A survey on impulsive dynamical systems

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Abstract. In this survey we provide an introduction to the theory of impulsive dynamical systems in both the autonomous and nonautonomous cases. In the former, we will show two different approaches which have been proposed to analyze such kind of dynamical systems which can experience some abrupt changes (impulses) in their evolution. But, unlike the autonomous framework, the nonautonomous one is being developed right now and some progress is being obtained over the recent years. We will provide some results on how the theory of autonomous impulsive dynamical systems can be extended to cover such nonautonomous situations, which are more often to occur in the real world.

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1 Introduction

The theory of impulsive differential equations (IDE, for short) describes the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. These systems are modeled by differential equations which describe the period of continuous variation of state and conditions which describe the discontinuities of first kind of the solution or of its derivatives at the moments of impulses. Many real world problems can experience abrupt external forces which can change completely their dynamics. For instance, an example of a real world problem that can be represented by an impulsive differential equation is a medicine intake, where the user must take regular doses of the medicine, which causes abrupt changes in the amount of medicine in their body, to control the disease or making it disappear. Examples that model real world problems in science and technology can be found

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in [1, 13, 19, 20]. The reader is also referred to [2, 3, 26] to obtain more details about the theory of IDEs, for instance, results concerning existence and uniqueness of solutions, dependence of solutions on initial values, variation of parameters, oscillation and stability.

As pointed out in [2, 26] there exist different kinds of impulses, for instance, systems with impulses at fixed times and systems with impulses at variable times. Impulses that vary in time are more attractive due to their complexity, applicability in real world problems, and, moreover, the impulses may occur due to conditions on the phase space and not in time. As an example, we may cite the billiard-type system which can be modeled by differential systems with impulses acting on the first derivatives of the solutions. Indeed, the positions of the colliding balls do not change at the moments of impact (impulse), but their velocities gain finite increments (the velocity will change according to the position of the ball).

Solutions of IDEs with impulses at variable time may generate "impulsive dynamical systems" (family of piecewise continuous functions that satisfy the identity and semigroup properties), for instance, when the differential equation is autonomous. As in the theory of IDEs, the case of impulsive dynamical systems with impulses that vary in time is more difficult to handle since we do not know previously the time of impulses. However, it provides us an effective tool to describe more types of discontinuous motions.

The theory of impulsive dynamical systems is a new chapter of the theory of topological dynamical systems and it was started by Rozko in the papers [27, 28], where he introduced several notions of impulsive systems with impulses at fixed times. In the early 90's Kaul (see [24, 25]) constructed the mathematical base for this theory with impulses at variable times, and has been followed by several authors in order to develop the theory which is known up to date. For instance, we would like to mention the papers by Ciesielski (see [16–18]), where it is analyzed the continuity of the function ϕ (see 2.2) that describes "the time of reaching impulse points", and recently the works by Bonotto and his collaborators (see [6–10]) where the theory has been investigated.

Throughout this work, an impulsive dynamical system is a dynamical system that possesses impulses depending on the state (and not on the time), that is, there is a set in the phase space which is responsible by the discontinuities of the solutions of the system. It is worth mentioning that the theory presented in this work provides a different approach from the theory presented in [21], where the author carries out a study of some types of discontinuous differential equations. Roughly speaking, Filippov considers in [21] the equation $x' = f(t, x)$, where the right-hand side function is discontinuous and it is assumed to satisfy some Carathéodory conditions. Also, the solutions in this framework have to be absolutely continuous, which is another relevant detail that makes Filippov's theory different from the one presented in [2, 3, 26] and the theory presented here, where the solutions can be (and usually are) discontinuous.

We aim to provide a survey on the theory of impulsive dynamical systems in both the autonomous and nonautonomous fields. We start with the autonomous framework which has being studied over the last years and, for the first part of this paper, we will recall some results established in the paper [5]. In this work the authors propose a new approach for the impulsive autonomous theory, by considering precompact attractors and pointing out several improvements that this precompact approach provides, when comparing with the previous theory in this framework. Examples to illustrate the impulsive autonomous theory are described in [5], one of them is reproduced in this survey, at the end of the section devoted to the autonomous case (see Example 2.22).

To start off, in Section 2 we include some basic definitions from the continuous au-

tonomous dynamical systems theory in order to introduce the definition of impulsive dynamical system. In the sequel we present some technical definitions and results, known as "tube conditions", that is important in the development of this theory. Then, before presenting an impulsive autonomous example, we introduce the concept of omega limit sets, which is the key to construct the global attractor, as well as some results on the invariance and attraction in order to obtain an existence result for the global attractor.

In Section 3, we analyze the nonautonomous case, taking into account that a complete description of the results and their proofs can be found in our paper [4], while in this survey we only intend to provide the main ideas of the new theory highlighting the difficulties that one can have in dealing with this much more complicated nonautonomous situation. Needless to say that most problems in the real world are, by their own nature, nonautonomous (or even stochastic) and, when we wish to mathematically analyze them, we usually approximate those problems by some autonomous models to simplify the study. However, even being the autonomous framework very useful, and providing a great amount of results, it does not take into account the whole richness of nonautonomous problems. In [11, 12], one can find examples to illustrate how different the autonomous and nonautonomous settings can be. Mentioning again the medicine intake example, we could not expect that the action of the medicine in the user body depends only on the elapsed time but also the initial and final times must play their role in the evolution of the system.

We follow the same structure than in the autonomous part, by starting with a brief introduction on the continuous nonautonomous dynamical systems in order to define the impulsive nonautonomous dynamical systems. We also state a result (see Theorem 3.9) that is important to transfer properties from the impulsive skew-product semiflow (autonomous) to the impulsive nonautonomous dynamical system. Next we present the nonautonomous version of the "tube conditions" and some convergence properties, which are more general than the first ones because take into account a second variable (the fibers). Then we define the notion of impulsive cocycle attractor and impulsive pullback omega limit, and also present some results about invariance and attraction. We would like to mention that the definition of impulsive pullback omega limit set introduced in [4] is a little different from the previous one, and this difference appears naturally when we start developing the impulsive nonautonomous theory, since in the impulsive scenario, the convergence results are obtained with some "correction times" (see Proposition 3.12). To conclude, we present, under suitable conditions, a result on the existence of impulsive cocycle attractor for an impulsive nonautonomous dynamical system and an example, borrowed from [4, Section 7], where a nonautonomous 2D-Navier–Stokes equation under impulses conditions is considered.

Finally, some conclusions, comments and future lines of research are included in Section 4.

2   Impulsive dynamical systems

To introduce the theory of impulsive dynamical system, we first recall, very briefly, the theory of *continuous autonomous dynamical systems* (or simply, *semigroups*).

Let (X, d) be a metric space and $\mathbb{R}_+$ be the set of nonnegative real numbers. A **semigroup** in X is a family of mappings $\{\pi(t)\colon t \geqslant 0\}$, indexed on $\mathbb{R}_+$, satisfying

  (i) $\pi(0)x = x$ for all $x \in X$;

 (ii) $\pi(t + s) = \pi(t)\pi(s)$ for all $t, s \geqslant 0$;

(iii) the map $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x$ is continuous.

A set $A \subset X$ is called π-**invariant** under $\{\pi(t) \colon t \geqslant 0\}$ if $\pi(t)A = A$ for all $t \geqslant 0$. Also A is π-**positively (negatively) invariant** if $\pi(t)A \subseteq A$ ($\pi(t)A \supseteq A$), for all $t \geqslant 0$.

Given two subsets $A, B \subseteq X$, we say that A π-**attracts** B if

$$\lim_{t \to +\infty} d_H(\pi(t)B, A) = 0,$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff semidistance between two sets, i.e.,

$$d_H(C, D) = \sup_{x \in C} \inf_{y \in D} d(x, y).$$

A set $\mathcal{A} \subset X$ is called a **global attractor** for the semigroup $\{\pi(t) \colon t \geqslant 0\}$ if it is compact, π-invariant and π-attracts all bounded subsets of X.

In this section, we present the definitions and basic properties of the impulsive dynamical systems theory (see [5–7, 16, 17] for more details).

Let $\{\pi(t) \colon t \geqslant 0\}$ be a semigroup in X. For each $D \subseteq X$ and $J \subseteq \mathbb{R}_+$ we define

$$F(D, J) = \bigcup_{t \in J} \pi(t)^{-1}(D).$$

A point $x \in X$ is called an **initial point** if $F(x, t) = \varnothing$ for all $t > 0$.

Now we are able to define the impulsive dynamical systems. An **impulsive dynamical system** (IDS, for short) (X, π, M, I) consists of a semigroup $\{\pi(t) \colon t \geqslant 0\}$ on a metric space (X, d), a nonempty closed subset $M \subseteq X$ such that for every $x \in M$ there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \varnothing \quad \text{and} \quad \bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \varnothing, \tag{2.1}$$

and a continuous function $I \colon M \to X$ whose action will be explained below in the description of the impulsive trajectory. Condition (2.1) is outlined in the next figure.
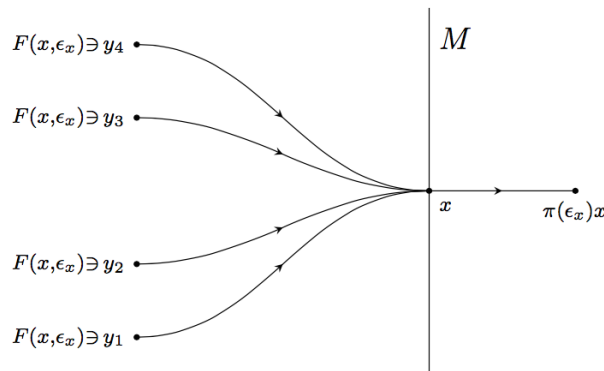

Figure 2.1: The flow of the semigroup $\{\pi(t) : t \geqslant 0\}$ is, in some sense, transversal to M.

The set M is called **impulsive set** and the function I is called **impulsive function**. We also define

$$M^+(x) = \left(\bigcup_{t > 0} \pi(t)x \right) \cap M$$

and the function $\phi \colon X \to (0, +\infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \varnothing. \end{cases} \tag{2.2}$$

If $M^+(x) \neq \varnothing$, the value $\phi(x)$ represents the first positive time such that the trajectory of x meets M. In this case, we say that the point $\pi(\phi(x))x$ is the **impulsive point** of x.

Remark 2.1. The definition of the function ϕ above makes sense thanks to the following result. See [5, 24].

Proposition 2.2. *Let* (X, π, M, I) *be an IDS and* $x \in X$. *If* $M^+(x) \neq \varnothing$ *then there exists* $s > 0$ *such that* $\pi(s)x \in M$ *and* $\pi(t)x \notin M$ *for* $0 < t < s$.

Now let us construct the impulsive trajectory of the IDS.

Definition 2.3. The **impulsive trajectory** of $x \in X$ by the IDS (X, π, M, I) is a map $\tilde{\pi}(\cdot)x$ defined in an interval $J_x \subseteq \mathbb{R}_+$, $0 \in J_x$, taking values in X which is given inductively by the following rule: if $M^+(x) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. However, if $M^+(x) \neq \varnothing$ then we denote $x = x_0^+$ and define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if } 0 \leqslant t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if } t = \phi(x_0^+). \end{cases}$$

Now let $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$ and $x_1^+ = I(\pi(s_0)x_0^+)$. In this case $s_0 < +\infty$ and the process can go on, but now starting at x_1^+. If $M^+(x_1^+) = \varnothing$, then we define $\tilde{\pi}(t)x = \pi(t - s_0)x_1^+$ for $s_0 \leqslant t < +\infty$ and in this case $\phi(x_1^+) = +\infty$. However, if $M^+(x_1^+) \neq \varnothing$ we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0 + \phi(x_1^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - s_0)x_1^+, & \text{if } s_0 \leqslant t < s_0 + \phi(x_1^+), \\ I(\pi(\phi(x_1^+))x_1^+), & \text{if } t = s_0 + \phi(x_1^+). \end{cases}$$

Now let $s_1 = \phi(x_1^+)$, $x_2 = \pi(s_1)x_1^+$ and $x_2^+ = I(\pi(s_1)x_1^+)$. Assume now that $\tilde{\pi}(\cdot)x$ is defined on the interval $[t_{n-1}, t_n]$ and that $\tilde{\pi}(t_n)x = x_n^+$, where $t_0 = 0$ and $t_n = \sum_{i=0}^{n-1} s_i$ for $n \in \mathbb{N}$. If $M^+(x_n^+) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t - t_n)x_n^+$ for $t_n \leqslant t < +\infty$ and $\phi(x_n^+) = +\infty$. However, if $M^+(x_n^+) \neq \varnothing$, then we define $\tilde{\pi}(\cdot)x$ on $[t_n, t_n + \phi(x_n^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - t_n)x_n^+, & \text{if } t_n \leqslant t < t_n + \phi(x_n^+), \\ I(\pi(\phi(x_n^+))x_n^+), & \text{if } t = t_n + \phi(x_n^+). \end{cases}$$

Now let $s_n = \phi(x_n^+)$, $x_{n+1} = \pi(s_n)x_n^+$ and $x_{n+1}^+ = I(\pi(s_n)x_n^+)$. This process ends after a finite number of steps if $M^+(x_n^+) = \varnothing$ for some $n \in \mathbb{N}$, or it may proceed indefinitely, if $M^+(x_n^+) \neq \varnothing$ for all $n \in \mathbb{N}$ and in this case $\tilde{\pi}(\cdot)x$ is defined in the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} s_i$.
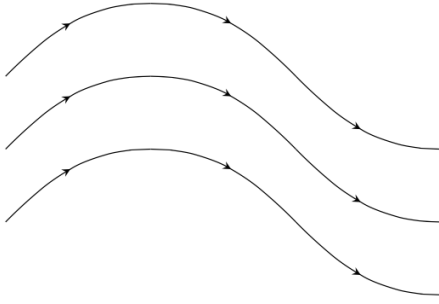
Figure 2.2: System (X, π) with
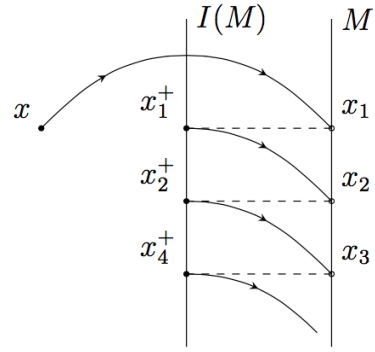continuous trajectories.

Figure 2.3: Impulsive trajectory of x
in the system (X, π, M, I).

Remark 2.4.

- We will always assume that all impulsive trajectories exist for all time $t \geqslant 0$, i.e., $T(x) = +\infty$ for all $x \in X$, since we are interested in the asymptotic behavior of impulsive dynamical systems.

- A simple consequence of the definition of impulsive trajectories is that if we assume that $I(M) \cap M = \varnothing$, then no point $x \in M$ is in any impulsive $\tilde{\pi}$-trajectory, except if the trajectory starts at x.

The definitions of $\tilde{\pi}$-invariance and $\tilde{\pi}$-attraction are analogous to the notions of π-invariance and π-attraction, respectively, simply replacing π by $\tilde{\pi}$.

2.1 Tube conditions on impulsive dynamical systems

In order to obtain some results in the impulsive theory of dynamical systems (for example, invariance and attraction results), we must ensure that the continuous semiflow possesses a nice behavior near the impulsive set M and, for this purpose we introduce the so-called "tube conditions". They are important to deduce a result ensuring the negative invariance of impulsive ω-limits. For more details and proofs see also [5, 16, 18].

Definition 2.5. Let $\{\pi(t) : t \geqslant 0\}$ be a semigroup on X. A closed set S containing $x \in X$ is called a **section** through x if there exists $\lambda > 0$ and a closed subset L of X such that:

(a) $F(L, \lambda) = S$;

(b) $F(L, [0, 2\lambda])$ contains a neighborhood of x;

(c) $F(L, \nu) \cap F(L, \zeta) = \varnothing$, if $0 \leqslant \nu < \zeta \leqslant 2\lambda$.

We say that the set $F(L, [0, 2\lambda])$ is a **λ-tube** (or simply a **tube**) and the set L is a **bar**.
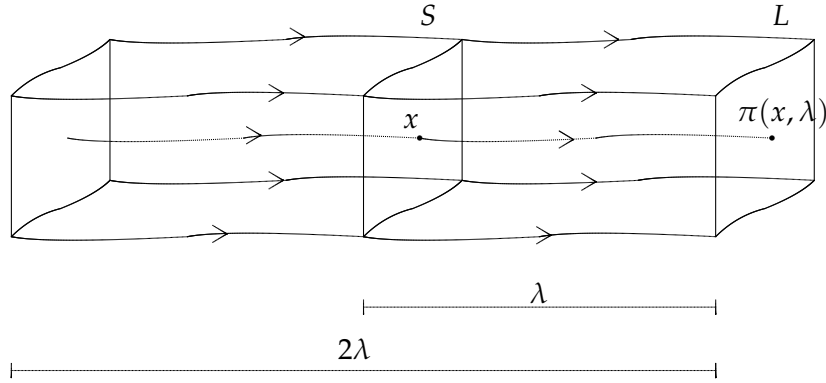
Figure 2.4: Tube $F(L, [0, 2\lambda])$.

Definition 2.6. Let (X, π, M, I) be an IDS. We say that a point $x \in M$ satisfies the **strong tube condition** (**STC**), if there exists a section S through x such that $S = F(L, [0, 2\lambda]) \cap M$. Also, we say that a point $x \in M$ satisfies the **special strong tube condition** (**SSTC**) if it satisfies STC and the λ-tube $F(L, [0, 2\lambda])$ is such that $F(L, [0, \lambda]) \cap I(M) = \varnothing$.

We finish this part presenting two proposition. The first one yields to a better understanding about the behavior of impulsive trajectories near the impulsive set M and will be useful to obtain some results later. It states that the impulsive flow $\tilde{\pi}(t)$ cannot reach the "right side" of the impulsive set M for large values of t. The second proposition summarizes some important convergence results that also will be useful to obtain further results. For details and proofs the reader may see [5].

Proposition 2.7 ([5])**.** *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and let $y \in M$ satisfy SSTC with λ-tube $F(L, [0, 2\lambda])$. Then $\tilde{\pi}(t)X \cap F(L, [0, \lambda]) = \varnothing$ for all $t > \lambda$.*

Proposition 2.8. *Let (X, π, M, I) be an IDS.*

(i) *Suppose that $I(M) \cap M = \varnothing$ and each point of M satisfies STC. Let $x \in X \setminus M$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \stackrel{n \to +\infty}{\longrightarrow} x$. Then, given $t \geqslant 0$, there exists a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subseteq [0, +\infty)$ such that $\eta_n \stackrel{n \to +\infty}{\longrightarrow} 0$ and $\tilde{\pi}(t + \eta_n)x_n \stackrel{n \to +\infty}{\longrightarrow} \tilde{\pi}(t)x$.*

(ii) *Suppose that each point in M satisfies STC. Let $x \in X \setminus M$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $X \setminus M$ such that $x_n \stackrel{n \to +\infty}{\longrightarrow} x$. Then if $\alpha_n \stackrel{n \to +\infty}{\longrightarrow} 0$ and $\alpha_n \geqslant 0$, for all $n \in \mathbb{N}$, we have $\tilde{\pi}(\alpha_n)x_n \stackrel{n \to +\infty}{\longrightarrow} x$.*

(iii) *Let $z \in M$ satisfy STC with λ-tube $F(L, [0, 2\lambda])$. Assume that there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ such that $z_n \in F(L, (\lambda, 2\lambda])$ and $z_n \stackrel{n \to +\infty}{\longrightarrow} z$. Then there exist a subsequence $\{z_{n_k}\}_{k \in \mathbb{N}}$ of $\{z_n\}_{n \in \mathbb{N}}$ and a sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ such that $\epsilon_k > 0$ and $\epsilon_k \to 0$ as $k \to +\infty$, $y_k = \pi(\epsilon_k)z_{n_k} \in M$, $\phi(z_{n_k}) = \epsilon_k$ and $y_k \stackrel{k \to +\infty}{\longrightarrow} z$.*

2.2 Attractors

We start with a first approach about attractors for the IDS. In [6], the authors propose the following definition of global attractor for an impulsive dynamical system.

Definition 2.9. A compact subset $\mathcal{A}$ of X is a **global attractor** for an IDS (X, π, M, I) if the following conditions are fulfilled:

  (i) $\mathcal{A} \cap M = \varnothing$;

  (ii) $\mathcal{A}$ is $\tilde{\pi}$-invariant;

 (iii) $\mathcal{A}$ $\tilde{\pi}$-attracts all bounded subsets of X.

Remark 2.10.

    1. This definition is consistent with the notion of a global attractor for semigroups, that is, when $M = \varnothing$, both definitions coincide; and in fact, this notion of a global attractor is useful to describe the asymptotic dynamics of $\tilde{\pi}$ in many cases.

    2. Since $\mathcal{A}$ is a compact set and M is a closed set, condition (i) implies that there exists a positive distance between $\mathcal{A}$ and M. Then the asymptotic behavior of the impulsive dynamical systems is qualitatively not different from the asymptotic behavior of the original dynamical system, thus, this notion does not consider some IDS. Let us see an example borrowed from [5] to illustrate these facts.

Example 2.11. Consider the following continuous differential equation

$$\dot{x} = \begin{cases} 1, & \text{if } x < 0, \\ 1 - x, & \text{if } x \geq 0, \end{cases} \tag{2.3}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}$ and consider the action of the impulsive function $I(0) = -1$. The solutions of (2.3) without the action of I are given by

$$\pi(t)x_0 = \begin{cases} t + x_0, & x_0 < 0, \ t \in [0, -x_0), \\ -e^{-t-x_0} + 1, & x_0 < 0, \ t \in [-x_0, +\infty), \\ (x_0 - 1)e^{-t} + 1, & x_0 \geqslant 0, \ t \in [0, +\infty). \end{cases}$$

This problem has only one bounded invariant set; namely the asymptotically stable equilibrium solution $\{1\}$, and it is also the global attractor for (2.3). Now, the solutions of (2.3) with the action of I, are given by

$$\tilde{\pi}(t)x_0 = \begin{cases} t + x_0, & x_0 < 0, \ t \in [0, -x_0), \\ t + x_0 - n, & x_0 < 0, \ t \in [-x_0 + n - 1, -x_0 + n), \ n \in \mathbb{N}, \\ (x_0 - 1)e^{-t} + 1, & x_0 \geq 0, \ t \in [0, +\infty). \end{cases} \tag{2.4}$$

    We can see that the dynamics is quite different, since there appeared the "impulsive periodic orbit" $[-1, 0)$. Note that in this case there is no subset of $\mathbb{R}$ satisfying all the conditions of Definition 2.9. But we can distinguish some interesting sets:

- The set $\mathcal{A}_1 = [-1, 0) \cup \{1\}$ is $\tilde{\pi}$-invariant and $\tilde{\pi}$-attracting bounded sets, $\mathcal{A}_1 \cap M = \varnothing$, but $\mathcal{A}_1$ is not compact.

- The set $\mathcal{A}_2 = [-1, 0] \cup \{1\}$ $\tilde{\pi}$-attracts bounded sets, $\mathcal{A}_2$ is compact, but $\mathcal{A}_2 \cap M \neq \varnothing$ and $\mathcal{A}_2$ is neither $\tilde{\pi}$-positively nor $\tilde{\pi}$-negatively invariant.

- The set $\mathcal{A}_3 = [-1, 1]$ $\tilde{\pi}$-attracts bounded sets, $\mathcal{A}_3$ is compact, it is $\tilde{\pi}$-positively invariant, but it is not $\tilde{\pi}$-negatively invariant and $\mathcal{A}_3 \cap M \neq \varnothing$.

Inspired by the ideas from this last example, in [5] the authors provide another definition of global attractor, in order to cover a larger class of impulsive dynamical systems. Their definition is the following.

Definition 2.12. A subset $\mathcal{A} \subset X$ will be called a **global attractor** for the IDS (X, π, M, I) if it satisfies the following conditions:

  (i) $\mathcal{A}$ is precompact and $\mathcal{A} = \overline{\mathcal{A}} \setminus M$;

 (ii) $\mathcal{A}$ is $\tilde{\pi}$-invariant;

(iii) $\mathcal{A}$ $\tilde{\pi}$-attracts bounded subsets of X.

Remark 2.13.

- The main difference between Definition 2.12 and Definition 2.9 is the compactness. In Definition 2.12, the global attractor does not need to be compact and now the attractor can "touch" the impulsive set M, while compact sets which do not intersect M have to be at a positive distance from M.

- It is easy to see that, with Definition 2.12, if $\mathcal{A}$ exists, it is unique.

- We recall now that a function $\psi \colon \mathbb{R} \to X$ is a **global solution** of $\tilde{\pi}$ if

$$\tilde{\pi}(t)\psi(s) = \psi(t + s), \quad \text{for all } t \geqslant 0 \text{ and } s \in \mathbb{R}.$$

  Moreover, if $\psi(0) = x$ we say that ψ is a **global solution through** x. Then, with Definition 2.12, if the IDS (X, π, M, I) possesses a global attractor $\mathcal{A}$ and $I(M) \cap M = \varnothing$ we have
$$\mathcal{A} = \{x \in X \colon \text{there exists a bounded global solution of } \tilde{\pi} \text{ through } x\}.$$

Coming back to Example 2.11, we can see that set $\mathcal{A}_1$ is the global attractor for the IDS, according to Definition 2.12. This example shows how different the continuous and the impulsive dynamics can be, as well as that a very large amount of impulsive dynamical systems, which do not fit the theory in [6], can now be considered.

In what follows we will present some definitions and results to ensure the existence of a global attractor for an IDS (X, π, M, I) as defined in Definition 2.12. We will include a sketch of some proofs and for all the details the reader may see [5].

We start giving the definition of impulsive ω-limit.

Definition 2.14. We represent the **impulsive positive orbit** of $x \in X$ starting at $s \geqslant 0$ by the set
$$\tilde{\gamma}_s^+(x) = \{\tilde{\pi}(t)x \colon t \geqslant s\}.$$

Also we set $\tilde{\gamma}^+(x) = \gamma_0^+(x)$.

Given a subset $B \subseteq X$ we define $\tilde{\gamma}_s^+(B) = \bigcup_{x \in B} \tilde{\gamma}_s^+(x)$ and we define the **impulsive ω-limit** of B as the set
$$\tilde{\omega}(B) = \bigcap_{t \geqslant 0} \overline{\tilde{\gamma}_t^+(B)}$$

which has a characterization analogous to the case of semigroups, i.e.,

$$\tilde{\omega}(B) = \Big\{ x \in X \colon \text{ there exist sequences } \{x_n\}_{n \in \mathbb{N}} \subseteq B \text{ and } \{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$$

$$\text{with } t_n \overset{n \to +\infty}{\longrightarrow} +\infty \text{ such that } \tilde{\pi}(t_n)x_n \overset{n \to +\infty}{\longrightarrow} x \Big\}$$

and $\tilde{\omega}(B)$ is closed for every subset $B \subseteq X$.

To continue with a more detailed description of the properties of impulsive ω-limits, we will need a dissipativity condition on the IDS (X, π, M, I).

Definition 2.15. An IDS (X, π, M, I) is called **bounded dissipative** if there exists a precompact set $K \subseteq X$ with $K \cap M = \varnothing$ that $\tilde{\pi}$-attracts all bounded subsets of X. Any set K satisfying these conditions will be called a **pre-attractor**.

In order to obtain the global attractor for the IDS we must obtain some properties on the impulsive omega limit. First we present one that guarantees its compactness and attraction.

Proposition 2.16 ([5], Proposition 3.4). *If (X, π, M, I) is a bounded dissipative IDS with a pre-attractor K, then for any nonempty bounded subset B of X the impulsive ω-limit $\tilde{\omega}(B)$ is nonempty, compact, $\tilde{\pi}$-attracts B and $\tilde{\omega}(B) \subseteq \overline{K}$.*

Now we aim to provide some results about the invariance of the impulsive ω-limits. We first present a positive invariance result that has a straightforward proof using item (i) of Proposition 2.8.

Proposition 2.17 ([5], Proposition 3.7). *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and each point of M satisfies STC. Then for any nonempty bounded subset B of X the set $\tilde{\omega}(B) \setminus M$ is positively $\tilde{\pi}$-invariant.*

Here we present the negative invariance result for the impulsive ω-limit set. This result is quite hard to obtain and we will include a sketch of its proof. For the detailed proof see [5, Proposition 3.12].

Proposition 2.18 ([5], Proposition 3.12). *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and each point from M satisfies SSTC and let $B \subseteq X$. If $\tilde{\omega}(B)$ is compact and $\tilde{\pi}$-attracts B, then $\tilde{\omega}(B) \setminus M$ is negatively $\tilde{\pi}$-invariant.*

Sketchy proof. Let $x \in \tilde{\omega}(B) \setminus M$ and $t \geqslant 0$. The compactness and attraction of $\tilde{\omega}(B)$ imply that $\tilde{\pi}(t_n - t)x_n \overset{n \to +\infty}{\longrightarrow} y \in \tilde{\omega}(B)$, for $\{x_n\}_{n \in \mathbb{N}} \subseteq B$ and $t_n \overset{n \to +\infty}{\longrightarrow} +\infty$ such that $\tilde{\pi}(t_n)x_n \overset{n \to +\infty}{\longrightarrow} x$. The proof is finished using Proposition 2.7 and Proposition 2.8, paying special attention to analyze separately the cases $y \in M$ and $y \notin M$. $\qquad\square$

Until now we have not shown any result saying that $\tilde{\omega}(B)$ does not intersect M, for a given subset B of X, and in fact, $\tilde{\omega}(B)$ can possess points in M. But according to Definition 2.12, the global attractor cannot intersect M and, to obtain this result, let us see that $\tilde{\omega}(B) \setminus M$ also $\tilde{\pi}$-attracts B. This result can be found in [5, Lemma 3.13 and Proposition 3.14].

Proposition 2.19 ([5]). *Let (X, π, M, I) be a bounded dissipative IDS with a pre-attractor K such that $I(M) \cap M = \varnothing$ and every point from M satisfies SSTC. Assume that there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. If B is a nonempty bounded subset of X, then $\tilde{\omega}(B) \cap M \subseteq \overline{\tilde{\omega}(B) \setminus M}$. Moreover, if $\tilde{\omega}(B)$ $\tilde{\pi}$-attracts B, then $\tilde{\omega}(B) \setminus M$ $\tilde{\pi}$-attracts B.*

To finish this section about the autonomous impulsive dynamical systems, we present a result on the existence of global attractors for IDS (according to Definition 2.12). A result on the existence of a compact global attractor, according to Definition 2.9, can be found in [6, Theorem 3.7].

Definition 2.20. An impulsive dynamical system (X, π, M, I) is called **strongly bounded dissipative** if there exists a nonempty precompact set K in X such that $K \cap M = \varnothing$ and $\tilde{\pi}$-absorbs all bounded subsets of X, i.e., for any bounded subset B of X there exists $t_B \geqslant 0$ such that $\tilde{\pi}(t)B \subseteq K$ for all $t \geqslant t_B$.

Note that if (X, π, M, I) is strongly bounded dissipative, then it is bounded dissipative.

Theorem 2.21 ([5], Theorem 4.7). *Let (X, π, M, I) be a strongly bounded dissipative IDS with $\tilde{\pi}$-absorbing set K, such that $I(M) \cap M = \varnothing$, every point in M satisfies SSTC and there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. Then (X, π, M, I) possesses a global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K) \setminus M$.*

Sketchy proof. By propositions 2.17 and 2.18, $\tilde{\omega}(K) \setminus M$ is $\tilde{\pi}$-invariant and by Proposition 2.16 $\tilde{\omega}(K) \subset \overline{K}$ is a nonempty compact set. Proposition 2.19 implies that $\tilde{\omega}(K) \setminus M$ is nonempty and

$$\overline{\tilde{\omega}(K) \setminus M} \subseteq \overline{\tilde{\omega}(K)} = \tilde{\omega}(K).$$

To finish the proof, note that the strong bounded dissipativity implies that $\tilde{\omega}(B) \subseteq \tilde{\omega}(K)$, and using Proposition 2.19 again we have $\tilde{\omega}(B) \setminus M$ $\tilde{\pi}$-attracts B for any bounded subset B of X, thus $\tilde{\omega}(K) \setminus M$ $\tilde{\pi}$-attracts all bounded subsets of X, which concludes the proof. $\square$

2.3 Example

We show now an example to illustrate the theory described above. This example is borrowed from [5, Example 4.8]

Example 2.22. Consider the impulsive dynamical system in $X = \mathbb{R}^2$ generated by the following impulsive differential equation

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = -y, \\ (x(0), y(0)) = (x_0, y_0), \\ I \colon M \to I(M), \end{cases} \tag{2.5}$$

where:

- $M = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$,

- $I(M) \subset \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 9\}$ and the function $I \colon M \to I(M)$ is defined as follows: given $(x, y) \in M$ we consider the line segment $\Gamma_{(x,y)}$ that connects the points (x, y) and $(3, y)$. The point $I(x, y)$ is the unique point in the intersection $\Gamma_{(x,y)} \cap I(M)$ (observe Figure 2.5).

Let $\{\pi(t) \colon t \geqslant 0\}$ be the semigroup in $\mathbb{R}^2$ generated by (2.5) with no impulse, that is, $\pi(t)(x_0, y_0) = (x_0 e^{-t}, y_0 e^{-t})$ and consider the IDS (X, π, M, I). It is not difficult to see that:

- each point of M satisfies SSTC;

- $I(M) \cap M = \varnothing$;

- there exists $\xi > 0$ such that $\phi(x,y) \geqslant \xi$ for all $(x,y) \in I(M)$.
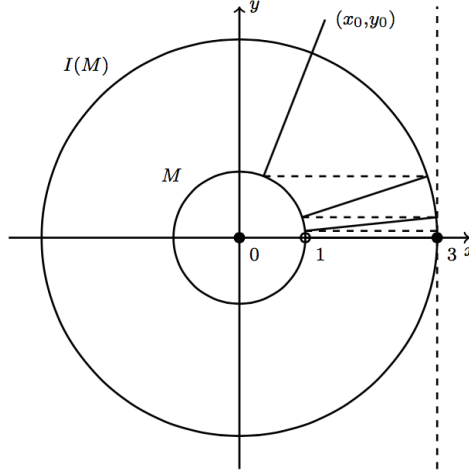

Figure 2.5: Impulsive trajectory of $(x_0, y_0) \in \mathbb{R}^2$.

If we let $K = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 9\} \setminus M$, it is clear that K is a precompact subset of $\mathbb{R}^2$, $K \cap M = \varnothing$ and K $\tilde{\pi}$-absorbs all bounded subsets of X, hence (X, π, M, I) is strongly bounded dissipative with $\tilde{\pi}$-absorbing set K and Theorem 2.21 ensures that (X, π, M, I) has a global attractor $\mathcal{A} = \tilde{\omega}(K) \setminus M$.

We can see that $\tilde{\omega}(K) = \{(0,0)\} \cup \{(x,0) \colon x \in [1,3]\}$ and hence $\mathcal{A} = \{(0,0)\} \cup \{(x,0) \colon x \in (1,3]\}$.

3   Impulsive nonautonomous dynamical systems

In this second part, we present some recent results on the impulsive nonautonomous theory. We will propose a definition for an *impulsive cocycle attractor*, which is one of the possible frameworks that we can choose when working with nonautonomous dynamical systems. The detailed proof for the results in this section can be found in [4].

In order to deal with cocycle dynamics, we introduce briefly the concept of (continuous) nonautonomous dynamical systems (see [14]).

Definition 3.1. Let X and Σ be two complete metric spaces and $\{\theta_t \colon t \in \mathbb{R}\}$ be a group in Σ. For each pair $(t, \sigma) \in \mathbb{R}_+ \times \Sigma$, let $\varphi(t, \sigma) \colon X \to X$ be a map satisfying the following properties:

(i) $\varphi(0, \sigma)x = x$ for all $x \in X$ and $\sigma \in \Sigma$;

(ii) $\varphi(t + s, \sigma) = \varphi(t, \theta_s \sigma)\varphi(s, \sigma)$ for all $t, s \in \mathbb{R}_+$ and $\sigma \in \Sigma$;

(iii) the map $\mathbb{R}_+ \times \Sigma \times X \ni (t, \sigma, x) \mapsto \varphi(t, \sigma)x \in X$ is continuous.

We say that $(\varphi, \theta)_{(X,\Sigma)}$ is a **nonautonomous dynamical system** (NDS, for short). The group $\{\theta_t \colon t \in \mathbb{R}\}$ in this context is called **driving group**, the map φ is called **cocycle** and the property **(ii)** is commonly known as the **cocycle property**.

Definition 3.2. A family $\hat{B} = \{B(\sigma)\}_{\sigma \in \Sigma}$ with $B(\sigma) \subseteq X$ for each $\sigma \in \Sigma$ is called a **nonautonomous set**. The nonautonomous set $\hat{B}$ is **open (closed/compact)** if each **fiber** $B(\sigma)$ is an open (closed/compact) subset of X.

Definition 3.3. Given an NDS $(\varphi, \theta)_{(X,\Sigma)}$, we say that a nonautonomous set $\hat{B}$ is φ-**invariant** if

$$\varphi(t, \sigma)B(\sigma) = B(\theta_t \sigma), \quad \text{for all } t \in \mathbb{R}_+ \text{ and for all } \sigma \in \Sigma.$$

We say that $\hat{B}$ is **positively (negatively) φ-invariant** if

$$\varphi(t, \sigma)B(\sigma) \subseteq (\supseteq)B(\theta_t \sigma), \quad \text{for all } t \in \mathbb{R}_+ \text{ and for all } \sigma \in \Sigma.$$

Definition 3.4. A collection $\mathfrak{D}$ of nonautonomous sets is called a **universe** in X if it is inclusion-closed, that is, if $\hat{D}_1 \in \mathfrak{D}$ and $D_2(\sigma) \subseteq D_1(\sigma)$, for all $\sigma \in \Sigma$, then $\hat{D}_2 \in \mathfrak{D}$.

Definition 3.5. Given a universe $\mathfrak{D}$ in X, we say that a nonautonomous set $\hat{A}$ is $(\varphi, \mathfrak{D})$-**pullback attracting** if

$$\lim_{t \to +\infty} d_H(\varphi(t, \theta_{-t}\sigma)D(\theta_{-t}\sigma), A(\sigma)) = 0,$$

for every family $\hat{D} \in \mathfrak{D}$ and for all $\sigma \in \Sigma$.

Definition 3.6. Given a universe $\mathfrak{D}$ in X, a compact nonautonomous set $\hat{A}$ is called a $\mathfrak{D}$-**cocycle attractor** for the NDS $(\varphi, \theta)_{(X,\Sigma)}$ if it is:

(i) φ-invariant;

(ii) $(\varphi, \mathfrak{D})$-pullback attracting;

(iii) minimal among the closed nonautonomous sets satisfying property **(ii)**.

Given an NDS $(\varphi, \theta)_{(X,\Sigma)}$, we can construct a semigroup, called the **skew-product semiflow**, $\{\Pi(t) : t \geqslant 0\}$ in $\mathbb{X} \doteq X \times \Sigma$, given by

$$\Pi(t)(x, \sigma) = (\varphi(t, \sigma)x, \theta_t \sigma) \quad \text{for all } (x, \sigma) \in \mathbb{X} \text{ and } t \geqslant 0.$$

In [4], the authors give a notion of impulsive nonautonomous dynamical systems. Let us introduce that. So first let $(\varphi, \theta)_{(X,\Sigma)}$ be a NDS. For each $D \subseteq X$, $J \subseteq \mathbb{R}_+$ and $\sigma \in \Sigma$ we define

$$F_\varphi(D, J, \sigma) = \{x \in X : \varphi(t, \sigma)x \in D, \text{ for some } t \in J\}.$$

A point $x \in X$ is called an **initial point** if $F_\varphi(x, \tau, \sigma) = \varnothing$ for all $\tau > 0$ and for all $\sigma \in \Sigma$.

Definition 3.7. An **impulsive nonautonomous dynamical system** (INDS, for short) $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ consists of a nonautonomous dynamical system $(\varphi, \theta)_{(X,\Sigma)}$, a nonempty closed subset $M \subseteq X$ such that for each $x \in M$ and each $\sigma \in \Sigma$ there exists $\epsilon_{x,\sigma} > 0$ such that

$$\bigcup_{t \in (0, \epsilon_{x,\sigma})} F_\varphi(x, t, \theta_{-t}\sigma) \cap M = \varnothing \quad \text{and} \quad \{\varphi(s, \sigma)x : s \in (0, \epsilon_{x,\sigma})\} \cap M = \varnothing,$$

and a continuous function $I : M \to X$ whose action is specified in the sequel. The set M is called the **impulsive set** and the function I is called the **impulse function**. We also define

$$M_\varphi^+(x, \sigma) = \{\varphi(\tau, \sigma)x : \tau > 0\} \cap M.$$

A nonautonomous version of Proposition 2.2 holds, that is, given an INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$, $x \in X$ and $\sigma \in \Sigma$, if $M_\varphi^+(x,\sigma) \neq \varnothing$ then there exists $t > 0$ such that $\varphi(t,\sigma)x \in M$ and $\varphi(\tau,\sigma)x \notin M$ for $0 < \tau < t$. Thus, we are able to define the function $\phi(\cdot,\sigma) \colon X \to (0,+\infty]$ by

$$\phi(x,\sigma) = \begin{cases} s, & \text{if } \varphi(s,\sigma)x \in M \text{ and } \varphi(t,\sigma)x \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \varphi(t,\sigma)x \notin M \text{ for all } t > 0. \end{cases} \tag{3.1}$$

As in the autonomous case, the value $\phi(x,\sigma)$ represents the smallest positive time such that the trajectory of x in the fiber σ meets M. In this case, we say that the point $\varphi(\phi(x,\sigma),\sigma)x$ is the **impulsive point** of x in the fiber σ.

Analogously to the autonomous case, let us explain the construction of the impulsive trajectory of the INDS in order to emphasize the main differences arising from the nonautonomous character of the problem.

Definition 3.8. Given $\sigma \in \Sigma$, the **impulsive semitrajectory** of $x \in X$ starting at fiber σ by the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ is a map $\tilde{\varphi}(\cdot,\sigma)x$ defined in an interval $J_{(x,\sigma)} \subseteq \mathbb{R}_+$, $0 \in J_{(x,\sigma)}$, with values in X given inductively by the following rule: if $M_\varphi^+(x,\sigma) = \varnothing$, then $\tilde{\varphi}(t,\sigma)x = \varphi(t,\sigma)x$ for all $t \in [0,+\infty)$ and in this case $\phi(x,\sigma) = +\infty$. However, if $M_\varphi^+(x,\sigma) \neq \varnothing$ then we denote $x = x_0^+$ and we define $\tilde{\varphi}(\cdot,\sigma)x$ on $[0, \phi(x_0^+,\sigma)]$ by

$$\tilde{\varphi}(t,\sigma)x = \begin{cases} \varphi(t,\sigma)x_0^+, & \text{if } 0 \leqslant t < \phi(x_0^+,\sigma), \\ I(\varphi(\phi(x_0^+,\sigma),\sigma)x_0^+), & \text{if } t = \phi(x_0^+,\sigma). \end{cases}$$

Now let $s_0 = \phi(x_0^+,\sigma)$, $x_1 = \varphi(s_0,\sigma)x_0^+$ and $x_1^+ = I(\varphi(s_0,\sigma)x_0^+)$. In this case, since $s_0 < +\infty$ then the process can go on, but now starting at x_1^+. If $M_\varphi^+(x_1^+,\theta_{s_0}\sigma) = \varnothing$ then we define $\tilde{\varphi}(t,\sigma)x = \varphi(t - s_0, \theta_{s_0}\sigma)x_1^+$ for $s_0 \leqslant t < +\infty$ and we get $\phi(x_1^+, \theta_{s_0}\sigma) = +\infty$. However, if $M_\varphi^+(x_1^+,\theta_{s_0}\sigma) \neq \varnothing$, we define $\tilde{\varphi}(\cdot,\sigma)x$ on $[s_0, s_0 + \phi(x_1^+,\theta_{s_0}\sigma)]$ by

$$\tilde{\varphi}(t,\sigma)x = \begin{cases} \varphi(t - s_0, \theta_{s_0}\sigma)x_1^+, & \text{if } s_0 \leqslant t < s_0 + \phi(x_1^+,\theta_{s_0}\sigma), \\ I(\varphi(\phi(x_1^+,\theta_{s_0}\sigma),\theta_{s_0}\sigma)x_1^+), & \text{if } t = s_0 + \phi(x_1^+,\theta_{s_0}\sigma). \end{cases}$$

Let $s_1 = \phi(x_1^+,\theta_{s_0}\sigma)$, $x_2 = \varphi(s_1,\theta_{s_0}\sigma)x_1^+$ and $x_2^+ = I(\varphi(s_1,\theta_{s_0}\sigma)x_1^+)$. Now, we assume that $\tilde{\varphi}(\cdot,\sigma)x$ is defined on the interval $[t_{n-1}, t_n]$ and that $\tilde{\varphi}(t_n,\sigma)x = x_n^+$, where $t_0 = 0$ and $t_n = \sum_{i=0}^{n-1} s_i$ for $n = 1,2,3,\ldots$ If $M_\varphi^+(x_n^+,\theta_{t_n}\sigma) = \varnothing$, then $\tilde{\varphi}(t,\sigma)x = \varphi(t - t_n, \theta_{t_n}\sigma)x_n^+$ for $t_n \leqslant t < +\infty$ and $\phi(x_n^+,\theta_{t_n}\sigma) = +\infty$. However, if $M_\varphi^+(x_n^+,\theta_{t_n}\sigma) \neq \varnothing$, then we define $\tilde{\varphi}(\cdot,\sigma)x$ on $[t_n, t_{n+1}]$ by

$$\tilde{\varphi}(t,\sigma)x = \begin{cases} \varphi(t - t_n, \theta_{t_n}\sigma)x_n^+, & \text{if } t_n \leqslant t < t_{n+1}, \\ I(\varphi(\phi(x_n^+,\theta_{t_n}\sigma),\theta_{t_n}\sigma)x_n^+), & \text{if } t = t_{n+1}. \end{cases}$$

Now let $s_n = \phi(x_n^+,\theta_{t_n}\sigma)$, $x_{n+1} = \varphi(s_n,\theta_{t_n}\sigma)x_n^+$ and $x_{n+1}^+ = I(\varphi(s_n,\theta_{t_n}\sigma)x_n^+)$. This process ends after a finite number of steps if $M_\varphi^+(x_n^+,\theta_{t_n}\sigma) = \varnothing$ for some $n \in \mathbb{N}$, or it may proceed indefinitely, if $M_\varphi^+(x_n^+,\theta_{t_n}\sigma) \neq \varnothing$ for all $n \in \mathbb{N}$ and in this case $\tilde{\varphi}(\cdot,\sigma)x$ is defined in the interval $[0, T(x,\sigma))$, where $T(x,\sigma) = \sum_{i=0}^{+\infty} s_i$.

From now on, we will always assume that $T(x,\sigma) = +\infty$, for all $x \in X$ and $\sigma \in \Sigma$. It is not difficult to see that this condition is satisfied when there exists $\delta = \delta(\sigma) > 0$ such that $\phi(x,\omega) \geqslant \delta$ for all $x \in I(M)$ and $\omega \in \{\theta_t\sigma : t \in \mathbb{R}\}$.

The definitions of $\tilde{\varphi}$-invariance and $\tilde{\varphi}$-attraction are analogous to the notions of φ-invariance and φ-attraction, respectively, simply replacing φ by $\tilde{\varphi}$.

As a consequence of the construction of the impulsive semitrajectory $\tilde{\varphi}$, we present a result that is useful to transfer some properties from the impulsive skew-product semiflow to the INDS [4, Theorem 2.9].

Theorem 3.9 ([4, Theorem 2.9]). *Let* $(\varphi, \theta)_{(X,\Sigma)}$ *be a nonautonomous dynamical system,* $\{\Pi(t) \colon t \geqslant 0\}$ *the associated skew-product semiflow in* $\mathbb{X} = X \times \Sigma$ *and* $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ *the associated INDS. Let* $\tilde{\Pi}^*$ *be defined by*

$$\tilde{\Pi}^*(t)(x, \sigma) = (\tilde{\varphi}(t, \sigma)x, \theta_t \sigma) \quad \text{for all} \ \ (x, \sigma) \in \mathbb{X} \ \ \text{and} \ \ t \geq 0,$$

and also let $\{\tilde{\Pi}(t) \colon t \geqslant 0\}$ *be the impulsive dynamical system* $(\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})$, *where* $\mathbb{M} = M \times \Sigma$ *and* $\mathbb{I} \colon \mathbb{M} \to \mathbb{X}$ *is given by* $\mathbb{I}(x, \sigma) = (I(x), \sigma)$, *for* $x \in M$. *Then*

$$\tilde{\Pi}^*(t) = \tilde{\Pi}(t) \quad \text{for all} \ \ t \geqslant 0.$$

Moreover, if ϕ *is the function defined in* (3.1)*, then it coincides with the function used to define the impulsive semitrajectory* $\{\tilde{\Pi}(t) \colon t \geqslant 0\}$.

The theorem above also says that the following diagram is commutative:

$$
\begin{array}{ccc}
(\varphi, \theta)_{(X,\Sigma)} & \longrightarrow & \{\Pi(t) \colon t \geqslant 0\} \\
\big\downarrow & \circlearrowleft & \big\downarrow \\
[(\varphi, \theta)_{(X,\Sigma)}, M, I] & \longrightarrow & (\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})
\end{array}
\qquad \textbf{(D)}
$$

Under the conditions of Theorem 3.9, for each $\sigma \in \Sigma$ and $t, s \in \mathbb{R}_+$, we have $\tilde{\varphi}(t + s, \sigma) = \tilde{\varphi}(t, \theta_s \sigma)\tilde{\varphi}(s, \sigma)$.

3.1 Tube conditions and convergence properties on INDS

As we have already mentioned, the "tube conditions" are very important for the theory of impulsive dynamical systems and let us see their version in the nonautonomous case. We also present some important convergence properties for the impulsive nonautonomous theory. The following results are obtained using Theorem 3.9 and the corresponding results in the autonomous case and their proofs can be found in [4].

Definition 3.10. Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS. We say that a point $x \in M$ satisfies the φ**-strong tube condition** (φ**-STC**), if for each $\sigma \in \Sigma$, the pair (x, σ) satisfies STC with respect to the impulsive skew-product $(\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})$. Also, we say that a point $x \in M$ satisfies the φ**-special strong tube condition** (φ**-SSTC**), if for each $\sigma \in \Sigma$, the pair (x, σ) satisfies SSTC with respect to the impulsive skew-product $(\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})$.

Proposition 3.11 ([4, Proposition 3.7]). *Let* $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ *be an INDS such that* $I(M) \cap M = \varnothing$ *and let* $y \in M$ *satisfy* φ*-SSTC. Then, for each* $\sigma \in \Sigma$, *the point* (y, σ) *satisfies SSTC with* λ*-tube* $\mathbb{F}_\Pi(\mathbb{L}, [0, 2\lambda])$ *such that* $\tilde{\Pi}(t)(X \times \Sigma) \cap F_\Pi(\mathbb{L}, [0, \lambda]) = \varnothing$ *for all* $t > \lambda$.

The next proposition summarizes the results in [4, Section 4].

Proposition 3.12. *Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS.*

(i) *Suppose that $I(M) \cap M = \varnothing$ and each point of M satisfies φ-STC. Let also $x \in X \setminus M$ and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \overset{n \to +\infty}{\longrightarrow} x$. Then, given $t \geqslant 0$, $\sigma \in \Sigma$ and a sequence $\{\sigma_n\}_{n \in \mathbb{N}} \subset \Sigma$ with $\sigma_n \overset{n \to +\infty}{\longrightarrow} \sigma$, there exists a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subseteq [0, +\infty)$ such that $\eta_n \overset{n \to +\infty}{\longrightarrow} 0$ and $\tilde{\varphi}(t + \eta_n, \sigma_n) x_n \overset{n \to +\infty}{\longrightarrow} \tilde{\varphi}(t, \sigma) x$.*

(ii) *Suppose that each point in M satisfies φ-STC, $x \in X \setminus M$, $\sigma \in \Sigma$, $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $X \setminus M$ such that $x_n \overset{n \to +\infty}{\longrightarrow} x$ and $\sigma_n \overset{n \to +\infty}{\longrightarrow} \sigma$. Then if $\alpha_n \overset{n \to +\infty}{\longrightarrow} 0$ and $\alpha_n \geqslant 0$, for all $n \in \mathbb{N}$, we have $\tilde{\varphi}(\alpha_n, \sigma_n) x_n \overset{n \to +\infty}{\longrightarrow} x$.*

(iii) *Assume that each $x \in M$ satisfies φ-SSTC and $I(M) \cap M = \varnothing$. Let $\hat{B}$ be a nonautonomous set, $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$, $\sigma \in \Sigma$, $\{\eta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{x_n\}_{n \in \mathbb{N}}$ be sequences such that $\eta_n \overset{n \to +\infty}{\longrightarrow} 0$, $x_n \in B(\theta_{-t_n} \sigma)$ for each $n \in \mathbb{N}$. If $\{\tilde{\varphi}(t_n + \eta_n, \theta_{-t_n} \sigma) x_n\}_{n \in \mathbb{N}}$ is convergent with limit $y \in M$ and $\{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence with $\epsilon_n \overset{n \to +\infty}{\longrightarrow} 0$, then there is a subsequence $\{\tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k}\}_{k \in \mathbb{N}}$ such that $\phi(\tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k}, \theta_{\eta_{n_k}} \sigma) \overset{k \to +\infty}{\longrightarrow} 0$ and either*

$$\tilde{\varphi}(\epsilon_{n_k}, \theta_{\eta_k} \sigma) \tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k} \overset{k \to +\infty}{\longrightarrow} y$$

or

$$\tilde{\varphi}(\epsilon_{n_k}, \theta_{\eta_k} \sigma) \tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k} \overset{k \to +\infty}{\longrightarrow} I(y).$$

In particular,

$$\tilde{\varphi}(\alpha_k, \theta_{\eta_k} \sigma) \tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k} \overset{k \to +\infty}{\longrightarrow} I(y),$$

where $\alpha_k = \phi(\tilde{\varphi}(t_{n_k} + \eta_{n_k}, \theta_{-t_{n_k}} \sigma) x_{n_k}, \theta_{\eta_{n_k}} \sigma)$.

3.2   Impulsive cocycle attractors

Here we will present the notion of attractor for an INDS (impulsive cocycle attractor), define and establish some properties of the impulsive omega limit sets in order to obtain an existence result of impulsive cocycle attractors. We will see that this notion of attractor is not a natural generalization of the global attractor given in [5] (see Definition 2.12), since the results on the invariance in the impulsive case cannot be obtained as a natural generalization of the continuous case. A more complete analysis can be found in [4] and some of the proofs will be reproduced here to illustrate the techniques.

Let us introduce the notion of attractor for an INDS (with respect to a universe).

Definition 3.13. Given a universe $\mathfrak{D}$, a compact nonautonomous set $\hat{A}$ is called a $\mathfrak{D}$-**impulsive cocycle attractor** for the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ if:

(i) $\hat{A} \setminus M = \{A(\sigma) \setminus M\}_{\sigma \in \Sigma}$ is $\tilde{\varphi}$-invariant;

(ii) $\hat{A}$ is $(\tilde{\varphi}, \mathfrak{D})$-pullback attracting;

(iii) $\hat{A}$ is minimal, that is, if $\hat{C}$ is a closed nonautonomous set satisfying (ii), then $A(\sigma) \subseteq C(\sigma)$ for each $\sigma \in \Sigma$.

Remark 3.14. Note that, in the trivial case (i.e., $\Sigma = \{\sigma\}$), the definition of the cocycle attractor reduces to a compact set A such that $A \setminus M$ is invariant and attracts bounded sets of X

which is not the definition of a global attractor for the autonomous case, as given in Definition 2.12. We again emphasize that the nonautonomous framework is more challenging than the autonomous one, and so, it is reasonable that we find more restrictive conditions in the definition of impulsive cocycle attractors.

Now we state the definition of the impulsive omega limit set along with its characterization.

Definition 3.15. Given a nonautonomous set $\hat{B} \doteq \{B(\sigma)\}_{\sigma \in \Sigma}$ and $\sigma \in \Sigma$ we define the **impulsive pullback omega-limit of $\hat{B}$ at the fiber** σ as the set

$$\tilde{\omega}(\hat{B}, \sigma) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \bigcup_{\epsilon \in [0, s^{-1})} \tilde{\varphi}(t + \epsilon, \theta_{-t}\sigma) B(\theta_{-t}\sigma)}$$

and the **impulsive pullback omega-limit** of $\hat{B}$ as the nonautonomous set

$$\tilde{\omega}(\hat{B}) \doteq \{\tilde{\omega}(\hat{B}, \sigma)\}_{\sigma \in \Sigma}.$$

Lemma 3.16. *It follows that*

$$\tilde{\omega}(\hat{B}, \sigma) = \Big\{ x \in X : \text{there exist sequences } \{t_n\}_{n \in \mathbb{N}}, \{\epsilon_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \text{ and } \{x_n\}_{n \in \mathbb{N}} \subseteq B(\theta_{-t_n}\sigma)$$

$$\text{with } t_n \xrightarrow{n \to +\infty} +\infty, \ \epsilon_n \xrightarrow{n \to +\infty} 0 \text{ such that } \tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \xrightarrow{n \to +\infty} x \Big\}$$

and $\tilde{\omega}(\hat{B}, \sigma)$ is closed.

Remark 3.17. Note that if $M = \varnothing$, then $\tilde{\omega}(\hat{B}, \sigma) = \omega(\hat{B}, \sigma)$, for each nonautonomous set $\hat{B}$.

Definition 3.18. An INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is said to be **pullback $\mathfrak{D}$-asymptotically compact**, if for each $\sigma \in \Sigma$, $\hat{D} \in \mathfrak{D}$ and sequences $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$, $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $t_n \xrightarrow{n \to +\infty} +\infty$ and $x_n \in D(\theta_{-t_n}\sigma)$, implies that the sequence $\{\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)x_n\}_{n \in \mathbb{N}}$ possesses a convergent subsequence.

Definition 3.19. A nonautonomous set $\hat{B}$ is said to be **pullback $\mathfrak{D}$-absorbing** for the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$, if for each $\sigma \in \Sigma$ and $\hat{D} \in \mathfrak{D}$, there exists $t_0 = t_0(\sigma, \hat{D}) \geqslant 0$ such that

$$\tilde{\varphi}(t, \theta_{-t}\sigma)D(\theta_{-t}\sigma) \subseteq B(\sigma) \quad \text{for all } t \geqslant t_0.$$

Following the same scheme as the autonomous case, we will present results on the impulsive pullback omega limit and finish this section giving a result on the existence of a impulsive cocycle attractor. The results can be found in [4, Section 4 and Section 5] and we will include some of the proofs in order to illustrate the techniques.

Proposition 3.20 ([4, Proposition 4.8]). *If the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is pullback $\mathfrak{D}$-asymptotically compact, each point of M satisfies φ-SSTC, $I(M) \cap M = \varnothing$, $\hat{B} \in \mathfrak{D}$ and $\sigma \in \Sigma$, then the nonautonomous set $\tilde{\omega}(\hat{B})$ is nonempty, compact and pullback attracts $\hat{B}$, that is, for each $\sigma \in \Sigma$*

$$\lim_{t \to +\infty} \mathrm{d}_H(\tilde{\varphi}(t, \theta_{-t}\sigma)B(\theta_{-t}\sigma), \tilde{\omega}(\hat{B}, \sigma)) = 0.$$

Proof. Let $\sigma \in \Sigma$ and take sequences $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ $(t_n \geq 0)$ and $x_n \in B(\theta_{-t_n}\sigma)$, $n \in \mathbb{N}$. By the asymptotic compactness, the sequence $\{\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence for a point $x \in X$. It is easy to verify that $x \in \tilde{\omega}(\hat{B}, \sigma)$, which proves that $\tilde{\omega}(\hat{B}, \sigma)$ is nonempty.

Now, since $\tilde{\omega}(\hat{B}, \sigma)$ is closed, to show its compactness it is sufficient to prove that if $\{z_n\}_{n\in\mathbb{N}} \subseteq \tilde{\omega}(\hat{B}, \sigma)$ is a sequence, then we can obtain a convergent subsequence. So, let $\{z_n\}_{n\in\mathbb{N}} \subseteq \tilde{\omega}(\hat{B}, \sigma)$, then for each $n \in \mathbb{N}$, one can obtain sequences $\{t_k^n\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\epsilon_k^n\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{x_k^n\}_{k\in\mathbb{N}} \subset B(\theta_{-t_k^n}\sigma)$ such that $t_k^n \overset{k\to+\infty}{\longrightarrow} +\infty$, $\epsilon_k^n \overset{k\to+\infty}{\longrightarrow} 0$ and

$$\tilde{\varphi}(t_k^n + \epsilon_k^n, \theta_{-t_k^n}\sigma)x_k^n \overset{k\to+\infty}{\longrightarrow} z_n.$$

Thus, there is a natural $k_n \geq n$ such that

$$\mathrm{d}\big(\tilde{\varphi}(t_{k_n}^n + \epsilon_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n, z_n\big) \leq \frac{1}{n}.$$

Note that

$$\tilde{\varphi}(t_{k_n}^n + \epsilon_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n = \tilde{\varphi}(\epsilon_{k_n}^n, \sigma)\tilde{\varphi}(t_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n,$$

for $n, k \in \mathbb{N}$.

Since $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is pullback $\mathfrak{D}$-asymptotically compact, we may assume without loss of generality that there is $w \in X$ such that

$$\tilde{\varphi}(t_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n \overset{n\to+\infty}{\longrightarrow} w.$$

If $w \notin M$, then using item (ii) of Proposition 3.12, we get

$$\tilde{\varphi}(\epsilon_{k_n}^n, \sigma)\tilde{\varphi}(t_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n \overset{n\to+\infty}{\longrightarrow} w,$$

which shows that $z_{k_n} \overset{n\to+\infty}{\longrightarrow} w$.

If $w \in M$, we may assume by item (iii) of Proposition 3.12 that either

$$\tilde{\varphi}(\epsilon_{k_n}^n, \sigma)\tilde{\varphi}(t_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n \overset{n\to+\infty}{\longrightarrow} w$$

or

$$\tilde{\varphi}(\epsilon_{k_n}^n, \sigma)\tilde{\varphi}(t_{k_n}^n, \theta_{-t_{k_n}^n}\sigma)x_{k_n}^n \overset{n\to+\infty}{\longrightarrow} I(w),$$

which shows that $\{z_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence.

Now, assume that the last statement does not hold, that is, there exist $\sigma \in \Sigma$, $\epsilon_0 > 0$ and sequences $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $z_n \in B(\theta_{-t_n}\sigma)$ such that

$$\mathrm{d}\big(\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)z_n, \tilde{\omega}(\hat{B}, \sigma)\big) \geq \epsilon_0, \qquad n \in \mathbb{N}.$$

But $\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)z_n \overset{n\to+\infty}{\longrightarrow} x$ for some $x \in X$ along some subsequence. Clearly $x \in \tilde{\omega}(\hat{B}, \sigma)$ and

$$0 = \mathrm{d}(x, \tilde{\omega}(\hat{B}, \sigma)) \geq \epsilon_0,$$

which gives us a contradiction and proves the result. □

Proposition 3.21 ([4, Proposition 4.9]). *Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS such that $I(M) \cap M = \varnothing$ and each point of M satisfies φ-STC. Then for any nonempty nonautonomous set $\hat{B}$, its impulsive ω-limit $\tilde{\omega}(\hat{B}) \setminus M \doteq \{\tilde{\omega}(\hat{B}, \sigma) \setminus M\}_{\sigma\in\Sigma}$ is positively $\tilde{\varphi}$-invariant.*

Proof. Fix $\sigma \in \Sigma$ and $t \geqslant 0$. Let $x \in \tilde{\omega}(\hat{B}, \sigma) \setminus M$. Then there exist $\{t_n\}_{n \in \mathbb{N}}, \{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{x_n\}_{n \in \mathbb{N}} \subseteq B(\theta_{-t_n}\sigma)$ with $t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty$, $\epsilon_n \stackrel{n \to +\infty}{\longrightarrow} 0$ such that $\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \stackrel{n \to +\infty}{\longrightarrow} x$. Since $x \notin M$ and M is closed, we may assume that $\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \notin M$ for all $n \in \mathbb{N}$. Therefore, by item (i) of Proposition 3.12, there exists a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $\eta_n \stackrel{n \to +\infty}{\longrightarrow} 0$ and

$$\tilde{\varphi}(t_n + t + \eta_n + \epsilon_n, \theta_{-(t+t_n)}\theta_t \sigma)x_n = \tilde{\varphi}(t + \eta_n, \theta_{\epsilon_n}\sigma)\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \stackrel{n \to +\infty}{\longrightarrow} \tilde{\varphi}(t, \sigma)x.$$

Hence, $\tilde{\varphi}(t, \sigma)x \in \tilde{\omega}(\hat{B}, \theta_t \sigma)$. If $t = 0$, there is nothing to do. If $t > 0$ observe that $\tilde{\varphi}(t, \sigma)x \notin M$, since any impulsive trajectory starting at a point of $X \setminus M$ never reaches M in finite time (note that $I(M) \cap M = \varnothing$). This shows the positive $\tilde{\varphi}$-invariance of $\tilde{\omega}(\hat{B}) \setminus M$. □

Before establishing the negative invariance for impulsive pullback ω-limit sets, we need an auxiliary result.

Lemma 3.22 ([4, Lemma 4.10]). *Let $[(\varphi, \theta)_{(X, \Sigma)}, M, I]$ be an INDS with $I(M) \cap M = \varnothing$. Assume that every point from M satisfies φ-SSTC and let $\hat{B}$ be a nonautonomous set. If $y \in \tilde{\omega}(\hat{B}, \sigma) \cap M$ then $I(y) \in \tilde{\omega}(\hat{B}, \sigma) \setminus M$.*

Proposition 3.23 ([4, Proposition 4.11]). *Let $[(\varphi, \theta)_{(X, \Sigma)}, M, I]$ be an INDS with $I(M) \cap M = \varnothing$. Assume that every point from M satisfies φ-SSTC and let $\hat{B}$ be a nonautonomous set. If $\tilde{\omega}(\hat{B})$ is compact and pullback attracts $\hat{B}$, then $\tilde{\omega}(\hat{B}) \setminus M$ is negatively $\tilde{\varphi}$-invariant.*

Proof. Let $t \geqslant 0$, $\sigma \in \Sigma$ and $x \in \tilde{\omega}(\hat{B}, \theta_t \sigma) \setminus M$. Then there exist sequences $\{t_n\}_{n \in \mathbb{N}}, \{\epsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{x_n\}_{n \in \mathbb{N}} \subseteq B(\theta_{-t_n+t}\sigma)$ with $t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty$ and $\epsilon_n \stackrel{n \to +\infty}{\longrightarrow} 0$ such that

$$\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n+t}\sigma)x_n \stackrel{n \to +\infty}{\longrightarrow} x.$$

Now, since $\tilde{\omega}(\hat{B})$ is compact and pullback attracts $\hat{B}$ and we have item (ii) of Proposition 3.12, we can assume that $\{\tilde{\varphi}(t_n - t + \epsilon_n, \theta_{-t_n+t}\sigma)x_n\}_{n \in \mathbb{N}}$ possesses a convergent subsequence (which we denote by the same notation and we already assumed that $t_n > t$, since $t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty$ and t is fixed). Thus $y_n \doteq \tilde{\varphi}(t_n - t + \epsilon_n, \theta_{-t_n+t}\sigma)x_n \stackrel{n \to +\infty}{\longrightarrow} y \in \tilde{\omega}(\hat{B}, \sigma)$.

Case 1: $y \in X \setminus M$.

By item (i) of Proposition 3.12, there exists a nonnegative sequence $\eta_n \stackrel{n \to +\infty}{\longrightarrow} 0$ such that

$$\tilde{\varphi}(t + \eta_n, \theta_{\epsilon_n}\sigma)y_n \stackrel{n \to +\infty}{\longrightarrow} \tilde{\varphi}(t, \sigma)y.$$

But $\tilde{\varphi}(t + \eta_n, \theta_{\epsilon_n}\sigma)y_n = \tilde{\varphi}(t_n + \epsilon_n + \eta_n, \theta_{-t_n+t}\sigma)x_n$, and using item (ii) of Proposition 3.12 we know that $\tilde{\varphi}(t + \eta_n, \theta_{\epsilon_n}\sigma)y_n \stackrel{n \to +\infty}{\longrightarrow} x$. Therefore, $x = \tilde{\varphi}(t, \sigma)y \in \tilde{\varphi}(t, \sigma)(\tilde{\omega}(\hat{B}, \sigma) \setminus M)$.

Case 2: $y \in M$.

In this case, using item (iii) of Proposition 3.12 and Lemma 3.22, we obtain a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ such that $\gamma_k = \phi(y_{n_k}, \theta_{\epsilon_{n_k}}\sigma) \stackrel{k \to +\infty}{\longrightarrow} 0$ and

$$z_k^+ \doteq \tilde{\varphi}(\gamma_k, \theta_{\epsilon_{n_k}}\sigma)y_{n_k} \stackrel{k \to +\infty}{\longrightarrow} I(y) \doteq z \in \tilde{\omega}(\hat{B}, \sigma) \setminus M.$$

Now, by item (i) of Proposition 3.12, there exists a non-negative sequence $\alpha_k \stackrel{k \to +\infty}{\longrightarrow} 0$ such that

$$\tilde{\varphi}(t + \alpha_k, \theta_{\gamma_k+\epsilon_{n_k}}\sigma)z_k^+ \stackrel{k \to +\infty}{\longrightarrow} \tilde{\varphi}(t, \sigma)z.$$

But $\tilde{\varphi}(t + \alpha_k, \theta_{\gamma_k + \epsilon_k}\sigma)z_k^+ = \tilde{\varphi}(t_{n_k} + \epsilon_{n_k} + \gamma_k + \alpha_k, \theta_{-t_{n_k}+t}\sigma)x_{n_k}$ and again, using item (ii) of Proposition 3.12, we have $\tilde{\varphi}(t + \alpha_k, \theta_{\gamma_k + \epsilon_{n_k}}\sigma)z_{n_k}^+ \overset{k\to+\infty}{\longrightarrow} x$. Therefore,

$$x = \tilde{\varphi}(t, \sigma)z \in \tilde{\varphi}(t, \sigma)(\tilde{\omega}(\hat{B}, \sigma) \setminus M). \qquad \square$$

Now we present a result which guarantees the existence of an impulsive cocycle attractor.

Theorem 3.24 ([4, Theorem 5.1]). *Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS pullback $\mathfrak{D}$-asymptotically compact such that $I(M) \cap M = \varnothing$ and every point from M satisfies φ-SSTC. Assume that there exists a pullback $(\tilde{\varphi}, \mathfrak{D})$-absorbing nonautonomous set $\hat{K} \in \mathfrak{D}$. Then, the nonautonomous set $\hat{A}$ defined by*

$$A(\sigma) = \tilde{\omega}(\hat{K}, \sigma)$$

is a $\mathfrak{D}$-impulsive cocycle attractor for the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$.

Proof. By Proposition 3.20 we have $\hat{A}$ is nonempty, compact and pullback $\mathfrak{D}$-attracts $\hat{K}$. The invariance of $\hat{A} \setminus M$ follows from Proposition 3.21 and Proposition 3.23. Since $\hat{K}$ is pullback $(\tilde{\varphi}, \mathfrak{D})$-absorbing then we deduce that $\hat{K}$ is $(\tilde{\varphi}, \mathfrak{D})$-pullback attracting. Suppose there exists a nonautonomous closed set $\hat{C}$ that pullback $\mathfrak{D}$-attracts every nonautonomous set $\hat{B} \in \mathfrak{D}$. Since $\tilde{\omega}(\hat{B}) \setminus M$ is $\tilde{\varphi}$-invariant, we have

$$d_H(\tilde{\omega}(\hat{B}, \sigma) \setminus M, C(\sigma)) = d_H(\tilde{\varphi}(t, \theta_{-t}\sigma)\tilde{\omega}(\hat{B}, \theta_{-t}\sigma) \setminus M, C(\sigma)) \overset{t\to+\infty}{\longrightarrow} 0,$$

that is, $\tilde{\omega}(\hat{B}, \sigma) \setminus M \subseteq C(\sigma)$, for every $\hat{B} \in \mathfrak{D}$ and $\sigma \in \Sigma$.

Now, let $x \in \tilde{\omega}(\hat{B}, \sigma) \cap M$. Then there exist sequences $\{t_n\}_{n\in\mathbb{N}}, \{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $x_n \in B(\theta_{-t_n}\sigma)$ with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, $\epsilon_n \overset{n\to+\infty}{\longrightarrow} 0$ such that $\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \overset{n\to+\infty}{\longrightarrow} x$. Let $z_n \doteq \tilde{\varphi}(t_n, \theta_{-t_n}\sigma)x_n$, $n \in \mathbb{N}$. We may assume that $z_n \overset{n\to+\infty}{\longrightarrow} z \in \tilde{\omega}(\hat{B}, \sigma)$. By items (ii) and (iii) of Proposition 3.12, we have (possibly, taking subsequences) either

(1) $\tilde{\varphi}(\epsilon_n, \sigma)z_n \overset{n\to+\infty}{\longrightarrow} z$ or

(2) $\tilde{\varphi}(\epsilon_n, \sigma)z_n \overset{n\to+\infty}{\longrightarrow} I(z),$

and since $\tilde{\varphi}(\epsilon_n, \sigma)z_n = \tilde{\varphi}(t_n + \epsilon_n, \theta_{-t}\sigma)x_n \overset{n\to+\infty}{\longrightarrow} x$ and $I(M) \cap M = \varnothing$, item (2) cannot happen and we must have $z = x \in M$ and $\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)x_n \overset{n\to+\infty}{\longrightarrow} x$. Since $\hat{C}$ pullback $\mathfrak{D}$-attracts nonautonomous sets, we have $x \in C(\sigma)$. Then $\tilde{\omega}(\hat{B}, \sigma) \subseteq C(\sigma)$ for every $\hat{B} \in \mathfrak{D}$ and $\sigma \in \Sigma$, which implies in particular that $\tilde{\omega}(\hat{K}, \sigma) \subset C(\sigma)$ and therefore $A(\sigma) \subseteq C(\sigma)$ and ends the proof. $\qquad \square$

Remark 3.25. With Definition 3.13, if $\hat{A}$ exists, it is uniquely determined.

To finish this section, we state an important characterization of the impulsive cocycle attractor.

Definition 3.26. We say that a function $\psi \colon \mathbb{R} \to X$ is a **global solution** of $\tilde{\varphi}$ at σ if

$$\tilde{\varphi}(t - s, \theta_s\sigma)\psi(s) = \psi(t) \quad \text{for all } t \geqslant s, \ s \in \mathbb{R}.$$

Moreover, if $\psi(0) = x$ we say that ψ is a **global solution through** x. We say that a global solution is **bounded** if $\psi(\mathbb{R})$ is a bounded subset of X.

Proposition 3.27 ([4, Proposition 5.5]). *At light of Definition 3.13, if the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ has an impulsive cocycle attractor $\hat{A} \in \mathfrak{D}$ with universe $\mathfrak{D}$ consisting of all nonautonomous sets $\hat{B}$ such that $\bigcup_{\sigma \in \Sigma} B(\sigma)$ is bounded in X and $I(M) \cap M = \varnothing$, then*

$$A(\sigma) \setminus M = \{x \in X \colon \psi \text{ is a bounded global solution of } \tilde{\varphi} \text{ at } \sigma \text{ through } x\}.$$

Proof. If $\psi(\cdot)$ is a bounded global solution of $\tilde{\varphi}$ at σ through x then $\psi(\mathbb{R}) \cap M = \varnothing$, since if $\psi(t_0) \in M$ for some $t_0 \in \mathbb{R}$ then $\tilde{\varphi}(t_0 - s, \theta_s \sigma)\psi(s) = \psi(t_0) \in M$ for each s such that $t_0 - s > 0$ which cannot happen (the impulsive cocycle from x cannot reach M in positive time for any $x \in X$, because $I(M) \cap M = \varnothing$). Hence, $\psi(\mathbb{R}) \cap M = \varnothing$. By its invariance we can see that $x \in A(\sigma)$ and therefore, $x \in A(\sigma) \setminus M$.

For the reverse inclusion, if $x \in A(\sigma) \setminus M$ then $x \in \tilde{\varphi}(1, \theta_{-1}\sigma)A(\theta_{-1}\sigma)$ and there exists $x_{-1} \in A(\theta_{-1}\sigma)$ such that $\tilde{\varphi}(1, \theta_{-1}\sigma)x_{-1} = x$. Again, since $x_{-1} \in A(\theta_{-1}\sigma)$ there exists $x_{-2} \in A(\theta_{-2}\sigma)$ such that $\tilde{\varphi}(1, \theta_{-2}\sigma)x_{-2} = x_{-1}$. Inductively, we can construct a sequence $\{x_{-n}\}_{n \in \mathbb{N}}$ such that $\tilde{\varphi}(1, \theta_{-n-1}\sigma)x_{-n-1} = x_{-n}$ for all $n \geqslant 0$, with $x_0 = x$. Then we can define

$$\psi(t) = \begin{cases} \tilde{\varphi}(t + n, \theta_{-n}\sigma)x_{-n}, & \text{if } t \in [-n, -n+1], \ n \in \mathbb{N}, \\ \tilde{\varphi}(t, \sigma)x_0, & \text{if } t \geqslant 0. \end{cases}$$

Since $\hat{A} \in \mathfrak{D}$, it is clear that this global solution is bounded and completes the proof. $\qquad \square$

3.3 Nonautonomous 2D-Navier–Stokes equations with impulses

Here we present an example to illustrate the impulsive nonautonomous theory. A more detailed description can be found in [4, Section 6].

The Navier–Stokes equations model fluid flow and are obtained used the conservation of linear momentum, which is

$$u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = g(t), \tag{3.2}$$

together with an incompressibility condition

$$\nabla \cdot u = 0,$$

where $u(t, x)$ denotes the vector velocity, $\nu > 0$ is the kinematic viscosity, g is a body force and p is a scalar pressure.

Now, we will consider this model with impulses in the state space, that can be imagined as forced changes on the vector velocity of the fluid, in order to prevent problems that may occur when the fluid reaches certain speeds. One can modify this model a little and add a component $v(t, x)$ for the position of the fluid, and we could imagine impulses as a way to avoid barriers and obstacles along the fluid trajectory.

We use here the approached adopted in [15], and we treat this problem in $\Omega = [0, 2\pi]^2$ (a periodic domain) and we require zero total momentum, that is, if $\int_\Omega u_0 = 0$ and $\int_\Omega g(t) = 0$ for all $t \geqslant 0$, then $\int_\Omega u(t) = 0$ for all $t \geqslant 0$.

Writing $\dot{\mathbb{Z}}^2 = \mathbb{Z}^2 \setminus \{0, 0\}$, let $\dot{H}^s$ be the subspace of the Sobolev space H^s which consists of all divergence-free, zero average, periodic real functions

$$\dot{H}^s \doteq \left\{ u = \sum_{k \in \dot{\mathbb{Z}}^2} \hat{u}_k e^{ik \cdot x} \colon \hat{u}_k = \overline{\hat{u}_k}, \ \sum_{k \in \dot{\mathbb{Z}}^2} |k|^{2s} |\hat{u}_k|^2 < \infty, \ k \cdot \hat{u}_k = 0 \right\},$$

with the norm

$$\|u\|_s^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2s} |\hat{u}_k|^2.$$

We have that $H \doteq \dot{H}^0$ is the natural phase space for this problem, and we write $\|\cdot\|$ for the norm in H (the usual L^2-norm). Remember also that the space of divergence-free functions is perpendicular (in $L^2(\Omega)$) to the space of gradients, since integrating by parts, we have

$$\int_\Omega u \cdot \nabla p = -\int_\Omega (\nabla \cdot u)p = 0,$$

and so we use the Leray projector P, which is the orthogonal projection of $L^2(\Omega)$ into the space os divergence-free fields. Applying this projector to (3.2) we obtain

$$\frac{du}{dt} + \nu Au + B(u,u) = f(t), \tag{3.3}$$

where $A = -P\Delta$ is the Stokes operator, $B(u,u) = P[(u \cdot \nabla)u]$ and $f(t) = Pg(t)$. In the periodic case, we have that $Au = -\Delta Pu$ and so $Au = -\Delta u$, for $u \in \dot{H}^s$.

We define the fractional power $A^{s/2}$ of A by $D(A^{s/2}) = \dot{H}^s$ and

$$A^{s/2}\left(\sum_{k \in \dot{\mathbb{Z}}^2} \hat{u}_k e^{ik \cdot x}\right) = \sum_{k \in \dot{\mathbb{Z}}^2} |k|^s \hat{u}_k e^{ik \cdot x}.$$

We note that the norms $\|\cdot\|_1$ and the norm $\|A^{1/2} \cdot\|$ are equivalent, and also that $\dot{H}^1$ is compactly embedded in H. Also, we denote the dual space of $\dot{H}^1$ by H^{-1}.

A simple integration by parts leads to the following antisymmetric identity

$$(B(u,v),w) = -(B(u,w),v),$$

which implies in particular that

$$(B(u,v),v) = 0. \tag{3.4}$$

Also, with a little more effort and using the incompressibility condition, one can prove that in the two-dimensional periodic case, we have

$$(B(u,u),Au) = 0.$$

Then, we can summarize the results of [15, Section 11.1] in the next proposition.

Proposition 3.28. *Assume that $\|f(t)\| \leqslant \alpha$ for all $t \geqslant 0$, then we have:*

(i) *equation (3.3) defines a nonautonomous dynamical system $(\varphi, \theta)_{(H, \mathbb{R})}$, where $\theta_t s = t + s$ for all $t \geqslant 0$ and $s \in \mathbb{R}$ and*

$$\varphi(t,s)u_0 = u(t+s,s,u_0)$$

is the unique solution in H of (3.3), with $u(s,s,u_0) = u_0 \in H$;

(ii) *$\varphi(\cdot,s)u_0 \in L^\infty(0,T;H) \cap L^2(0,T;D(A^{1/2}))$ and $\varphi_t(\cdot,s)u_0 \in L^2(0,T;D(A^{-1/2}))$ for every $T > 0$;*

(iii) *for $u_0 \in H$ and $s \in \mathbb{R}$*

$$\|\varphi(t,s)u_0\|^2 \leqslant e^{-\nu\lambda_1 t}\|u_0\|^2 + \frac{\alpha^2}{\nu^2\lambda_1^2}, \quad \text{for all } t \geqslant 0,$$

where λ_1 is the first eigenvalue of A.

Now we assume that M is an impulsive set in H for $(\varphi, \theta)_{(H, \mathbb{R})}$, and assume that every point of M satisfies $\boldsymbol{\varphi}$**-SSTC**. Also, let $I : M \to H$ be an impulsive function such that

(H1) $I(M) \cap M = \varnothing$;

(H2) $\|I(v)\|^2 \leqslant \mu$, for all $v \in M$.

(H3) Assume that there exists $\xi > 0$ such that $\phi(v, s) \geqslant 2\xi$, for all $v \in I(M)$ and $s \in \mathbb{R}$.

Let $\tilde{\varphi}(t, s)u_0$ be the associated impulsive solution of

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u, u) = f(t), \\ u(0) = u_0 \in H, \\ I : M \to H. \end{cases} \tag{3.5}$$

We assume that $\|f(t)\| \leq \alpha$ for all $t \geq 0$.

Now we summarize some results (see [4, Section 6]) which are useful to obtain an existence result of impulsive cocycle attractor for this example.

Proposition 3.29.

   (i) *([4, Lemma 6.2]) For each $t > 0$ and $s \in \mathbb{R}$, the map $\varphi(t, s) : H \to H$ is compact.*

  (ii) *([4, Lemma 6.3]) We have $\|\tilde{\varphi}(t, s)u_0\|^2 \leqslant \mu + \frac{\alpha^2}{\nu^2 \lambda_1^2}$, for all $u_0 \in I(M)$, $t \geqslant 0$ and $s \in \mathbb{R}$.*

 (iii) *([4, Proposition 6.4]) If $B \subset H$ is a bounded subset then there exists $t_0 = t_0(B) \geqslant 0$ such that $\|\tilde{\varphi}(t, s)u_0\|^2 \leqslant \mu + \frac{\alpha^2}{\nu^2 \lambda_1^2}$, if $t \geqslant t_0$, for all $u_0 \in B$ and $s \in \mathbb{R}$.*

 (iv) *([4, Lemma 6.5]) If G is a precompact subset of H and $\tau \in [0, \xi)$, then $\tilde{\varphi}(\tau, s)G$ is precompact in H for each $s \in \mathbb{R}$.*

Using the results in Proposition 3.29, we can construct a compact nonautonomous set $\hat{K} = \{K(s)\}_{s \in \mathbb{R}}$ which $\tilde{\varphi}$-pullback absorbs all bounded subsets of H. We will reproduce its proof here.

Theorem 3.30 ([4, Theorem 6.6]). *There exists a compact nonautonomous set $\hat{K} = \{K(s)\}_{s \in \mathbb{R}}$ which $\tilde{\varphi}$-pullback absorbs all nonautonomous sets $\hat{D}$ with $\bigcup_{s \in \mathbb{R}} D(s)$ bounded in H, and such that $\bigcup_{s \in \mathbb{R}} K(s)$ is bounded in H.*

Proof. Let $B_0 = \left\{ u \in H : \|u\|^2 \leqslant \mu + \frac{\alpha^2}{\nu^2 \lambda_1^2} \right\}$. Firstly, we fix $\tau \in (\xi, 2\xi)$. We claim that $G(s) = \tilde{\varphi}(\tau, \theta_{-\tau}s)B_0$ is precompact for each $s \in \mathbb{R}$. Indeed, we can write $B_0 = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{u \in B_0 : \phi(u, \theta_{-\tau}s) \geqslant 2\xi\}, \qquad C_2 = \{u \in B_0 : \xi < \phi(u, \theta_{-\tau}s) \leqslant 2\xi\} \quad \text{and}$$
$$C_3 = \{u \in B_0 : \phi(u, \theta_{-\tau}s) \leqslant \xi\}.$$

Then we have

$$G(s) = \varphi(\tau, \theta_{-\tau}s) C_1 \cup \tilde{\varphi}(\tau - \xi, \theta_{-\tau+\xi}s) \varphi(\xi, \theta_{-\tau}s) C_2 \cup \varphi(\tau - \xi, \theta_{-\tau+\xi}s) \tilde{\varphi}(\xi, \theta_{-\tau}s) C_3,$$

since $\phi(v, s) \geqslant 2\xi$ for all $v \in I(M)$ and $s \in \mathbb{R}$, and $\tau - \xi \in (0, \xi)$.

By Proposition 3.29, since C_1 and $\tilde{\varphi}(\xi, \theta_{-\tau}s) C_3$ are bounded (see item (ii)), it follows that sets $\varphi(\tau, \theta_{-\tau}s) C_1$ and $\varphi(\tau - \xi, \theta_{-\tau+\xi}s) \tilde{\varphi}(\xi, \theta_{-\tau}s) C_3$ are precompact in H (see item (i)). Also,

since $\varphi\left(\xi, \theta_{-\tau}s\right)C_2$ is precompact in H, it follows that $\tilde{\varphi}\left(\tau - \xi, \theta_{-\tau+\xi}s\right)\varphi\left(\xi, \theta_{-\tau}s\right)C_2$ is also precompact in H (item (iv)).

Therefore, $K(s) \doteq \overline{G(s)}$ is compact in H, for each $s \in \mathbb{R}$. Clearly, we have that

$$\sup_{v \in K(s)} \|v\|^2 \leqslant \beta + \frac{\alpha^2}{\nu^2 \lambda_1^2},$$

where $\beta = \max\{\mu, L_0\}$ and $L_0 = \sup_{u \in B_0} \|u\|^2$.

Now it remains to prove that $\hat{K}$ $\tilde{\varphi}$-pullback absorbs nonautonomous bounded sets $\hat{D}$ with $\bigcup_{s \in \mathbb{R}} D(s)$ bounded in H. To this end, let $\hat{D}$ a nonautonomous set in H with $B \doteq \bigcup_{s \in \mathbb{R}} D(s)$ bounded in H and fix $s \in \mathbb{R}$.

We know, by item (iii) of Proposition 3.29, that there exists $t_0 = t_0(B) > 0$ such that

$$\tilde{\varphi}(t, \theta_{-t-\tau}s)B \subset B_0, \quad \text{for all } t \geqslant t_0.$$

Thus

$$\tilde{\varphi}(t + \tau, \theta_{-t-\tau}s)B = \tilde{\varphi}(\tau, \theta_{-\tau}s)\tilde{\varphi}(t, \theta_{-t-\tau}s)B \subset \tilde{\varphi}(\tau, \theta_{-\tau}s)B_0 \subset K(s),$$

which shows that if $t \geqslant t_0 + \tau$

$$\tilde{\varphi}(t, \theta_{-t}s)D(\theta_{-t}s) \subset \tilde{\varphi}(t, \theta_{-t}s)B \subset K(s),$$

and proves that $\hat{K}$ is a $\tilde{\varphi}$-pullback absorbs $\hat{D}$. $\qquad\square$

As a consequence of this last theorem we obtain that the INDS $[(\varphi, \theta)_{(H,\mathbb{R})}, M, I]$ defined by (3.5) has an impulsive cocycle attractor (see [4, Corollary 6.7]).

4　Conclusion, comments and future directions

In this survey paper we described the theories of impulsive dynamical systems in both autonomous and nonautonomous frameworks. In the first part of this survey we presented two different approaches to study the asymptotic dynamical behavior of autonomous systems, proposed by Bonotto and Demuner (see [6, 7]) and Bonotto *et al.* (see [5]), respectively. In [6, 7], the definition of *global attractors for impulsive autonomous dynamical systems* was first introduced, where the attractor is invariant, consists of a compact set which does not intersect the impulsive set M and attracts bounded sets. This definition is consistent with the notion of global attractors for semigroups (they coincide when $M = \varnothing$) and describes the asymptotic behavior of many impulsive dynamical systems. However, it is not suitable for a large class of impulsive dynamical systems. For example, when the global attractor is compact and is disjoint with the closed set M, the compactness of the global attractor implies a separation between them and hence the asymptotic behavior of the impulsive dynamical system is not qualitatively different from the asymptotic behavior of the original system without impulse (see, e.g., Example 2.11). Later in [5] the notion of *precompact global attractors* was introduced, where the global attractor can "touch" the impulsive set M, *i.e.*, the boundary of the global attractor can have points which belong to M. The simplicity of autonomous framework allows us to study various types of impulsive dynamical systems, along with many interesting new applications. In this survey we illustrated one of the three interesting applications presented in [5].

In the second part of this survey we described the recently developed theories of nonautonomous impulsive dynamical systems, with multiple lines of prospective research. In particular, we recalled the main results of our recent work [4], where we proposed the first approach in the nonautonomous theory to study impulsive dynamical systems. This is done by defining the notion of impulsive nonautonomous dynamical systems, in which the trajectories have to be defined in a careful manner to obtain their relationship with the associated impulsive skew-product semiflow (see Theorem 3.9). The main goal is to construct a proper notion of impulsive cocycle attractors and develop their existence. To this end, we introduced a different notion of omega limit set (see Definition 3.15), to overcome the difficulties encountered in proving the usual properties such as invariance and pullback attraction in the nonautonomous theory.

It is worth mentioning again that the theory of impulsive dynamical systems is still in the early stage of investigation and has many interesting topics to be discovered, especially in the nonautonomous framework. We have made an initial step toward establishing the modern theory of impulsive dynamical systems, by developing a definition of impulsive nonautonomous dynamical systems and presenting an existence result of impulsive cocycle attractor. Yet there are many other interesting and important problems along this direction to be investigated, even in the autonomous framework. For example, on the one hand, there are no studies to date on the semi-continuity and geometrical structures of attractors for impulsive dynamical systems, and on the other hand, there are not many examples from applications analyzed in a detailed way. The main reasons are the difficulties in order to check some of the hypotheses ensuring the generation of an impulsive system, as well as the conditions required for the existence of attractors. Therefore, this is a field to be explored in a more detailed way in the future and we plan to work on this direction. Another major research direction would be developing a set of analog theories for impulsive dynamical systems where the nonautonomous character involves uncertainty, i.e., noise. This leads to a framework for random impulsive dynamical systems, a brand new area of research.

Acknowledgements

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