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Degenerate parabolic equations appearing in atmospheric dispersion of pollutants

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Abstract. Linear and nonlinear degenerate abstract parabolic equations with variable coefficients are studied. Here the equation and boundary conditions are degenerated on all boundary and contain some parameters. The linear problem is considered on the moving domain. The separability properties of elliptic and parabolic problems in mixed $L_{\mathbf{p}}$ spaces are obtained. Moreover, the existence and uniqueness of optimal regular solution of mixed problem for nonlinear parabolic equation is established. Note that, these problems arise in fluid mechanics and environmental engineering.

Keywords: differential-operator equations, degenerate PDE, semigroups of operators, nonlinear problems, separable differential operators, positive operators in Banach spaces.

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1 Introduction

In this work, the boundary value problems (BVPs) for parameter dependent degenerate differential-operator equations (DOEs) are considered. Namely, equations and boundary conditions contain small parameters. These problems have numerous applications in PDE, pseudo DE, mechanics and environmental engineering. The BVP for DOEs have been studied extensively by many researchers (see e.g. [1, 3, 4, 7–10, 12–17, 19–26, 28, 29] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [3, 10, 14–16, 29]. The maximal regularity properties for DOEs have been studied e.g. in [1, 4, 11, 19–22, 24, 25, 28, 29]. DOEs in Banach space valued function class are investigated e.g. in [2, 4, 13, 14, 20, 23, 25, 28, 29]. Nonlinear DOEs are studied e.g. in [3, 20, 24, 25]. The Fredholm property of BVP for elliptic equations are studied e.g. in [2, 3, 7].

The main objective of the present paper is to discusse the initial and BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + B\Big(\big(t, x, u, D^{[1]} u\big)\Big) u = F\big(t, x, u, D^{[1]} u\big), \tag{1.1}$$

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where $a_k(x)$ are complex valued functions, B and F are nonlinear operators in a Banach space E and

$$D^{[1]}u = \left(\frac{\partial^{[1]}u}{\partial x_1}, \frac{\partial^{[1]}u}{\partial x_2}, \dots, \frac{\partial^{[1]}u}{\partial x_n}\right), \qquad x = (x_1, x_2, \dots, x_n) \in G = \prod_{k=1}^{n}(0, b_k),$$

$$D_k^{[i]}u = \frac{\partial^{[i]}u}{\partial x_k^i} = \left[x^{\alpha_k}(b_k - x_k)^{\beta_k}\frac{\partial}{\partial x_k}\right]^i u(x), \qquad 0 \le \alpha_k, \beta_k < 1.$$

First, we consider the BVP for the degenerate elliptic DOE with small parameters

$$\sum_{k=1}^{n} \varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2} + A(x)u + \lambda u + \sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} A_k(x)\frac{\partial^{[1]}u}{\partial x_k} = f(x), \tag{1.2}$$

where a_k are complex-valued functions, ε_k are small parameters, $A(x)$ and $A_k(x)$ are linear operators, λ is a complex parameter.

Namely we prove that, for $f \in L_{\mathbf{p}}(G; E)$, $|\arg \lambda| \le \varphi$, $0 < \varphi \le \pi$ and sufficiently large $|\lambda|$, problem (1.2) has a unique solution $u \in W_{\mathbf{p}}^{[2]}(G; E(A), E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{k}{2}} \varepsilon_k^{\frac{i}{2}} \left\|\frac{\partial^{[i]}u}{\partial x_k^i}\right\|_{L_{\mathbf{p}}(G;E)} + \|Au\|_{L_{\mathbf{p}}(G;E)} \le C\|f\|_{L_{\mathbf{p}}(G;E)}.$$

Especially, it is shown that the corresponding differential operator is positive and also is a generator of an analytic semigroup. Then by using this result, we prove the well-posedeness in $L_{\mathbf{p}}(G; E)$ to initial and BVP for the following degenerate abstract parabolic equation with parameters

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} \varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2} + A(x)u = f(x, t), \qquad t \in (0, T),\ x \in G. \tag{1.3}$$

Finally, via maximal regularity properties of (0.3) and contaction mapping argument we derive the existence and uniqueness of solution of the problem (1.1).

Note that, the equation and boundary conditions are degenerated on all edges of boundary G. Moreover, it happened with the different rate at both boundary edges.

In application, the system of degenerate nonlinear parabolic equations is presented. Particularly, we consider the system that serves as a model of systems used to describe photochemical generation and atmospheric dispersion of ozone and other pollutants. The model of the process is given by initial and BVP for the atmospheric reaction–advection–diffusion system having the form

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{3}\left[a_{ki}(x)\frac{\partial^{[2]}u_i}{\partial x_k^2} + b_{ki}(x)\frac{\partial^{[1]}}{\partial x_k}(u_i\omega_k)\right] + \sum_{k=1}^{3} d_k u_k + f_i(u) + g_i, \tag{1.4}$$

where

$$x \in G_3 = \{x = (x_1, x_2, x_3),\ 0 < x_k < b_k\},$$

$$u_i = u_i(x, t), \quad i, k = 1, 2, 3, \quad u = u(x, t) = (u_1, u_2, u_3), \quad t \in (0, T)$$

and the state variables u_i represent concentration densities of the chemical species involved in the photochemical reaction. The relevant chemistry of the chemical species involved in the photochemical reaction and appears in the nonlinear functions $f_i(u)$, with the terms g_i, representing elevated point sources, $a_{ki}(x), b_{ki}(x)$ are real-valued functions. The advection terms

$\omega = \omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$, describe transport from the velocity vector field of atmospheric currents or wind. In this direction the work [11] and references there can be mentioned. The existence and uniqueness of solution of the problem (1.4) is established by the theoretic-operator method, i.e., this problem reduced to degenerate differential-operator equation.

Modern analysis methods, particularly abstract harmonic analysis, the operator theory, interpolation of Banach spaces, semigroups of linear operators, microlocal analysis, embedding and trace theorems in vector-valued Sobolev–Lions spaces are the main tools implemented to carry out the analysis.

2 Notations, definitions and background

Let $\gamma = \gamma(x)$ be a positive measurable function on $\Omega \subset R^n$ and E be a Banach space. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable E-valued functions defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x) dx\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. $L_{\mathbf{p},\gamma}(G; E)$, $G = \prod_{k=1}^{n}(0, b_k)$ will denote the space of all E-valued $\mathbf{p}$-summable functions with mixed norm, i.e., the space of all measurable functions f defined on G equipped with norm

$$\|f\|_{L_{\mathbf{p},\gamma}(G;E)} = \left(\left(\int_0^{b_n}\left(\cdots\int_0^{b_2}\left(\int_0^{b_1}\|f(x)\|_E^{p_1}\gamma(x)dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}}\cdots\right)^{\frac{p_n}{p_{n-1}}} dx_n\right)^{\frac{1}{p_n}} < \infty.$$

For $\gamma(x) \equiv 1$ we will denote these spaces by $L_p(\Omega; E)$ and $L_{\mathbf{p}}(G; E)$, respectively (see e.g. [5] for $E = \mathbb{C}$).

The Banach space E is called an *UMD*-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (see e.g. [6]). *UMD* spaces include e.g. L_p, l_p spaces, Lorentz spaces L_{pq} and Lorentz–Morrey spaces $R^{p,q,\lambda}$, when $p, q \in (1, \infty)$, $\lambda \in [0, n)$ [18].

Let E_1 and E_2 be two Banach spaces continuously embedding in a locally convex space. By $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \le p \le \infty$ we will denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the K-method [27, §1.3.2].

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embeds into E. Let us consider the Sobolev–Lions-type space $W_{p,\gamma}^m(a, b; E_0, E)$, consisting of all functions $u \in L_{p,\gamma}(a, b; E_0)$ that have generalized derivatives $u^{(m)} \in L_{p,\gamma}(a, b; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^m} = \|u\|_{W_{p,\gamma}^m(a,b;E_0,E)} = \|u\|_{L_{p,\gamma}(a,b;E_0)} + \left\|u^{(m)}\right\|_{L_{p,\gamma}(a,b;E)} < \infty.$$

Let

$$W_{p,\gamma}^{[m]} = W_{p,\gamma}^{[m]}(0, 1; E_0, E) = \left\{u : u \in L_p(0, 1; E_0),\ u^{[m]} \in L_p(0, 1; E), \|u\|_{W_{p,\gamma}^{[m]}} = \|u\|_{L_p(0,1;E_0)} + \left\|u^{[m]}\right\|_{L_p(0,1;E)} < \infty\right\}.$$

Now, let we define E-valued Sobolev–Lions-type spaces with mixed $L_{\mathbf{p}}$ and $L_{\mathbf{p},\gamma}$ norms. Let

$$\alpha_k(x) = x^{\alpha_{1k}}(b_k - x_k)^{\alpha_{2k}}, \quad \alpha = (\alpha_1, \alpha_2,), \quad \mathbf{p} = (p_1, p_2, \ldots, p_n).$$

Consider E-valued weighted space defined by

$$W^{[m]}_{\mathbf{p},\alpha}(G, E(A), E) = \left\{ u; u \in L_{\mathbf{p}}(G; E_0),\ \tfrac{\partial^{[m]} u}{\partial x_k^m} \in L_{\mathbf{p}}(G; E),\ \|u\|_{W^{[m]}_{\mathbf{p},\alpha}} = \|u\|_{L_{\mathbf{p}}(G;E_0)} + \sum_{k=1}^{n} \left\| \tfrac{\partial^{[m]} u}{\partial x_k^m} \right\|_{L_{\mathbf{p}}(G;E)} < \infty \right\}.$$

Let ε_k be small parameters and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. We denote by $W^m_{\mathbf{p},\gamma}(\Omega; E_0, E)$ the space of all functions $u \in L_{\mathbf{p},\gamma}(\Omega; E_0)$ possessing generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L_{\mathbf{p},\gamma}(\Omega; E)$ with the parametrized norm

$$\|u\|_{W^m_{\mathbf{p},\gamma,\varepsilon}(\Omega;E_0,E)} = \|u\|_{L_{\mathbf{p},\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \varepsilon_k \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L_{\mathbf{p},\gamma}(\Omega;E)} < \infty.$$

For definition of R-sectorial operator see e.g. [7, p. 39]

In a similar way as in [21, Theorems 2.3, 2.4] we have the following result.

Theorem 2.1. *Assume the following conditions be satisfied:*

(1) $\gamma = \gamma(x)$ is a weight function defined on domain $\Omega \subset R^n$ satisfying A_p condition;

(2) E is a UMD space, A is a R-sectorial operator in E and $p_k \in (1, \infty)$; $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$;

(3) there exists a bounded linear extension operator from $W^m_{\mathbf{p},\gamma}(\Omega; E(A), E)$ to $W^m_{\mathbf{p},\gamma}(R^n; E(A), E)$.

Then, the embedding

$$D^\beta W^m_{\mathbf{p},\gamma}(\Omega; E(A), E) \subset L_{\mathbf{p},\gamma}\left(\Omega; E\left(A^{1-\frac{|\beta|}{m}-\mu}\right)\right)$$

is continuous and for $0 \le \mu \le 1 - \frac{|\beta|}{m}$, $0 < h \le h_0 < \infty$ the following uniform estimate holds

$$\prod_{k=1}^{n} \varepsilon_k^{\frac{|\beta|}{m}} \|D^\alpha u\|_{L_{\mathbf{p},\gamma}\left(\Omega; E\left(A^{1-\varkappa-\mu}\right)\right)} \le h^\mu \|u\|_{W^m_{\mathbf{p},\gamma,\varepsilon}(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_{\mathbf{p},\gamma}(\Omega;E)}$$

for all $u \in W^m_{\mathbf{p},\gamma}(\Omega; E(A), E)$.

Consider the BVP for the degenerate ordinary DOE with parameter

$$Lu = \varepsilon a(x) u^{[2]}(x) + (A(x) + \lambda) u(x) = f, \tag{2.1}$$

$$L_1 u = \sum_{i=0}^{m_1} \varepsilon^{\sigma_i} \delta_i u^{[i]}(0) = 0, \qquad L_2 u = \sum_{i=0}^{m_2} \varepsilon^{\sigma_i} \beta_i u^{[i]}(1) = 0, \qquad x \in (0, 1), \tag{2.2}$$

where $u^{[i]} = \left[x^{\gamma_1}(1-x)^{\gamma_2} \frac{d}{dx}\right]^i u(x)$, $0 \le \gamma_k < 1$, $\sigma_i = \frac{i}{2} + \frac{1}{2p(1-\gamma_0)}$, $\gamma_0 = \min\{\gamma_1, \gamma_2\}$, $m_k \in \{0, 1\}$, δ_i, β_i are complex numbers; $A(x)$ is a linear operator in a Banach space E for $x \in (0, 1)$, ε is a small positive and λ is a complex parameter.

We suppose $\delta_{m_1} \ne 0$, $\beta_{m_1} \ne 0$ and

$$\int_0^x z^{-\gamma_1}(1-z)^{-\gamma_2} dz < \infty.$$

Consider the operator B_ε generated by problem (1.1)–(1.2) for $\lambda = 0$, i.e.,

$$\begin{aligned} D(B_\varepsilon) &= W^{[2]}_{p,\gamma}(0,1;E(A),E,L_k) \\ &= \left\{u : u \in W^{[2]}_{p,\gamma}(0,1;E(A),E),\ L_k u = 0,\ k = 1,2\right\}, \\ B_\varepsilon u &= -\varepsilon a(x) u^{[2]} + A(x)u. \end{aligned}$$

Condition 2.2. *Assume the following conditions are satisfied:*

(1) E is a UMD space and $\gamma(x) = x^{\gamma_1}(1-x)^{\gamma_2}$, $0 \le \gamma_k < 1 - \frac{1}{p}$, $1 < p < \infty$, $a \in C([0,1])$ and $a(x) < 0$ for $x \in (0,1)$;

(2) A is a R positive operator in E and $A(x)A^{-1}(x_0) \in C([0,1];B(E))$ for x, $x_0 \in (0,1)$.

By reasoning as in [21, Theorem 5.1] and by using the method used in [22, Theorem 1] we get the following theorem.

Theorem 2.3. *Assume that Condition 2.2 holds. Then problem (2.1) has a unique solution $u \in W^{[2]}_{p,\gamma}(0,1;E(A),E)$ for $f \in L_p(0,1;E)$ and $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds*

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\| u^{[i]} \right\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \le C\|f\|_{L_p(0,1;E)}.$$

In a similar way as in [25, Theorem 3.1] we obtain the following theorem.

Theorem 2.4. *Suppose the Condition 2.2 is satisfied. Then, the operator B_ε is uniformly R-positive in $L_p(0,1;E)$.*

3 Degenerate elliptic equations with parameters

Consider the BVP for the following degenerate partial DOE with parameters

$$\sum_{k=1}^{n} \varepsilon_k a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x)u + \lambda u + \sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} A_k(x) \frac{\partial^{[1]} u}{\partial x_k} = f(x), \tag{3.1}$$

$$L_{k1} u = \sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}} \delta_{ki} u^{[i]}_{x_k}(G_{k0}) = 0, \qquad L_{k2} u = \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}} \beta_{ki} u^{[i]}_k(G_{kb}) = 0,$$

for $x^{(k)} \in G_k$, where $A(x)$ and $A_k(x)$ are linear operators, $u = u(x)$, ε_k are small parameters, δ_{ki}, β_{ki} are complex numbers, λ is a complex parameter, $m_{kj} \in \{0,1\}$ and

$$\frac{\partial^{[i]} u}{\partial x_k^i} = \left[x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}} \frac{\partial}{\partial x_k} \right]^i u(x), \qquad 0 \le \alpha_{1k}, \alpha_{2k} < 1,$$

$$\sigma_{ik} = \frac{i}{2} + \frac{1}{2p_k(1-\alpha_{0k})}, \qquad \alpha_{0k} = \min\{\alpha_{1k}, \alpha_{2k}\},$$

a_k are complex-valued functions and

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n) \in G = \prod_{k=1}^{n} (0, b_k), \\ G_{k0} &= (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n), \qquad p_k \in (1, \infty), \\ G_{kb} &= (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n), \\ x^{(k)} &= (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in G_k = \prod_{j \neq k} (0, b_j). \end{aligned}$$

Let

$$\alpha = \alpha(x) = \prod_{k=1}^{n} x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}}.$$

Remark 3.1. Under the substitutions

$$\tau_k = \int_0^{x_k} x_k^{-\alpha_k} (b_k - x_k)^{-\alpha_k} dx_k, \qquad k = 1, 2, \dots, n$$

the spaces $L_p(G; E)$ and $W^{[2]}_{p,\alpha}(G; E(A), E)$ are mapped isomorphically onto the weighted spaces $L_{p,\tilde{\alpha}}(\tilde{G}; E)$ and $W^2_{p,\tilde{\alpha}}(\tilde{G}; E(A), E)$, respectively, where

$$\tilde{G} = \prod_{k=1}^{n} (0, \tilde{b}_k), \, \tilde{b}_k = \int_0^{b_k} x_k^{-\alpha_{1k}} (b_k - x_k)^{-\alpha_{2k}} dx_k, \tilde{\alpha}(\tau) = \alpha(x_1(\tau_1), x_2(\tau_2), \dots, x_n(\tau_n)).$$

Consider the principal part of (3.1), i.e., consider the problem

$$\sum_{k=1}^{n} \varepsilon_k a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u + \lambda u = f(x), \tag{3.2}$$

$$\sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}} \delta_{ki} u_{x_k}^{[i]}(G_{k0}) = 0, \qquad \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}} \beta_{ki} u_k^{[i]}(G_{kb}) = 0.$$

Condition 3.2. *Assume*

1. *E is a UMD space,* $\gamma(x) = \prod_{k=1}^{n} x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}}$*, where* $0 \leq \alpha_{1k}, \alpha_{2k} < 1 - \frac{1}{p_k}$*,* $p_k \in (1, \infty)$*,* $\delta_{km_{k1}} \neq 0$*,* $\beta_{km_{k2}} \neq 0$*;*
2. $A(x)$ *is a uniformly R-positive operator in E,* $A(x) A^{-1}(\bar{x}) \in C(\bar{G}; L(E))$*,* $x \in G$*;*
3. $a_k(x) \in C^{(m)}(\bar{G})$ *and* $a_k(x_k) < 0$ *for* $x_k \in (0, b_k)$*.*

First, we prove the separability properties of the problem (3.2).

Theorem 3.3. *Assume that Condition 3.2 holds. Then problem (3.2) has a unique solution* $u \in W^{[2]}_{\mathbf{p},\alpha}(G; E(A), E)$ *for* $f \in L_{\mathbf{p}}(G; E)$*,* $|\arg \lambda| \leq \varphi$ *with sufficiently large* $|\lambda|$ *and the following coercive uniform estimate holds*

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1 - \frac{i}{2}} \varepsilon_k^{\frac{i}{2}} \left\| \frac{\partial^{[i]} u}{\partial x_k^i} \right\|_{L_{\mathbf{p}}(G;E)} + \|A u\|_{L_{\mathbf{p}}(G;E)} \leq C \|f\|_{L_{\mathbf{p}}(G;E)}. \tag{3.3}$$

Proof. Consider the BVP

$$(L+\lambda)u = a_1(x_1)\varepsilon_1 D_{x_1}^{[2]} u(x_1) + (A(x_1)+\lambda)u(x_1) = f(x_1), \tag{3.4}$$

$$L_{1j}u = 0, \qquad j = 1,2, \qquad x_1 \in (0,b_1),$$

where L_{1j} are boundary conditions of type (3.2) on $(0,b_1)$. By virtue of Theorem 2.3, problem (3.4) has a unique solution $u \in W^{[2]}_{p_1,\alpha_1}(0,b_1;E(A),E)$ for $f \in L_{p_1}(0,b_1;E)$, $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \varepsilon_1^{\frac{j}{2}} \left\| u^{[j]} \right\|_{L_{p_1}(0,b_1;E)} + \|Au\|_{L_{p_1}(0,b_1;E)} \le C\|f\|_{L_{p_1}(0,b_1;E)}.$$

Now, let us consider the following BVP

$$\sum_{k=1}^{2} \varepsilon_k a_k(x_k) D_k^{[2]} u(x_1,x_2) + A(x_1,x_2)u(x_1,x_2) + \lambda u(x_1,x_2) = f(x_1,x_2), \tag{3.5}$$

$$L_{k1}u = 0, \quad L_{k2}u = 0, \quad k = 1,2, \quad x_1,x_2 \in G_2 = (0,b_1)\times(0,b_2).$$

Let $\mathbf{p}_2 = (p_1,p_2)$ and $\alpha(2) = (\alpha_1,\alpha_2)$. Since $L_{p_2}(0,b_2;L_{p_1}(0,b_1);E) = L_{\mathbf{p}_2}(G_2;E)$, the BVP (3.5) can be expressed as

$$a_2\varepsilon_2 D_2^{[2]} u(x_2) + (B_{\varepsilon_1}(x_2)+\lambda)u(x_2) = f(x_2), \quad L_{2j}u = 0, \quad j = 1,2,$$

for $x_1 \in (0,b_1)$, where B_{ε_1} is a differential operator in $L_{p_1}(0,b_1;E)$ for $x_2 \in (0,b_2)$, generated by problem (3.4). By virtue of [3, Theorem 4.5.2], $L_{p_1}(0,b_1;E) \in UMD$ for $p_1 \in (1,\infty)$. Hence, by [28, Corollary 4.1] the space $L_{p_1}(0,b_1;E)$ satisfies the multiplier condition. Moreover, the Theorem 2.4 implies the uniform R-positivity of operator B_{ε_1}. Hence, by Theorem 2.3, problem (3.5) has a unique solution $u \in W^{[2]}_{\mathbf{p}_2,\alpha(2)}(G_2;E(A);E)$ for $f \in L_{\mathbf{p}_2}(G_2;E)$, $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$ and (3.3) holds for $n = 2$. By continuing this we obtain the assertion. □

Theorem 3.4. *Let the Condition 3.2 hold and let $A_k(x)A^{-(\frac{1}{2}-\nu)}(x) \in C(\bar{G};L(E))$ for $0 < \nu < \frac{1}{2}$. Then, problem* (2.1) *has a unique solution $u \in W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)$ for $f \in L_{\mathbf{p}}(G;E)$, $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds*

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon_k^{\frac{i}{2}} \left\| \frac{\partial^{[i]}u}{\partial x_k^i} \right\|_{L_{\mathbf{p}}(G;E)} + \|Au\|_{L_{\mathbf{p}}(G;E)} \le C\|f\|_{L_{\mathbf{p}}(G;E)}. \tag{3.6}$$

Proof. By assumption and by Theorem 2.1, for all $h > 0$ we have the following Ehrling–Nirenberg–Gagliardo-type estimate

$$\|L_1 u\|_{L_{\mathbf{p}}(G;E)} \le h^{\mu}\, \|u\|_{W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_{\mathbf{p}}(G;E)}. \tag{3.7}$$

Let $\mathbf{O}_\varepsilon$ denote the operator generated by the problem (3.2) and

$$L_1 u = \sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} A_k(x) \frac{\partial^{[1]}u}{\partial x_k}.$$

By using the estimate (3.7) we obtain that there is a $\delta \in (0,1)$ such that

$$\left\| L_1(\mathbf{O}_\varepsilon+\lambda)^{-1} \right\|_{B(X)} < \delta.$$

Hence, from perturbation theory of linear operators we obtain the assertion. □

4 Abstract Cauchy problem for degenerate parabolic equation with parameter

Consider the initial and BVP for degenerate parabolic equation with parameter:

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}\varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+du=f(x,t), \qquad t\in(0,T),\, x\in G. \tag{4.1}$$

$$\sum_{i=0}^{m_{k1}}\varepsilon_k^{\sigma_{ik}}\delta_{ki}u_{x_k}^{[i]}(G_{k0},t)=0, \qquad \sum_{i=0}^{m_{k2}}\varepsilon_k^{\sigma_{ik}}\beta_{ki}u_k^{[i]}(G_{kb},t)=0,$$

$$u(x,0)=0, \quad t\in(0,T), \quad x^{(k)}\in G_k, \tag{4.2}$$

where $u=u(x,t)$ is a solution, δ_{ki}, β_{ki} are complex numbers, ε_k are positive parameters, a_k are complex-valued functions on G, $A(x)$ is a linear operator in a Banach space E, domains G, G_k, G_{k0}, G_{kb}, σ_{ik} and $x^{(k)}$ are defined in Section 2 and

$$\frac{\partial^{[i]}u}{\partial x_k^i}=\left[x^{\alpha_{1k}}(b_k-x_k)^{\alpha_{2k}}\frac{\partial}{\partial x_k}\right]^i u(x,t), \qquad d>0.$$

For $\bar{\mathbf{p}}=(p_0,\mathbf{p})$, $\mathbf{p}=(p_1,p_2,\ldots,p_n)$, $G_T=(0,T)\times G$, $L_{\tilde{\mathbf{p}},\boldsymbol{\alpha}}(G_T;E)$ will denote the space of all E-valued weighted $\tilde{\mathbf{p}}$-summable functions with mixed norm.

Theorem 4.1. *Suppose the Condition 3.2 holds for $\varphi>\frac{\pi}{2}$. Then, for $f\in L_{\mathbf{p}}(G_T;E)$ and sufficiently large $d>0$ problem (4.1)–(4.2) has a unique solution belonging to $W_{\bar{\mathbf{p}},\alpha}^{1,[2]}(G_T;E(A),E)$ and the following coercive estimate holds*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\bar{\mathbf{p}}}(G_T;E)}+\sum_{k=1}^{2}\varepsilon_k\left\|\frac{\partial^{[2]}u}{\partial x_k^2}\right\|_{L_{\bar{\mathbf{p}}}(G_T;E)}+\|Au\|_{L_{\bar{\mathbf{p}}}(G_T;E)}\leq C\|f\|_{L_{\bar{\mathbf{p}}}(G_T;E)}.$$

Proof. The problem (4.1) can be expressed as the following abstract Cauchy problem

$$\frac{du}{dt}+(\mathbf{O}_\varepsilon+d)u(t)=f(t), \qquad u(0)=0. \tag{4.3}$$

From Theorems 2.4, 3.3 we get that $\mathbf{O}_\varepsilon$ is R-sectorial in $F=L_{\mathbf{p}}(G;E)$. By [18, §1.14], $\mathbf{O}_\varepsilon$ is a generator of an analytic semigroup in F. Then by virtue of [28, Theorem 4.2], problem (4.3) has a unique solution $u\in W_{p_0}^1(0,T;D(\mathbf{O}_\varepsilon),F)$ for $f\in L_{p_0}(0,T;F)$ and sufficiently large $d>0$. Moreover, the following uniform estimate holds

$$\left\|\frac{du}{dt}\right\|_{L_{p_0}(0,T;F)}+\|\mathbf{O}_\varepsilon u\|_{L_{p_0}(0,T;F)}\leq C\|f\|_{L_{p_0}(0,T;F)}.$$

Since $L_{p_0}(G_T;F)=L_{\bar{\mathbf{p}}}(G_T;E)$, by Theorem 3.3 we have

$$\|(\mathbf{O}_\varepsilon+d)u\|_{L_{p_0}((0,T);F)}=D(\mathbf{O}_\varepsilon).$$

Hence, the assertion follows from the above estimate. □

5 Degenerate parabolic DOE on the moving domain

Consider the degenerate problem (4.1)–(4.2) on the moving domain $G(s) = \prod_{k=1}^{n}(0, b_k(s))$:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x)u + du = f(x,t), \tag{5.1}$$

$$L_{k1}u = \sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}} \delta_{ki} u_{x_k}^{[i]}(G_{k0}(s), t) = 0, \qquad L_{k2}u = \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}} \beta_{ki} u_k^{[i]}(G_{kb}(s), t) = 0,$$

$$u(x,0) = 0, \qquad t \in (0,T), \quad x \in G(s), \tag{5.2}$$

where the end points $b_k(s)$ depend of a parameter s, $x_k \in (0, b_k(s))$ and $b_k(s)$ are positive continues function, $G_{k0}(s)$, $G_{kb}(s)$ are domains defined in Section 2, replacing $(0, b_k)$ by $(0, b_k(s))$ and

$$\sigma_{ik} = \frac{i}{2} + \frac{1}{2p(1-\alpha_{0k})}, \qquad \alpha_{0k} = \min\{\alpha_{1k}, \alpha_{2k}\},$$

$$\frac{\partial^{[i]} u}{\partial x_k^i} = \left[x^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}} \frac{\partial}{\partial x_k} \right]^i u(x,t).$$

Let

$$G_T = G_T(s) = (0,T) \times G(s).$$

Theorem 4.1 implies the following.

Proposition 5.1. *Assume the Condition 3.2 hold for $\varphi > \frac{\pi}{2}$. Then, problem (5.1)–(5.2) has a unique solution $u \in W_{\tilde{\mathbf{p}},\alpha}^{1,[2]}((G(s)); E(A), E)$ for $f \in L_{\mathbf{p}}(G_T(s); E)$ and sufficiently $d > 0$. Moreover, the following coercive uniform estimate holds*

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} + \sum_{k=1}^{2} \varepsilon_k \left\| \frac{\partial^{[2]} u}{\partial x_k^2} \right\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} + \|Au\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} \leq C \|f\|_{L_{\tilde{\mathbf{p}}}(G_T;E)}. \tag{5.3}$$

Proof. Under the substitution $\tau_k = x_k b_k(s)$ the problem (5.1)–(5.2) reduced to the following BVP in fixed domain G:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} b_k^{-2}(s) \tilde{a}_k(\tau_k) \frac{\partial^{[2]} u}{\partial \tau_k^2} + \tilde{A}(\tau)u = \tilde{f}(\tau,t), \qquad t \in R_+, \tau \in G. \tag{5.4}$$

$$\sum_{i=0}^{m_{k1}} b_k^{\sigma_{ik}}(s) \delta_{ki} u_{x_k}^{[i]}(G_{k0}, t) = 0, \qquad \sum_{i=0}^{m_{k2}} b_k^{\sigma_{ik}}(s) \beta_{ki} u_k^{[i]}(G_{kb}, t) = 0,$$

$$u(x,0) = 0, \quad t \in (0,T), \quad x \in G = \prod_{k=1}^{n}(0, b_k), \tag{5.5}$$

where

$$\tilde{a}_k(\tau) = a_k(x(\tau)), \qquad \tilde{A}(\tau) = A((x(\tau))), \qquad \tilde{f}(\tau) = f((x(\tau))),$$

$$x(\tau) = (x_1(\tau_1), x_2(\tau_2), \ldots, x_n(\tau_n)).$$

The problem (5.4)–(5.5), is a particular case of (4.1)–(4.2). So, by virtue of Theorem 4.1 we obtain the required assertion. □

6 Nonlinear degenerate abstract parabolic problem

In this section, we consider initial and BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+B\Big(\big(t,x,u,D^{[1]}u\big)\Big)u=F\big(t,x,u,D^{[1]}u\big), \tag{6.1}$$

$$\sum_{i=0}^{m_{k1}}\delta_{ki}u_{x_k}^{[i]}(G_{k0},t)=0,\qquad \sum_{i=0}^{m_{k2}}\beta_{ki}u_k^{[i]}(G_{kb},t)=0,$$
$$u(x,0)=0,\quad t\in(0,T),\quad x\in G,\quad x^{(k)}\in G_k, \tag{6.2}$$

where $u=u(x,t)$ is a solution, δ_{ki}, β_{ki} are complex numbers, a_k are complex-valued functions on G; domains G, G_k, G_{k0}, G_{kb} and σ_{ik}, $x^{(k)}$ are defined in Section 2 and

$$D_k^{[i]}u=\frac{\partial^{[i]}u}{\partial x_k^i}=\left[x_k^{\alpha_k}(b_k-x_k)^{\alpha_k}\frac{\partial}{\partial x_k}\right]^i u(x,t),\qquad 0\le\alpha_k<1.$$

Let $G_T=(0,T)\times G$, where $G=\prod_{k=1}^n(0,b_k)$. Moreover, let

$$b_k\in(0,b_{0k}),\ G_0=\prod_{k=1}^n(0,b_{0k}),T\in(0,T_0),$$
$$B_{ki}=\big(W^{2,p}(G_k,E(A),E),L^p(G_k;E)\big)_{\eta_{ik},p},$$
$$\eta_{ik}=\frac{m_{kj}+\frac{1}{p(1-\alpha_k)}}{2},\qquad B_0=\prod_{k=1}^n\prod_{i=0}^1 B_{ki}.$$

Remark 6.1. By virtue of [27, § 1.8.] and the Remark 3.1, operators $u\to\frac{\partial^{[i]}u}{\partial x_k^i}\big|_{x_{k=0}}$ are continuous from $W_{p,\alpha}^{[2]}(G;E(A),E)$ onto B_{ki} and there are the constants C_1 and C_0 such that for $w\in W_{p,\alpha}^{[2]}(G;E(A),E)$, $W=\{w_{ki}\}$, $w_{ki}=\frac{\partial^{[i]}w}{\partial x_k^i}$, $i=0,1$, $k=1,2,\ldots,n$

$$\left\|\frac{\partial^{[i]}w}{\partial x_k^i}\right\|_{B_{ki},\infty}=\sup_{x\in G}\left\|\frac{\partial^{[i]}w}{\partial x_k^i}\right\|_{B_{ki}}\le C_1\|w\|_{W_{p,\alpha}^{[2]}(G;E(A),E)},$$
$$\|W\|_{0,\infty}=\sup_{x\in G}\sum_{k,i}\|w_{ki}\|_{B_{ki}}\le C_0\|w\|_{W_{p,\alpha}^{[2]}(G;E(A),E)}.$$

Condition 6.2. *Suppose the following hold:*

(1) E is an UMD space and $0\le\alpha_1,\alpha_2<1-\frac{1}{p}$, $p\in(1,\infty)$;

(2) a_k are continuous functions on $\bar G$, $a_k(x)<0$, for all $x\in G$, $\delta_{km_{k1}}\ne0$, $\beta_{km_{k1}}\ne0$, $k=1,2,\ldots,n$;

(3) there exist $\Phi_{ki}\in B_{ki}$ such that the operator $B(t,x,\Phi)$ for $\Phi=\{\Phi_{ki}\}\in B_0$ is R-sectorial in E uniformly with respect to $x\in G_0$ and $t\in[0,T_0]$; moreover,

$$B(t,x,\Phi)B^{-1}\big(t^0,x^0,\Phi\big)\in C(\bar G;L(E)),\qquad t^0\in(0,T),\ x^0\in G;$$

(4) $A=B\big(t^0,x^0,\Phi\big)$: $G_T\times B_0\to L(E(A),E)$ is continuous. Moreover, for each positive r there is a positive constant $L(r)$ such that $\|[B(t,x,U)-B(t,x,\bar U)]v\|_E\le L(r)\|U-\bar U\|_{B_0}\|Av\|_E$ for $t\in(0,T)$, $x\in G$, $U,\bar U\in B_0$, $\bar U=\{\bar u_{ki}\}$, $\bar u_{ki}\in B_{ki}$, $\|U\|_{B_0}$, $\|\bar U\|_{B_0}\le r$, $v\in D(A)$;

(5) *the function $F : G_T \times B_0 \to E$ such that $F(\cdot, U)$ is measurable for each $U \in B_0$ and $F(t, x, \cdot)$ is continuous for a.a. $t \in (0, T)$, $x \in G$. Moreover, $\|F(t, x, U) - F(t, x, \bar{U})\|_E \leq \Psi_r(x)\|U - \bar{U}\|_{B_0}$ for a.a. $t \in (0, T)$, $x \in G$, $U, \bar{U} \in B_0$ and $\|U\|_{B_0}, \|\bar{U}\|_{B_0} \leq r$; $f(\cdot) = F(\cdot, 0) \in L_p(G_T; E)$.*

The main result of this section is the following.

Theorem 6.3. *Let the Condition 6.2 be satisfied. Then there is $T \in (0, T_0)$ and $b_k \in (0, b_{0k})$ such that problem (6.1)–(6.2) has a unique solution belonging to $W^{1,[2]}_{p,\alpha}(G_T; E(A), E)$.*

Proof. Consider the following linear problem

$$\frac{\partial w}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} w}{\partial x_k^2} + du = f(x, t), \qquad x \in G,\ t \in (0, T),$$

$$\sum_{i=0}^{m_{k1}} \delta_{ki} w^{[i]}_{x_k}(G_{k0}, t) = 0, \qquad \sum_{i=0}^{m_{k2}} \beta_{ki} w^{[i]}_{k}(G_{kb}, t) = 0, \tag{6.3}$$

$$w(x, 0) = 0, \quad t \in (0, T), \quad x \in G, \quad x^{(k)} \in G_k, \quad d > 0. \qquad \square$$

By Theorem 4.1 and in view of Proposition 5.1 there exists a unique solution $w \in W^{1,[2]}_{p,\alpha}(G_T; E(A), E)$ of the problem (6.3) for $f \in L_p(G_T; E)$ and sufficiently large $d > 0$ and it satisfies the following coercive estimate

$$\|w\|_{W^{1,[2]}_{p,\alpha}(G_T;E(A),E)} \leq C_0 \|f\|_{L_p(G_T;E)},$$

uniformly with respect to $b \in (0, b_0]$, i.e., the constant C_0 does not depends on $f \in L_p(G_T; E)$ and $b \in (0\ b_0]$ where

$$A(x) = B(x, 0), \quad f(x) = F(x, 0), \quad x \in (0, b).$$

We want to solve the problem (6.1)–(6.2) locally by means of maximal regularity of the linear problem (6.3) via the contraction mapping theorem. For this purpose, let w be a solution of the linear BVP (6.3). Consider a ball

$$B_r = \{v \in Y, v - w \in Y_1, \|v - w\|_Y \leq r\}.$$

For given $v \in B_r$, consider the following linearized problem

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) = F(x, V) + [B(x, 0) - B(x, V)]v,$$

$$\sum_{i=0}^{m_{k1}} \delta_{ki} w^{[i]}_{x_k}(G_{k0}, t) = 0, \qquad \sum_{i=0}^{m_{k2}} \beta_{ki} w^{[i]}_{k}(G_{kb}, t) = 0, \tag{6.4}$$

$$w(x, 0) = 0, \quad t \in (0, T), \quad x \in G, \quad x^{(k)} \in G_k.$$

where $V = \{v_{ki}\}$, $v_{ki} \in B_{ki}$. Define a map Q on B_r by $Qv = u$, where u is solution of (6.4). We want to show that $Q(B_r) \subset B_r$ and that Q is a contraction operator provided T and b_k are sufficiently small, and r is chosen properly. In view of separability properties of the problem (6.3) we have

$$\|Qv - w\|_Y = \|u - w\|_Y \leq C_0\{\|F(x, V) - F(x, 0)\|_X + \|[B(0, W) - B(x, V)]v\|_X\}.$$

By assumption (4) of condition 6.2 we have

$$\begin{aligned}
&\|[B(0,W)v - B(x,V)]v\|_X \\
&\leq \sup_{x\in[0,b]} \left\{ \|[B(0,W) - B(x,W)]v\|_{L(E_0,E)} + \|B(x,W) - B(x,V)\|_{L(E_0,E)} \|v\|_Y \right\} \\
&\leq \left[\delta(b) + L(R)\|W - V\|_{\infty,E_0} \right] [\|v - w\|_Y + \|w\|_Y] \\
&\leq \delta(b) + L(R)[C_1\|v - w\|_Y + \|v - w\|_Y][\|v - w\|_Y + \|w\|_Y] \\
&\leq \delta(b) + L(R)[C_1 r + r][r + \|w\|_Y],
\end{aligned}$$

where

$$\delta(b) = \sup_{x\in[0,b]} \|[B(0,W) - B(x,W)]\|_{B(E_0,E)}.$$

By assumption (5) of condition 6.2 we get

$$\begin{aligned}
\|F(x,V) - F(x,0,)\|_E &\leq \delta(b) + \|F(x,V) - F(x,W)\|_E + \|F(x,W) - F(x,0)\|_E \\
&\leq \delta(b) + \mu_R[\|v - w\|_Y + \|w\|_Y],
\end{aligned}$$

$$\mu_R C_1[\|v - w\|_Y + \|w\|_Y] \leq \mu_R[C_1 r + \|w\|_Y],$$

where $R = C_1 r + \|w\|_Y$ is a fixed number. In view of above estimates, by suitable choice of μ_R, L_R and for sufficiently small $T \in (0, T_0)$ and $b_k \in (0, b_{0k}]$ we have

$$\|Qv - w\|_Y \leq r,$$

i.e.

$$Q(B_r) \subset B_r.$$

Moreover, in a similar way we obtain

$$\begin{aligned}
&\|Qv - Q\bar{v}\|_Y \\
&\quad \leq C_0\{\mu_R C_1 + M_a + L(R)[\|v - w\|_Y + C_1 r] + L(R)C_1[r + \|w\|_Y]\|v - \bar{v}\|_Y\} + \delta(b).
\end{aligned}$$

By suitable choice of μ_R, L_R and for sufficiently small $T \in (0, T_0)$ and $b_k \in (0, b_{0k})$ we obtain $\|Qv - Q\bar{v}\|_Y < \eta\|v - \bar{v}\|_Y$, $\eta < 1$, i.e. Q is a contraction operator. Eventually, the contraction mapping principle implies a unique fixed point of Q in B_r which is the unique strong solution $u \in W^{1,[2]}_{p,\alpha}(G_T; E(A), E)$.

7 Cauchy problem for nonlinear system of degenerate parabolic equations

Consider the initial and BVP for the system of nonlinear parabolic equations

$$\frac{\partial u_m}{\partial t} = \sum_{k=1}^{n} a_k(x) \frac{\partial^{[2]} u_m}{\partial x_k^2} + \sum_{j=1}^{N} d_{mj}(x) u_j(x,t) + \sum_{k=1}^{n}\sum_{j=1}^{N} b_{kj}(x) \frac{\partial^{[1]} u_j}{\partial x_k} + F_m(x,t,u), \tag{7.1}$$

$$\sum_{i=0}^{m_{k1}} \delta_{ki} D_k^{[i]} u_m(G_{k0}, t) = 0, \qquad \sum_{i=0}^{m_{k2}} \beta_{ki} D_k^{[i]} u_m(G_{kb}, t) = 0,$$

$$u_m(x,0) = 0, \quad x \in G, \quad t \in (0,T), \quad m = 1,2,\ldots,N, \quad N \in \mathbb{N}, \tag{7.2}$$

where $u = (u_1, u_2, \ldots, u_N)$, $m_{kj} \in \{0,1\}$, δ_{ki}, β_{ki} are complex numbers, a_k are complex valued functions,

$$D_k^{[i]} u = \frac{\partial^{[i]} u}{\partial x_k^i} = \left[x_k^{\alpha_k} (b_k - x_k)^{\alpha_k} \frac{\partial}{\partial x_k} \right]^i u(x,t), \qquad 0 \le \alpha_k < 1,$$

$$x = (x_1, x_2, \ldots, x_n) \in G = \prod_{k=1}^{n} (0, b_k), \qquad m_{kj} \in \{0,1\},$$

$$G_{k0} = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n), \qquad q \in (1, \infty),$$

$$G_{kb} = (x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n);$$

and

$$\theta_{ki} = \frac{m_{ki} + \frac{1}{p(1-\alpha_k)}}{2}, \quad s_{ki} = s(1 - \theta_{ki}), \quad s > 0, B_{ki} = l_q^{s_{ki}}, \quad i = 0, 1,$$

$$B_0 = \prod_{k,i} B_{ki}, \quad \alpha_{km_{k1}} \neq 0, \quad \beta_{km_{k2}} \neq 0, \quad k = 1, 2, \ldots, n.$$

Let A be the operator in $l_q(N)$ defined by

$$D(A) = l_q(N), \quad A = \left[d_{mj}(x) \right], \quad d_{mj}(x) = g_m(x) 2^{sj}, \quad m, j = 1, 2, \ldots, N,$$

where

$$l_q(N) = \left\{ u = \{u_j\},\ j = 1, 2, \ldots, N, \|u\|_{l_q(N)} = \left(\sum_{j=1}^{N} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_q(A) = \left\{ u \in l_q(N), \|u\|_{l_q(A)} = \|Au\|_{l_q(N)} = \left(\sum_{j=1}^{N} \left| 2^{sj} u_j \right|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in G, \quad 1 < q < \infty, \quad N = 1, 2, \ldots, \infty.$$

Let $b_{kj}(x) = M_{kj}(x) 2^{\sigma j}$ and

$$B = L\big(L_p(G; l_q(N))\big).$$

From Theorem 6.3 we obtain the following result.

Theorem 7.1. *Let the following conditions hold:*

(1) a_k are continuous functions on $\bar{G}$ and $a_k(x) < 0$;

(2) $s \ge \frac{2np(2-q)}{q(p-1)}, 0 < \sigma < s_0$, $s_0 = \frac{s(p-1)}{2p}$, and

$$0 \le \alpha_k < 1 - \frac{1}{p}, \qquad p, q \in (1, \infty);$$

(3) $g_j \in C(\bar{G})$, $N_{kj} \in C(\bar{G})$; $d_{ii}(x) > 0$ and eigenvalues of the matrix $[d_{mi}(x)]$ are positive for all $x \in \bar{G}$, $m, i = 1, 2, \ldots, N$; there is a positive constant C such that

$$\sum_{k=1}^{n} \sum_{j=1}^{N} M_{kj}^{q_1}(x) \le C \sum_{j=1}^{N} g_j^{q_1}(x) < \infty, \qquad x \in G, \quad \frac{1}{q} + \frac{1}{q_1} = 1;$$

(4) *the function* $F(\cdot,v) = (F_1(\cdot,v),\dots,F_N(\cdot,v))$ *is measurable for each* $v \in B_{0p}$ *and the function* $F(x,\cdot)$ *for a.a.* $x \in G$ *is continuous and* $f(\cdot) = F(.,0) \in L_p(G;l_q)$*; for each* $R > 0$ *there is a function* $\Psi_R \in L_\infty(G)$ *such that*

$$\|F(x,U) - F(x,\bar{U})\|_{l_q} \le \Psi_R(x)\|U - \bar{U}\|_{l_q(A)}$$

a.a. $x \in G$ *and*

$$U, \bar{U} \in B_{0p}, \quad \|U\|_{B_{0p}} \le R, \quad \|U\|_{B_{0p}} \le R, \quad U = \{u_{kj}\}, \quad \bar{U} = \{\bar{u}_{kj}\}, \quad u_{kj}, \bar{u}_{kj} \in B_{0p}.$$

Then there is $T \in (0, T_0)$ and $b_k \in (0, b_{0k})$ such that problem (7.1)–(7.2) has a unique solution $u = \{u_m(x)\}_1^N$ that belongs to space $W_p^{1,2}(G_T, l_q(A), l_q)$.

Proof. By virtue of [26], the $l_q(N)$ is a *UMD* space. It is easy to see that the operator A is R-positive in $l_q(N)$. Then by using the conditions (1)–(3) we get that the condition (5) of Theorem 6.3 is hold. So in view of the Theorem 6.3 we obtain the assertion. □

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