GRAPHS WITH NO GRID OBSTACLE REPRESENTATION

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Abstract

A graph $G = (V, E)$ admits a *grid obstacle representation*, if there exist a subset Ω of the planar integer grid $\mathbb{Z}^2$ and an embedding $f : V \to \mathbb{Z}^2$ such that no vertex of G is mapped into a point of Ω, and two vertices $u, v \in V$ are connected by an edge of G if and only if there is a shortest path along the edges of $\mathbb{Z}^2$ that connects $f(u)$ and $f(v)$ and avoids all other elements of $\Omega \cup f(V)$. We answer a question of Bishnu, Ghosh, Mathew, Mishra, and Paul, by showing that there exist graphs that do not admit a grid obstacle representation.

1 Introduction

Let $\mathbb{Z}^2$ be the infinite grid in the plane, that is, a *graph* whose vertex set consists of all points with integer coordinates, and two vertices, (i, j) and (i', j'), are connected by an edge if and only if $|i - i'| + |j - j'| = 1$. The *(Manhattan) distance* between two grid points is the length of the shortest path between them in the grid graph $\mathbb{Z}^2$. With a slight abuse of notation, sometimes the set of integer points will also be denoted by $\mathbb{Z}^2$.

Let $G = (V, E)$ be a graph with vertex set V and edges set E. The following notion was introduced by Bishnu, Ghosh, Mathew, Mishra, and

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Paul [BGMMP16]. A *grid obstacle representation* of G consists of a set Ω of integer points, called *obstacles* and an embedding f of V into the set of integer points so that $uv \in E$ *if and only if* $f(u)$ and $f(v)$ can be connected by a shortest path in the graph $\mathbb{Z}^2$ that avoids all elements of Ω and all points representing vertices of G other than $f(u)$ and $f(v)$. Such a path will be called an *obstacle avoiding shortest path*.

A similar notion with respect to the whole Euclidean plane equipped with the Euclidean distance was studied in [AKL10], [BCV15], [MPP12], [PS11].

Bishnu, Ghosh *et al.* [BGMMP16] raised a number of interesting questions concerning grid obstacle representations. The first and most basic question they asked was whether every finite graph admits such a representation. They formulated the same question for bipartite graphs. Our following theorem provides a negative answers to both of these questions.

Theorem. *There is a bipartite graph that admits no grid obstacle representation.*

It is easy to see that every complete graph (and every complete bipartite graph) admits a grid obstacle representation. Hence, our Theorem immediately implies that the property that G admits such a representation is not *monotone*. In particular, there exists a (bipartite) graph $G = (V, E)$ and an edge $e \in E$ such that G admits a grid obstacle representations, but the graph obtained from G be deleting e does not. This answers Problem 6 in [BGMMP16].

2 Proof of Theorem

According to a well known construction of Reiman and Kővári-Sós-Turán, there exist bipartite graphs with n vertices and roughly $(n/2)^{3/2}$ edges that are C_4*-free*, i.e., that contain no cycle of length 4. This bound is asymptotically tight, as n tends to infinity.

Lemma 1. [KST54, R58] *For every n, there exists a C_4-free bipartite graph with n vertices and more than $n^{3/2}/4$ edges.* $\square$

Let $H = (V, E)$ be a graph with the properties guaranteed in Lemma 1. We are going to show that H does not admit a grid obstacle representation, provided that $|V| = n$ is sufficiently large. Suppose, for contradiction that (Ω, f) is such a representation, for a suitable set of obstacles $\Omega \subset \mathbb{Z}^2$ and an embedding $f : V \to \mathbb{Z}^2$. For any edge $uv \in E$, let $f(uv) = f(vu)$

denote an arbitrarily *fixed* obstacle avoiding shortest path between $f(u)$ and $f(v)$, that is, a shortest path that does not meet $\Omega \cup \{f(w) : w \neq u, v\}$. The x-coordinate and the y-coordinate of a point $p \in \mathbb{Z}^2$ will be denoted by $x(p)$ and $y(p)$, respectively.

Traveling along an obstacle avoiding shortest path $f(uv) = f(vu)$ from left to right, if this expression makes sense, that is, if $y(f(u)) \neq y(f(v))$, the path will have at least one of the following two types.

Type U: At each step it goes *upwards* or to the *right*.

Type D: At each step it goes *downwards* or to the *right*.

If $f(uv)$ is a *horizontal* segment, it has both types.

If $y(f(u)) = y(f(v))$, that is, $f(uv)$ is a *vertical* segment, then traversing it in one direction it has type U, and traversing it in the opposite direction it has type D. Therefore, vertical paths also have both types. Apart from horizontal and vertical paths, all obstacle avoiding shortest paths have a unique type.

A graph G drawn in the plane so that its vertices and edges are represented by points and continuous arcs between the corresponding point pairs, respectively, is called a *topological graph* and is denoted by $T = T(G)$. The (abstract) graph G is said to be the *underlying graph* of T. It is always assumed that an arc representing an edge uv of G does not pass through any point representing a vertex other than u and v. See [P03], for a survey.

We need the following well known result that can be regarded as a simple special case of the Hanani-Tutte theorem [H34, T70].

Lemma 2. *If a topological graph has no two edges with distinct endpoints that have a point in common, then its underlying graph is planar.* $\square$

We may assume, by symmetry, that for at least $|E|/2$ edges $uv \in E$, the paths $f(uv)$ are of type U. Let T denote the topological graph on the vertex set $\{f(v) : v \in V\}$ whose edge set consists of all paths $f(uv)$ that are of type U. Of course, T, as well as H, is bipartite and C_4-free.

In order to apply Lemma 2 to T, we need to verify

Lemma 3. *T contains no two vertex-disjoint edges that share a point.*

Proof. Suppose for contradiction that T has two vertex-disjoint edges, $f(uv)$ and $f(u'v')$, that intersect at a grid point $p \in \mathbb{Z}^2$. Suppose without loss of generality that $x(u) \leq x(v)$, $y(u) \leq y(v)$, $x(u') \leq x(v')$, and $y(u') \leq y(v')$. Traversing the arc $f(uv)$ from $f(u)$ towards $f(v)$ up to its intersection point p, and then switching to the path $f(u'v')$ and following it all the way to $f(v')$, we obtain an obstacle avoiding shortest path of type U connecting $f(u)$ and $f(v')$. By symmetry, $f(u')$ and $f(v)$ can also be connected with an obstacle avoiding shortest path of type U. Therefore, $f(u)f(v')f(u')f(v)$ is a C_4 in T, contradicting the definition. $\square$

By Lemma 1, T has at least $|E|/2 > n^{3/2}/8$ edges. If H admits a grid obstacle representation, then combining Lemmas 2 and 3, we obtain that (the underlying abstract graph of) T is planar. A bipartite planar graph with $n \geq 4$ vertices has at most $2n - 4$ edges. Thus, as long as $n^{3/2}/8 \geq 2n - 4$, no graph H with n vertices satisfying the conditions in Lemma 1 admits a grid obstacle representation. $\square$ $\square$

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