{k+1} - \lambda_{k+1},\end{aligned}$$

and hence $\|T\sigma_k - T\sigma_{k-1}\| > \|\sigma_k - \sigma_{k-1}\|$ if and only if $\kappa_{k+1} > \lambda_{k+1}$ for all $k \in \mathbb{N}$.

So, if we add the assumption that $\kappa_{k+1} > \lambda_{k+1}$ for all $k \in \mathbb{N}$, then we have that the usual right shift mapping on C is not nonexpansive with respect to $\|\cdot\|$. However, it is possible to adapt the underlying set and the right shift map T to make some iterate T^k $\|\cdot\|$-nonexpansive. It isn't especially important for the remainder of this chapter, so we will not explicitly prove it, but rather give an example to convince you that this is true.

Example 4.1.2. For any $p \in \mathbb{N}$ $(p \geq 2)$, let

$$C_p := \left\{ v = \sum_{n=1}^{\infty} t_n \sigma_{p^{n-1}} : 0 \leq t_n \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\}.$$

Note that C_p is closed, bounded, and convex for all p. Define an analogue of the right shift, $T_p : C_p \to C_p$ as usual,

$$T_p v = T_p \left(\sum_{n=1}^{\infty} t_n \sigma_{p^{n-1}} \right) := \sum_{n=1}^{\infty} t_n \sigma_{p^n}.$$

We can easily compute T_p^k,

$$T_p^k\left(\sum_{n=1}^{\infty} t_n \sigma_{p^{n-1}}\right) = \sum_{n=1}^{\infty} t_n \sigma_{p^{n-1+k}}.$$

Note that $\sigma_1 = e_1, \sigma_p \in C_p$ for all p. Consider

$$\begin{aligned}\left\|T_p^k\sigma_p - T_p^k\sigma_1\right\| - \|\sigma_p - \sigma_1\| &= \|\underbrace{0,\ldots,0}_{p^{k-1}},\underbrace{1,\ldots,1}_{p^k-p^{k-1}},0,0,\ldots\| - \|0,\underbrace{1,\ldots,1}_{p-1},0,0,\ldots)\| \\ &= \kappa_1 + \cdots + \kappa_{p^{k-1}} + \lambda_{p^k-p^{k-1}+1} + \cdots \\ &\quad - (\kappa_1 + \cdots + \kappa_p + \lambda_3 + \cdots) \\ &= \kappa_{p+1} + \cdots + \kappa_{p^{k-1}} - \left(\lambda_3 + \cdots + \lambda_{p^k-p^{k-1}}\right).\end{aligned}$$

If we want $\left\|T_p^k\sigma_p - T_p^k\sigma_1\right\| > \|\sigma_p - \sigma_1\|$ for all $k \in \mathbb{N}$, then we will require

$$\sum_{j=p+1}^{\infty} \kappa_j > \sum_{j=3}^{\infty} \lambda_j,$$

but since $(\kappa_n)_n$ is summable, we have that

$$\lim_{p\to\infty} \sum_{j=p+1}^{\infty} \kappa_j = 0,$$

which means that, for some p large enough,

$$\sum_{j=p+1}^{\infty} \kappa_j \leq \sum_{j=3}^{\infty} \lambda_j$$

So $(c_0, \|\cdot\|)$ fails to have the fixed point property for nonexpansive mappings, but, interestingly, $(c_0, \|\cdot\|)$ turns out to contain no asymptotically isometric copies of c_0 as we see in the next section.

4.2 A CLASS OF NORMS FAILING TO CONTAIN A.I. C_0'S

Dowling, Lennard, and Turett [15] gave an example of an equivalent renorming of $(\ell^\infty, \|\cdot\|_\infty)$ (and hence of $(c_0, \|\cdot\|_\infty)$) which contains no asymptotically isometric copies of c_0. Recall that, given a Banach space $(X, \|\cdot\|)$ with dual space $(X^*, \|\cdot\|_*)$, we say a sequence $(\varphi_n)_n$ in X^* converges *weak-** to ψ if, for all $x \in X$ we have

$$\lim_{n\to\infty} \varphi_n(x) = \psi(x).$$

Note that we will use "$\rightharpoonup^*$" to denote weak-* convergence. Also note that weak-* convergence in X^* depends on the choice of predual (e.g. both $(c, \|\cdot\|_\infty)$ and $(c_0, \|\cdot\|_\infty)$ have dual space $(\ell^1, \|\cdot\|_1)$, but $(c, \|\cdot\|_\infty)$ and $(c_0, \|\cdot\|_\infty)$ are not isometrically isomorphic Banach spaces).

Theorem 4.2.1 (Dowling, Lennard, Turett)**.** *$(\ell^\infty, \|\cdot\|)$ contains no a.i. c_0's, where*

$$\|x\| := \|x\|_\infty + \sum_{k=1}^{\infty} \frac{|x_k|}{2^k}.$$

First, we give the original proof of this theorem.

Original proof. Suppose for a contradiction that ℓ^∞ does contain an a.i. c_0 sequence, denoted $(y^{(n)})_n$. That is, there exists $(\varepsilon_n)_n$ null in $(0,1)$ such that for all $t \in c_0$,

$$\sup_n (1-\varepsilon_n)|t_n| \le \left\| \sum_{n=1}^{\infty} t_n y^{(n)} \right\|_\sim \le \sup_n |t_n|.$$

Since $1-\varepsilon_1 \le \left\|y^{(1)}\right\|_\sim$, we have that there is some $\ell \in \mathbb{N}$ minimal such that $y^{(1)}_\ell \neq 0$. Let

$$\alpha := \frac{1}{3\cdot 2^\ell} |y^{(1)}_\ell|$$

and choose $N_1 \ge \ell$ such that $\sum_{k>N_1} 2^{-k} < \alpha/4$. Further, choose N_2 s.t. $\varepsilon_n < \alpha$ for all $n \ge N_2$.

Without loss of generality, $y^{(n)} \rightharpoonup_n^* 0$, and thus $y_k^{(n)} \to_n 0$ for all k. With this in mind, choose $N \geq N_2$ such that $|y_k^{(n)}| < \alpha/4$ for all $k = 1, \dots, N_1$ and for all $n \geq N$. Now for $n \geq N$,

$$\begin{aligned}
\|y^{(n)}\|_\sim &= \|y^{(n)}\|_\infty + \sum_{j=1}^{\infty} \frac{|y_j^{(n)}|}{2^j} \\
&\leq \|y^{(n)}\|_\infty + \sum_{j=1}^{N_1} |y_j^{(n)}| \, 2^{-j} + \sum_{j=N_1+1}^{\infty} 2^{-j}(1) \\
&< \|y^{(n)}\|_\infty + \frac{\alpha}{2}
\end{aligned}$$

By the triangle inequality, $\|y^{(n)}\|_\infty \leq \frac{1}{2}\left(\|y^{(1)} + y^{(n)}\|_\infty + \|y^{(1)} - y^{(n)}\|_\infty\right)$ for all n, so $\|y^{(1)} + \delta_n y^{(n)}\|_\infty \geq \|y^{(n)}\|_\infty$ for some $\delta_n = \pm 1$. Since $(y^{(n)})_n$ is an a.i. c_0 sequence, we have

$$\begin{aligned}
1 \geq \|y^{(1)} + \delta_n y^{(n)}\|_\sim &= \|y^{(1)} + \delta_n y^{(n)}\|_\infty + \sum_{j=1}^{\infty} \frac{|y_j^{(1)} + \delta_n y_j^{(n)}|}{2^j} \\
&\geq \|y^{(n)}\|_\infty + \frac{|y_\ell^{(1)} + \delta_n y_\ell^{(n)}|}{2^\ell} \\
&\geq \|y^{(n)}\|_\sim - \frac{\alpha}{2} + \frac{|y_\ell^{(1)}| - |y_\ell^{(n)}|}{2^\ell} \\
&\geq \|y^{(n)}\|_\sim - \alpha - \frac{|y_\ell^{(1)}|}{2^\ell} \\
&\geq 1 - \varepsilon_n - \alpha + 2^{-\ell}\,|y_\ell^{(1)}| \\
&> 1 + \alpha.
\end{aligned}$$

Contradiction. Thus, $(\ell^\infty, \|\cdot\|)$ contains no asymptotically isometric copies of c_0. $\square$

Now a streamlined proof.

New proof. Let ε_n and $y^{(n)}$ be as above. Note that $1 - \varepsilon_1 \leq \|y^{(1)}\|_\sim \implies y^{(1)} \neq 0$. We still have that $\|y^{(n)}\|_\infty \leq \|y^{(1)} + \delta_n y^{(n)}\|_\infty$ for some $\delta_n = \pm 1$, and since $(y^{(n)})_n$ forms an a.i. c_0

basis, we also still have that $\left\|y^{(1)}+\delta_n y^{(n)}\right\|_\sim \le 1$. Now

$$\begin{aligned}
1 \ge \left\|y^{(1)}+\delta_n y^{(n)}\right\|_\sim &= \left\|y^{(1)}+\delta_n y^{(n)}\right\|_\infty + \sum_{j=1}^{\infty} \frac{|y_j^{(1)}+\delta_n y_j^{(n)}|}{2^j} \\
&\ge \left\|y^{(n)}\right\|_\infty + \sum_{j=1}^{\infty} \frac{|y_j^{(1)}+\delta_n y_j^{(n)}|}{2^j} \\
&= \left\|y^{(n)}\right\|_\sim - \sum_{j=1}^{\infty} \frac{|y_j^{(n)}|}{2^j} + \sum_{j=1}^{\infty} \frac{|y_j^{(1)}+\delta_n y_j^{(n)}|}{2^j} \\
&\ge \left\|y^{(n)}\right\|_\sim + \sum_{j=1}^{\infty} \frac{|y_j^{(1)}|}{2^j} - 2\sum_{j=1}^{\infty} \frac{|y_j^{(n)}|}{2^j} \\
&\ge 1-\varepsilon_n + \sigma\left(y^{(1)}\right) - 2\sigma\left(y^{(n)}\right) \\
&\to_n 1+\sigma\left(y^{(1)}\right) > 1,
\end{aligned}$$

which is a contradiction. Note that we define $\sigma(x) := \sum_j |x_j|/2^j$, and we implicitly use the facts that

1. $\sigma(x) > 0$ if $x \neq 0$, and
2. $\sigma(x^{(n)}) \to_n 0$ if $x_k^{(n)} \to_n 0$.

□

This leads us to a theorem for our norm on c_0 (or ℓ^∞), given by

$$\|x\| := \sum_{k=1}^{\infty} \kappa_k \sup_{j\ge k} |x_j| + \sum_{k=2}^{\infty} \lambda_k \sup_{1\le j\le k-1} |x_j|$$

where $\sum_k \kappa_k, \sum_k \lambda_k < \infty$ and all $\kappa_k, \lambda_k > 0$ except $\lambda_1 := 0$ since it doesn't appear in the norm. Let

$$\eta(x) := \sum_{k=1}^{\infty} \kappa_k \sup_{j\ge k} |x_j| \qquad \textit{and} \qquad \sigma(x) := \sum_{k=2}^{\infty} \lambda_k \sup_{1\le j\le k-1} |x_j|$$

Note:

1. σ and η are both seminorms (indeed they are both *norms*).
2. $\|x\| = \eta(x) + \sigma(x)$.
3. $\sigma(x) > 0$ if $x \neq 0$.
4. $\sigma(x^{(n)}) \to_n 0$ if $x_k^{(n)} \to_n 0$.

We have a theorem.

Theorem 4.2.2. $(\ell^\infty, \|\cdot\|)$ *contains no a.i.* c_0*'s.*

Proof. For a contradiction, let ε_n and $y^{(n)}$ be as above. Since η satisfies the triangle inequality, we still have that $\eta\left(y^{(n)}\right) \leq \eta\left(y^{(1)} + \delta_n y^{(n)}\right)$ for some $\delta_n = \pm 1$. Furthermore, we know that $1 - \varepsilon_1 \leq \left\|y^{(1)}\right\| \implies y^{(1)} \neq 0 \implies \sigma\left(y^{(1)}\right) > 0$, and that $\sigma\left(y^{(n)}\right) \to_n 0$ since $y^{(n)} \rightharpoonup_n^* 0 \implies y_k^{(n)} \to_n 0$ for all k. Again, as above, we also know that $\left\|y^{(1)} + \delta_n y^{(n)}\right\| \leq 1$. Now, very similarly to the above proof, we see that

$$\begin{aligned}
1 \geq \left\|y^{(1)} + \delta_n y^{(n)}\right\| &= \eta\left(y^{(1)} + \delta_n y^{(n)}\right) + \sigma\left(y^{(1)} + \delta_n y^{(n)}\right) \\
&\geq \eta\left(y^{(n)}\right) + \sigma\left(y^{(1)} + \delta_n y^{(n)}\right) \\
&= \left\|y^{(n)}\right\| - \sigma\left(y^{(n)}\right) + \sigma\left(y^{(1)} + \delta_n y^{(n)}\right) \\
&\geq \left\|y^{(n)}\right\| - 2\sigma\left(y^{(n)}\right) + \sigma\left(y^{(1)}\right) \\
&\geq 1 - \varepsilon_n - 2\sigma\left(y^{(n)}\right) + \sigma\left(y^{(1)}\right) \\
&\to_n 1 + \sigma\left(y^{(1)}\right) > 1
\end{aligned}$$

which is a contradiction. □

This proof generalizes quite easily to a class of norms on ℓ^∞. Say a norm $\|\cdot\|$ on ℓ^∞ is *dissociative* if it can be written as $\|x\| = \eta(x) + \sigma(x)$, where

1. η and σ satisfy the triangle inequality,
2. $\sigma(x) > 0$ if $x \neq 0$, and $\eta(x), \sigma(x) \geq 0$ for all x,
3. $\sigma\left(x^{(n)}\right) \to_n 0$ if $x_k^{(n)} \to_n 0$ for all k.

Following the last proof almost exactly yields the following theorem.

Theorem 4.2.3. *If* $\|\cdot\|$ *is a dissociative norm on* ℓ^∞*, then* $(\ell^\infty, \|\cdot\|)$ *contains no a.i. copies of* c_0*.*

Proof. Suppose for a contradiction that $(\ell^\infty, \|\cdot\|)$ contains an a.i. c_0 sequence $(y^{(n)})_n$ (corresponding to a null sequence $(e_n)_n$ in $(0,1)$). Without loss of generality, $y^{(n)} \rightharpoonup_n^* 0$, so $y_k^{(n)} \to_n 0$ for all k and $\sigma(y^{(n)}) \to_n 0$. We know that $1 - \varepsilon_1 \leq \left\|y^{(1)}\right\| \implies y^{(1)} \neq 0$, so $\sigma(y^{(1)}) > 0$. By the triangle inequality, $\eta(y^{(n)}) \leq \frac{1}{2}\left(\eta(y^{(1)} + y^{(n)}) + \eta(y^{(1)} - y^{(n)})\right)$ for all n, so $\eta(y^{(n)}) \leq \eta(y^{(1)} + \delta_n y^{(n)})$ for some $\delta_n = \pm 1$. Finally, we know that $\left\|y^{(1)} + \delta_n y^{(n)}\right\| \leq 1$, and $1 - \varepsilon_n \leq \left\|y^{(n)}\right\|$ for all n. Hence,

$$\begin{aligned} 1 \geq \left\|y^{(1)} + \delta_n y^{(n)}\right\| &= \eta(y^{(1)} + \delta_n y^{(n)}) + \sigma(y^{(1)} + \delta_n y^{(n)}) \\ &\geq \eta(y^{(n)}) + \sigma(y^{(1)} + \delta_n y^{(n)}) \\ &= \left\|y^{(n)}\right\| - \sigma(y^{(n)}) + \sigma(y^{(1)} + \delta_n y^{(n)}) \\ &\geq \left\|y^{(n)}\right\| - \sigma(y^{(n)}) - \sigma(\delta_n y^{(n)}) + \sigma(y^{(1)}) \\ &\geq 1 - \varepsilon_n - \sigma(y^{(n)}) - \sigma(\delta_n y^{(n)}) + \sigma(y^{(1)}) \\ &\to_n 1 + \sigma(y^{(1)}) > 1, \end{aligned}$$

which is a contradiction. □

At a glance, we see that $\eta \equiv 0$ satisfies all of the above requirements. That is, $\|\cdot\|$ is a dissociative norm on ℓ^∞ with $\eta \equiv 0$ if and only if $\left\|y^{(n)}\right\| \to_n 0$ whenever $y_k^{(n)} \to_n 0$ for all k, and we have an easy corollary:

Corollary 4.2.1. *Suppose $\|\cdot\|$ is a norm on ℓ^∞ such that $\left\|y^{(n)}\right\| \to_n 0$ whenever $y_k^{(n)} \to_n 0$ for all k. Then $(\ell^\infty, \|\cdot\|)$ contains no a.i. copies of c_0.*

Note that, even though they do the job, we don't need any of the aforementioned techniques to prove this corollary.

Direct proof of the corollary. If $(y^{(n)})_n$ is an a.i. c_0 sequence, then, without loss of generality, $y_k^{(n)} \to_n 0$ for all k, and by our only assumption on the norm we know that $\left\|y^{(n)}\right\| \to_n 0$. But since the y's form an a.i. c_0 sequence, we know that $1 - \varepsilon_n \leq \left\|y^{(n)}\right\| \leq 1$ for all $n \implies \left\|y^{(n)}\right\| \to_n 1$. Contradiction. □

Of course this begs the question *"can such an equivalent norm exist on ℓ^∞?"* The answer to this question is definitively **no**. Note that there are a few ways to prove this, but, in particular, such a norm would be a Schur norm, and ℓ^∞ is not a Schur space (since $c_0 \leq \ell^\infty$ and c_0 is not Schur). This could let us say a couple things.

1. First, perhaps we should call σ the "Schur part of $\|\cdot\|$."

2. Second, if $\|\cdot\|$ is a dissociative norm on ℓ^∞ equivalent to $\|\cdot\|_\infty$, then $\eta \not\equiv 0$. That is, there exists $x \in \ell^\infty$ for which $\eta(x) \neq 0$.

Of course it is still possible for a space to fail fpp(ne) despite containing no a.i. copies of c_0, so we have a few questions:

1. *Do any dissociative norms have the fixed point property?*
2. *"How many" (in the sense of category or porosity) norms can be seen as dissociative?*
3. *What properties do the dual norms have in ℓ^1?*
4. *What additional conditions can we place on η and σ so that, for instance, $(c_0, \|\cdot\|)$ has the weak fixed point property?*

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