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On sensitivity analysis of parameters for fractional differential equations with Caputo derivatives

Yuxiang Guo ✉ [1], **Baoli Ma** [1] and **Ranchao Wu** [2]

[1] The Seventh Research Division, Beijing University of Aeronautics and Astronautics
Haidian District, Beijing 100191, P. R. China
[2] School of Mathematics, Anhui University, Hefei, Anhui 230601, P. R. China

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Abstract. In this paper, we discuss the effect of parameter variations on the performance of fractional differential equations and give the concept of fractional sensitivity functions and fractional sensitivity equations. Meanwhile, by employing Laplace transform and the inverse Laplace transform, some main results on fractional differential equations are proposed. Finally, two simple examples with numerical simulations are provided to show the validity and feasibility of the proposed theorem.

Keywords: Caputo derivative, Laplace transform, Mittag-Leffler function, fractional sensitivity function, fractional sensitivity equation.

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1 Introduction

Fractional differential equations (FDEs) have become one of the most attractive topics in the last few years [4, 10, 13, 17, 19, 23, 26, 27]. The reason for this is that FDEs could well reflect the long-memory and non-local properties of many dynamical models, such as fractional oscillator equation in the coulomb damping vibration [25], fractional Schrödinger equation in quantum mechanics [21], fractional Langevin equation in the anomalous diffusion [3], fractional reaction-diffusion equation in biochemistry [11], fractional Lotka–Volterra equation in the historical biological systems [6], fractional Cattaneo equation in the laser short-pulse heating process [24], and so on.

Some fundamental theories of the solution of FDEs have been published [2, 7, 8, 14, 17, 18, 20, 22, 23, 28–30], like existence, uniqueness, continuous dependence on the order α and on the initial values, etc. These results are not introduced particularly in this paper. As we all know, the performance of many dynamical systems can be closely related to parameter variations. Actually, it is very important that there should be described the effect of small parameter variations on solutions before the dynamical analysis of systems. However, how to characterize this relation is not yet well established in the published literature. Motivated by the previous

✉ Corresponding author. Email: yuxiangguo@buaa.edu.cn

works, the paper studies the differentiability and continuous dependence of the solution of FDEs with respect to parameters. And this is to derive the concept of fractional sensitivity functions and fractional sensitivity equations that depict the relationship between the performance of systems and parameter variations. Meanwhile, by employing Laplace transform and the inverse Laplace transform, some main results on FDEs are proposed. Finally, two simple examples with numerical simulations are given to show the effectiveness of our theoretical results.

Throughout the paper, $\mathbb{R}^+$ and $\mathbb{Z}^+$ are the sets of positive real and integer numbers, respectively; while $\mathbb{R}$ and $\mathbb{C}$ denotes separately the sets of real and complex numbers. $\mathbb{R}^n$ represents the n-dimensional Euclidean space. For a vector $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, we use $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ and $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$ to denote respectively the 1-norm, 2-norm and infinity norm of vector x, while $\|x\|$ represents an arbitrary norm of vector x. ${}^CD$ and ${}_0I_t^\alpha$ denote the Caputo fractional derivative and Riemann–Liouville fractional integral of order α on $[0,t]$, respectively.

2 Preliminaries

In this section, we recall some basic definitions and properties related to fractional calculus which will be needed later. More detailed information on fractional calculus can be found in the literatures [17, 23]. In fractional calculus, the fractional integrals and derivatives usually employ the Riemann–Liouville definition and Caputo definition, respectively. Besides, without loss of generality, the lower limit of all fractional integrals and derivatives is supposed to be zero throughout the paper. We list their representation that will be used in our proofs.

Definition 2.1. Let $x(t)$ be a continuous function on an interval $[0,b]$. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ is defined as

$$ {}_0I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \qquad (t>0,\ \alpha>0), \tag{2.1}$$

where Γ is the Gamma function $\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$.

Definition 2.2. The Caputo fractional derivative with order $\alpha \in \mathbb{R}^+$ of function $x(t)$ is defined as follows

$$ {}_0^CD_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \qquad (t>0), \tag{2.2}$$

where n is an integer such that $0 < n-1 < \alpha \le n \in \mathbb{Z}^+$, $x^{(n)}(\tau)$ denotes the n-th derivative of x with respect to τ. For this formula (2.2), one has ${}_0^CD_t^\alpha x(t) = {}_0I_t^{n-\alpha} x^{(n)}(t)$. Moreover, when $0 < \alpha \le 1$, it holds

$$ {}_0^CD_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x'(\tau) d\tau = {}_0I_t^{1-\alpha} x'(t), \qquad (t>0). \tag{2.3}$$

The Laplace transform (LT) of the Caputo fractional derivative is given by

$$ LT\left\{ {}_0^CD_t^\alpha x(t) \right\} = s^\alpha \mathcal{X}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} x^{(k)}(0), \qquad (0 < n-1 < \alpha \le n \in \mathbb{Z}^+), $$

where $\mathcal{X}(s)$ denotes the Laplace transform of $x(t)$, t and s are the variable in the time domain and complex-frequency domain, respectively. Furthermore, the Laplace transform of ${}_0I_t^{\alpha}x(t)$ takes the particularly simple form

$$LT\left\{{}_0I_t^{\alpha}x(t)\right\} = s^{-\alpha}\mathcal{X}(s), \qquad (\alpha > 0).$$

In order to study the solution of FDEs, the Mittag-Leffler function is employed frequently. To this end, the Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0, \beta > 0),$$

where $z \in \mathbb{C}$. For $\beta = 1$, one has $E_{\alpha,1}(z) = E_{\alpha}(z)$. Moreover, $E_{1,1}(z) = e^z$.

The Laplace transform of the Mittag-Leffler function in two parameters is

$$LT\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \qquad (\mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}), \tag{2.4}$$

where $\lambda \in \mathbb{R}$, $\mathrm{Re}(s)$ denotes the real part of s and $LT\{\cdot\}$ stands for the Laplace transform.

Property 2.3. *Let $\alpha > 0$ and $\beta > 0$. If $x(t)$ is continuous functions on $[0, b]$, then for all $t \in [0, b]$*

$${}_0I_t^{\alpha}({}_0I_t^{\beta}x(t)) = {}_0I_t^{\alpha+\beta}x(t).$$

Property 2.4. *Let $x(t)$ be a continuously differentiable function defined on an interval $[0, b]$ of the real axis $\mathbb{R}$, then*

$${}_0I_t^{\alpha}({}_0^CD_t^{\alpha}x(t)) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}(t-0)^k, \qquad (t > 0),$$

where n is an integer such that $0 < n - 1 < \alpha \le n \in \mathbb{Z}^+$. In particular, if $0 < \alpha \le 1$, then

$${}_0I_t^{\alpha}({}_0^CD_t^{\alpha}x(t)) = x(t) - x(0), \qquad (t > 0).$$

3 Main results

3.1 Nonlinear fractional differential equations

Applying the Caputo derivative, a fractional-order differential equation with nonzero initial value can be defined by

$$\begin{aligned} {}_0^CD_t^{\alpha}x(t) &= f(t, x(t)), \qquad (t > 0), \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $\alpha \in (0, 1)$ is a real constant. By using the property of Riemann–Liouville fractional integral, one gets

$${}_0I_t^{\alpha}({}_0^CD_t^{\alpha}x(t)) = {}_0I_t^{\alpha}(f(t, x(t))).$$

From Property 2.4 and Definition 2.1, we have

$$x(t) = x_0 + {}_0I_t^{\alpha}(f(t, x(t))) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau, x(\tau))d\tau, \qquad (t > 0). \tag{3.2}$$

Remark 3.1. From (3.2), we can clearly see that the solution of Cauchy type problem (3.1) is fully determined by the initial value x_0 and the nonlinear function f in the usual sense, while does not consider information of the infinite time interval $x(\tau)$ $(-\infty < \tau \le 0)$. In this case, the solution $x(t)$ of Cauchy type problem (3.1) is well-defined on $t > 0$. It is thus assumed that the initial value of FDEs involving Caputo derivative is a constant function of time, and $x(t) = x(0^+)$ for all $t \le 0$ throughout the paper.

For the existence and uniqueness of the solution of Cauchy type problem (3.1), Ref. [17] has discussed extensively, which is stated in the following Lemma 3.2.

Lemma 3.2. *Let Ω be an open and connected set in $\mathbb{R}^n$, and assume that the nonlinear function $f(t,x) : [0,b] \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and satisfies the Lipschitz condition in x. Then there exists $x(t)$ such that the equation ${}_0^C D_t^\alpha x(t) = f(t, x(t))$ with $x(0) = x_0$ has a unique solution of the Volterra integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau, \qquad (t > 0),$$

where $\alpha \in (0,1)$ is a real constant.

From the above processes, it is recognized that this solution $x(t)$ depends continuously on the order α, the initial condition x_0, and the right-hand function $f(t,x)$. The corresponding theorem of such problems has been found in [7,8]. Here, a new criterion is devoted to the estimates of solutions of fractional-order equations.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^n$ is a domain that contains $x = 0$. Assume that $x(t) \in \Omega$ is a solution of Cauchy type problem (3.1), for all $t \ge 0$; and there exists a constant $l > 0$ such that $\|f(t,x(t))\| \le l\|x(t)\|$ on $[0,\infty) \times \Omega$. Then, for any $\alpha \in (0,1)$*

(i) $|{}_0^C D_t^\alpha [x^T(t)x(t)]| \le 2l\|x(t)\|^2$;

(ii) $\|x_0\|(E_\alpha(-2lt^\alpha))^{\frac{1}{2}} \le \|x(t)\| \le \|x_0\|(E_\alpha(2lt^\alpha))^{\frac{1}{2}}$.

Proof. (i) Based on ${}_0^C D_t^\alpha [x^T(t)x(t)] \le 2x^T(t){}_0^C D_t^\alpha x(t)$ [1, Remark 1], it can be concluded that

$${}_0^C D_t^\alpha [x^T(t)x(t)] \le 2x^T(t){}_0^C D_t^\alpha x(t) = 2x^T(t) f(t, x(t)).$$

That is,

$$|{}_0^C D_t^\alpha [x^T(t)x(t)]| \le 2\|x^T(t)\|\|f(t,x(t))\| \le 2l\|x(t)\|^2.$$

(ii) Let $L(t) = x^T(t)x(t)$ and $L_0 = x_0^T x_0$. According to the foregoing processes, one gets

$$-2lL(t) \le {}_0^C D_t^\alpha L(t) \le 2lL(t). \tag{3.3}$$

For the left-hand side of (3.3), there exists a non-negative function $N(t)$ satisfying

$${}_0^C D_t^\alpha L(t) = -2lL(t) + N(t). \tag{3.4}$$

The Laplace transform (LT) of Equation (3.4) yields

$$s^\alpha \mathcal{L}(s) - s^{\alpha-1} L_0 = -2l\mathcal{L}(s) + \mathcal{N}(s),$$

where $\mathcal{L}(s) = LT\{L(t)\}, \mathcal{N}(s) = LT\{N(t)\}$.
It follows that

$$\mathcal{L}(s) = \frac{s^{\alpha-1}L_0 + \mathcal{N}(s)}{s^\alpha + 2l}. \tag{3.5}$$

Taking the inverse Laplace transform of (3.5), one obtains

$$L(t) = L_0 E_\alpha(-2lt^\alpha) + N(t) * [t^{\alpha-1} E_{\alpha,\alpha}(-2lt^\alpha)],$$

where $*$ denotes the convolution operator, and since $t^{\alpha-1} E_{\alpha,\alpha}(-2lt^\alpha) \geq 0$, then it holds that

$$L_0 E_\alpha(-2lt^\alpha) \leq L(t).$$

The proof of the right-hand side of (3.3) is similar to the above procedure, so we omit it here. Therefore, we have

$$L_0 E_\alpha(-2lt^\alpha) \leq L(t) \leq L_0 E_\alpha(2lt^\alpha).$$

Taking the square root yields

$$\|x_0\| (E_\alpha(-2lt^\alpha))^{\frac{1}{2}} \leq \|x(t)\| \leq \|x_0\| (E_\alpha(2lt^\alpha))^{\frac{1}{2}}.$$

This completes the proof. □

In the next section, we give the closeness of solutions for FDEs involving Caputo derivative on a finite interval of the real axis in spaces of continuous functions.

Theorem 3.4. *Let Ω be an open and connected set in $\mathbb{R}^n$ and $f(t,x) : [0,b] \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and satisfies the Lipschitz condition in x with Lipschitz constant $l > 0$. Let $u(t)$ and $v(t)$ be solutions of ${}_0^C D_t^\alpha u(t) = g(t,u(t))$ with the initial value $u(0) = u_0$ and ${}_0^C D_t^\alpha v(t) = g(t,v(t)) + h(t,v(t))$ with the initial value $v(0) = v_0$, respectively. Assume that there exists a constant $\lambda > 0$ such that*

$$\|h(t,x)\| \leq \lambda, \qquad \forall (t,x) \in [0,b] \times \Omega.$$

Then,

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| E_\alpha(lt^\alpha) + \lambda t^\alpha E_{\alpha,\alpha+1}(lt^\alpha),$$

for all $t \in [0,b]$, where $\alpha \in (0,1)$ is a real constant.

Proof. The solutions $u(t)$ and $v(t)$ are given as, respectively.

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, u(\tau)) d\tau \\ v(t) &= v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [g(\tau, v(\tau)) + h(\tau, v(\tau))] d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} \|u(t) - v(t)\| \leq \|u_0 - v_0\| &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|g(\tau, u(\tau)) - g(\tau, v(\tau))\| d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|h(\tau, v(\tau))\| d\tau \\ &\leq \|u_0 - v_0\| + \frac{\lambda}{\Gamma(\alpha+1)} t^\alpha + \frac{l}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|u(\tau) - v(\tau)\| d\tau. \end{aligned}$$

Let $A(t) = \|u(t) - v(t)\|$, that is $A_0 = \|u_0 - v_0\|$ and $A(\tau) = \|u(\tau) - v(\tau)\|$. Then there exists a nonnegative function $M(t)$ satisfying

$$A(t) = A_0 + \frac{\lambda}{\Gamma(\alpha+1)} t^\alpha + \frac{l}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} A(\tau) d\tau - M(t). \tag{3.6}$$

Taking Laplace transform (LT) of (3.6), we obtain

$$\mathcal{A}(s) = \frac{A_0}{s} + \frac{\lambda}{s^{\alpha+1}} + \frac{l\mathcal{A}(s)}{s^{\alpha}} - \mathcal{M}(t), \tag{3.7}$$

where $\mathcal{A}(s) = LT\{A(t)\}$, $\mathcal{M}(s) = LT\{M(t)\}$.
It follows that

$$\mathcal{A}(s) = \frac{A_0 s^{\alpha-1} + \lambda s^{-1} - s^{\alpha}\mathcal{M}(t)}{s^{\alpha} - l}$$

and the inverse Laplace transform using the formula (2.4) gives the formula

$$A(t) = A_0 E_{\alpha}(lt^{\alpha}) + \lambda t^{\alpha} E_{\alpha,\alpha+1}(lt^{\alpha}) - M(t) * t^{-1} E_{\alpha,0}(lt^{\alpha}),$$

where $*$ denotes the convolution operator. And since

$$t^{-1} E_{\alpha,0}(lt^{\alpha}) = \frac{d}{dt}(E_{\alpha}(lt^{\alpha})) \geq 0,$$

then it yields that

$$A(t) \leq A_0 E_{\alpha}(lt^{\alpha}) + \lambda t^{\alpha} E_{\alpha,\alpha+1}(lt^{\alpha}).$$

That is

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| E_{\alpha}(lt^{\alpha}) + \lambda t^{\alpha} E_{\alpha,\alpha+1}(lt^{\alpha}).$$

This completes the proof. □

Remark 3.5. Theorem 3.4 here is an extension of Theorem 3.4 of the Ref. [16] about the closeness of solutions of integer-order to fractional-order differential equations.

Remark 3.6. This bound is useful only on a finite time interval $[0, b]$, since the right side of the inequality will be unbounded as b is large enough (i.e. the Mittag-Leffler function grows unbound as $t \to \infty$).

Corollary 3.7. *Let $x_1(t)$ and $x_2(t)$ be differentiable functions on an interval $[0, b]$ such that $\|x_1(0) - x_2(0)\| \leq \delta$ and $\|{}_0^C D_t^{\alpha} x_i(t) - f(t, x_i(t))\| \leq \lambda_i, (i = 1, 2)$ for $t \in [0, b]$. If the function f satisfies Lipschitz condition with Lipschitz constant $l > 0$, then*

$$\|x_1(t) - x_2(t)\| \leq \delta E_{\alpha}(lt^{\alpha}) + (\lambda_1 + \lambda_2) t^{\alpha} E_{\alpha,\alpha+1}(lt^{\alpha}),$$

for all $t \in [0, b]$, where $\alpha \in (0, 1)$ is a real constant.

Proof. Let $z(t) = x_1(t) - x_2(t)$, then

$$\begin{aligned} {}_0^C D_t^{\alpha} z(t) &= {}_0^C D_t^{\alpha} x_1(t) - {}_0^C D_t^{\alpha} x_2(t) \\ &= {}_0^C D_t^{\alpha} x_1(t) - f(t, x_1(t)) - {}_0^C D_t^{\alpha} x_2(t) + f(t, x_2(t)) + f(t, x_1(t)) - f(t, x_2(t)). \end{aligned}$$

It follows from the Property 2.4 that

$$\begin{aligned} {}_0 I_t^{\alpha}({}_0^C D_t^{\alpha} z(t)) = {}&{}_0 I_t^{\alpha}({}_0^C D_t^{\alpha} x_1(t) - f(t, x_1(t))) - {}_0 I_t^{\alpha}({}_0^C D_t^{\alpha} x_2(t) - f(t, x_2(t))) \\ &+ {}_0 I_t^{\alpha}(f(t, x_1(t)) - f(t, x_2(t))). \end{aligned}$$

Consequently,

$$\begin{aligned} \|z(t)\| - \|z(0)\| &\leq {}_0 I_t^{\alpha} \lambda_1 + {}_0 I_t^{\alpha} \lambda_2 + l {}_0 I_t^{\alpha} \|x_1(t) - x_2(t)\| \\ &= \frac{\lambda_1 + \lambda_2}{\Gamma(\alpha+1)} t^{\alpha} + \frac{l}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \|z(\tau)\| d\tau. \end{aligned}$$

That is,

$$\|z(t)\| \leq \|z(0)\| + \frac{\lambda_1+\lambda_2}{\Gamma(\alpha+1)}t^\alpha + \frac{l}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|z(\tau)\|d\tau.$$

Similar to the proof of Theorem 3.4, one gets

$$\|z(t)\| \leq \|z(0)\|E_\alpha(lt^\alpha) + (\lambda_1+\lambda_2)t^\alpha E_{\alpha,\alpha+1}(lt^\alpha).$$

Hence, it holds

$$\|x_1(t) - x_2(t)\| \leq \delta E_\alpha(lt^\alpha) + (\lambda_1+\lambda_2)t^\alpha E_{\alpha,\alpha+1}(lt^\alpha).$$

This completes the proof. □

Remark 3.8. For the case ($\delta = \lambda_1 = \lambda_2 = 0$), Corollary 3.7 shows that the initial value problem has only one solution as the right side of the differential equation satisfies a Lipschitz condition. The corresponding result for ordinary differential equations of integer-order has been found in [5, Theorem 5].

3.2 Fractional differential equations with perturbation

Let us consider the fractional differential equation with perturbation

$${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu),$$

where $\alpha \in (0,1)$ is a real constant, $\mu \in \mathbb{R}^P$ could represent physical parameters of the equation, and the study of these parameters explains the change of modeling errors, aging, or uncertainties and disturbances, which exist in any realistic problem.

With the help of Theorem 3.4, we should show that the next theorem related to the continuous dependence of solution in terms of parameters. Firstly, the conception of continuous dependence on parameters is introduced as follows.

Definition 3.9. Let $x(t,\mu_0)$ be a solution of the fractional-order differential equation ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu_0)$ defined on a finite time interval $[0, b]$, with the initial value $x(0, \mu_0) = x_{00}$. The solution is said to depend continuously on μ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all μ in the set $\{\mu \in \mathbb{R}^P \mid \|\mu - \mu_0\| < \delta\}$, the equation ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu)$ has a unique solution $x(t,\mu)$ defined on a finite time interval $[0, b]$, with the initial value $x(0,\mu) = x_{01}$, and satisfies $\|x(t,\mu) - x(t,\mu_0)\| < \varepsilon$, for all $t \in [0, b]$.

Remark 3.10. At the initial time $t = 0$, the initial values of equations ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu_0)$ and ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu)$ are the same thing, that is $x_{00} = x_{01} \triangleq x_0$.

Theorem 3.11. *Let Ω be an open and connected set in $\mathbb{R}^n$ and the nonlinear function $f(t, x, \mu)$: $[0, b] \times \Omega \times \{\|\mu - \mu_0\| \leq r\} \to \mathbb{R}^n$ is continuous in (t, x, μ) and satisfies the Lipschitz condition in x (uniformly in t and μ). Let $\phi(t,\mu_0)$ be a solution of the equation ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu_0)$ defined on a finite time interval $[0, b]$, with $\phi(0, \mu_0) = \phi_0 \in \Omega$. Assume that $\phi(t, \mu_0) \in \Omega$ is well-defined for all $t \in [0, b]$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{\|\mu - \mu_0\| < \delta\}$ then there exists a unique solution $\varphi(t, \mu)$ of the equation ${}_0^C D_t^\alpha x(t) = f(t, x(t), \mu)$ defined on a finite time interval $[0, b]$, with $\varphi(0, \mu) = \varphi_0$, and satisfying $\|\varphi(t, \mu) - \phi(t, \mu_0)\| < \varepsilon$.*

Proof. In order to prove this theorem, we need structure an invariant set D such that the solution of fractional-order equation starts in D at some time, and stays in D for all future time. This method is similar to the proof of Theorem 3.5 in [16]. Since $\phi(t,\mu_0)$ is continuous with respect to t, then $\phi(t,\mu_0)$ is bounded on the closed interval $[0,b]$. Consequently, there exists a small enough ε to ensure that the set $D \triangleq \{(t,x) \in [0,b] \times \mathbb{R}^n | \|x - \phi(t,\mu_0)\| \le \varepsilon\} \subset [0,b] \times \Omega$. Apparently, D is compact set, then $f(t,x,\mu)$ satisfies the Lipschitz condition with respect to x on D with Lipschitz constant $l > 0$. By the continuity of $f(t,x,\mu)$ concerning μ, for any $\zeta > 0$, there exists $\eta > 0$ (with $\eta < r$), such that

$$\|f(t,x,\mu) - f(t,x,\mu_0)\| < \zeta, \qquad \forall (t,x) \in D,\ \forall \|\mu - \mu_0\| < \eta.$$

Taking $\zeta < \varepsilon$ and $\|\phi_0 - \varphi_0\| < \zeta$, by the existence and uniqueness theorem, then there exists a unique solution $\varphi(t,\mu)$ of ${}_0^C D_t^\alpha x(t) = f(t,x(t),\mu)$ defined on $[0,b]$, with $\varphi(0,\mu) = \varphi_0$. We will prove that, by electing a small enough ζ, the solution stays in D for all $t \in [0,b]$. That is,

$$\begin{aligned}
&\|\varphi(t,\mu) - \phi(t,\mu_0)\| \\
&\quad = \left\| \varphi_0 - \phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau,\varphi,\mu) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau,\phi,\mu_0) d\tau \right\| \\
&\quad \le \|\varphi_0 - \phi_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau,\varphi,\mu_0) - f(\tau,\phi,\mu_0)\| d\tau \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau,\varphi,\mu) - f(\tau,\varphi,\mu_0)\| d\tau.
\end{aligned}$$

Based on Theorem 3.4, it follows that

$$\|\varphi(t,\mu) - \phi(t,\mu_0)\| < \|\varphi_0 - \phi_0\| E_\alpha(lt^\alpha) + \zeta t^\alpha E_{\alpha,\alpha+1}(lt^\alpha) < \zeta(E_\alpha(lt^\alpha) + t^\alpha E_{\alpha,\alpha+1}(lt^\alpha)).$$

By choosing $\zeta < \varepsilon(E_\alpha(lt^\alpha) + t^\alpha E_{\alpha,\alpha+1}(lt^\alpha))^{-1}$, one has $\|\varphi(t,\mu) - \phi(t,\mu_0)\| < \varepsilon$, where $\delta = \min(\zeta,\eta)$. From Definition 3.9, the solution of fractional-order equation is continuously dependent on parameter μ. This completes the proof. □

Corollary 3.12. *Let $f(t,x)$ is continuous in (t,x) and has continuous first partial derivative with respect to x, for all $(t,x) \in [0,b] \times \mathbb{R}^n$. Let $\psi(t,t_0,x_{t_0})$ be the solution of equation ${}_{t_0}^C D_t^\alpha x(t) = f(t,x(t))$ that starts at $x(t_0) = x_{t_0}$ for all $t_0 \ge 0$; Further let $\psi(t,t_0,x_{t_0})$ is fully determined by $x(t_0) = x_{t_0}$ in the usual sense and $\psi(t,t_0,x_{t_0})$ is well-defined on $[t_0,b]$, while does not consider knowledge of the infinite time interval $x(\tau)$ $(-\infty < \tau \le t_0)$. Then, for any real constant α $(0 < \alpha < 1)$,*

(i) $\psi(t,t_0,x_{t_0})$ is continuously differentiable with respect to t_0 and x_{t_0}.

(ii) Let $\psi_{t_0}(t)$ and $\psi_{x_{t_0}}(t)$ denote $\frac{\partial\psi}{\partial t_0}$ and $\frac{\partial\psi}{\partial x_{t_0}}$, respectively. Then

$$y(t) \triangleq \psi_{t_0}(t) + \psi_{x_{t_0}}(t) f(t_0,x_{t_0})\, {}_{t_0}I_t^\alpha (t-t_0)^{-1}$$

satisfies the fractional-order differential equation ${}_{t_0}^C D_t^\alpha y(t) = \frac{\partial f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))y(t)$ with the initial value $y(t_0) = 0$.

(iii) Under the assumptions of (ii), if the Jacobian matrix $\frac{\partial f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))$ is a real constant matrix, then $y(t) \equiv 0$, that is $\psi_{t_0}(t) + \psi_{x_{t_0}}(t) f(t_0,x_{t_0})\, {}_{t_0}I_t^\alpha (t-t_0)^{-1} \equiv 0$.

Proof. (i) By using the property of Riemann–Liouville fractional integral, one has

$${}_{t_0}I_t^\alpha ({}_{t_0}^C D_t^\alpha x(t)) = {}_{t_0}I_t^\alpha (f(t,x(t))).$$

From Property 2.4 and Definition 2.1, one gets

$$\begin{aligned}\psi(t,t_0,x_{t_0}) &= x_{t_0} + {}_{t_0}I_t^{\alpha}(f(t,\psi(t,t_0,x_{t_0})))\\ &= x_{t_0} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau,\psi(\tau,t_0,x_{t_0}))d\tau, \qquad (t>t_0). \end{aligned} \tag{3.8}$$

Apparently, $\psi(t,t_0,x_{t_0})$ is continuously differentiable with respect to t_0 and x_{t_0} since f and its partial derivative with respect to ψ are continuous in (t,ψ).
(ii) From (i), it yields that

$$\begin{aligned}\psi_{x_{t_0}}(t) &= \frac{\partial}{\partial x_{t_0}}\psi(t,t_0,x_{t_0})\\ &= I + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{\frac{f}{\partial\psi}(\tau,\psi(\tau,t_0,x_{t_0}))\frac{\partial}{\partial x_{t_0}}\psi(\tau,t_0,x_{t_0})}{(t-\tau)^{1-\alpha}}d\tau\\ &= I + {}_{t_0}I_t^{\alpha}\left[\frac{f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))\frac{\partial}{\partial x_{t_0}}\psi(t,t_0,x_{t_0})\right];\end{aligned}$$

and

$$\begin{aligned}\psi_{t_0}(t) &= \frac{\partial}{\partial t_0}\psi(t,t_0,x_{t_0})\\ &= -\frac{f(t_0,x_{t_0})}{\Gamma(\alpha)(t-t_0)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{\frac{f}{\partial\psi}(\tau,\psi(\tau,t_0,x_{t_0}))\frac{\partial}{\partial t_0}\psi(\tau,t_0,x_{t_0})}{(t-\tau)^{1-\alpha}}d\tau\\ &= -\frac{f(t_0,x_{t_0})}{\Gamma(\alpha)(t-t_0)^{1-\alpha}} + {}_{t_0}I_t^{\alpha}\left[\frac{f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))\frac{\partial}{\partial t_0}\psi(t,t_0,x_{t_0})\right].\end{aligned}$$

Note that

$${}_{t_0}I_t^{\alpha}(t-t_0)^{-1} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}(\tau-t_0)^{-1}d\tau = \frac{1}{\Gamma(\alpha)(t-t_0)^{1-\alpha}}.$$

Hence

$$\psi_{t_0}(t) = -f(t_0,x_{t_0})\,{}_{t_0}I_t^{\alpha}(t-t_0)^{-1} + {}_{t_0}I_t^{\alpha}\left[\frac{f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))\frac{\partial}{\partial t_0}\psi(t,t_0,x_{t_0})\right].$$

Consequently, denote that $y(t) \triangleq \psi_{t_0}(t) + \psi_{x_{t_0}}(t)\frac{f(t_0,x_{t_0})}{\Gamma(\alpha)(t-t_0)^{1-\alpha}}$, one gets

$$\begin{aligned}y(t) &= \psi_{t_0}(t) + \psi_{x_{t_0}}(t)\frac{f(t_0,x_{t_0})}{\Gamma(\alpha)(t-t_0)^{1-\alpha}}\\ &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{\frac{f}{\partial\psi}(\tau,\psi(\tau,t_0,x_{t_0}))}{(t-\tau)^{1-\alpha}}\left[\psi_{t_0}(\tau) + \psi_{x_{t_0}}(\tau)\frac{f(t_0,x_{t_0})}{\Gamma(\alpha)(t-t_0)^{1-\alpha}}\right]d\tau.\end{aligned}$$

Namely,

$$\begin{aligned}y(t) &= \psi_{t_0}(t) + \psi_{x_{t_0}}(t)f(t_0,x_{t_0})\,{}_{t_0}I_t^{\alpha}(t-t_0)^{-1}\\ &= {}_{t_0}I_t^{\alpha}\left\{\frac{f}{\partial\psi}(t,\psi(t,t_0,x_{t_0}))[\psi_{t_0}(t) + \psi_{x_{t_0}}(t)f(t_0,x_{t_0})\,{}_{t_0}I_t^{\alpha}(t-t_0)^{-1}]\right\}.\end{aligned} \tag{3.9}$$

Taking Caputo type fractional derivative of both sides of (3.9), it is clearly seen that $y(t) = \psi_{t_0}(t) + \psi_{x_{t_0}}(t) f(t_0, x_{t_0})_{t_0} I_t^\alpha (t-t_0)^{-1}$ satisfies the fractional-order equation

$$ {}_{t_0}^{C}D_t^\alpha y(t) = \frac{f}{\partial \psi}(t, \psi(t, t_0, x_{t_0}))y(t), \qquad (t > t_0, 0 < \alpha < 1), \tag{3.10}$$

with the initial value $y(t_0) = 0$.
(iii) Assume that the Jacobian matrix $\frac{\partial f}{\partial \psi}(t, \psi(t, t_0, x_{t_0}))$ is a real constant matrix A, then Equation (3.10) can be rewritten as ${}_{t_0}^{C}D_t^\alpha y(t) = Ay(t)$ with $y(t_0) = 0$. By applying Laplace transform and the inverse Laplace transform, it follows that

$$y(t) = E_\alpha(A(t-t_0)^\alpha)y(t_0) \equiv 0.$$

Therefore, $\psi_{t_0}(t) + \psi_{x_{t_0}}(t) f(t_0, x_{t_0})_{t_0} I_t^\alpha (t-t_0)^{-1} \equiv 0$.
This completes the proof. □

Remark 3.13. If the fractional-order differential equations are linear and autonomous (i.e. $f(t, x(t)) = f(x(t))$, and $f(x(t))$ is noted for the linear combination of the vector $x(t)$), then the result (iii) holds certainly.

3.3 Sensitivity equations of Caputo fractional derivative

Let the nonlinear function $f(t, x, \mu)$ and its partial derivative with respect to x and μ be continuous in (t, x, μ) for all $(t, x, \mu) \in [0, b] \times \mathbb{R}^n \times \mathbb{R}^p$. Before existing the perturbation of parameter μ, let us assume the nominal equation

$${}_{0}^{C}D_t^\alpha x(t) = f(t, x(t), \mu_0), \qquad x(0) = x_0,$$

has a unique solution $x(t, \mu_0)$ on a finite time interval $[0, b]$, where $\alpha \in (0, 1)$ is a real constant and μ_0 is a nominal value of μ. When there is the perturbation of parameter μ, the nominal equation can be written the following form

$${}_{0}^{C}D_t^\alpha x(t) = f(t, x(t), \mu), \qquad x(0) = x_0. \tag{3.11}$$

If $\|\mu - \mu_0\|$ is chosen as sufficiently small constant, then it follows from Theorem 3.11 that the equation (3.11) has a unique solution $x(t, \mu)$ on a finite time interval $[0, b]$ and it is close to the nominal solution $x(t, \mu_0)$. Therefore, the equation (3.11) is equivalent to a Volterra integral equation

$$x(t, \mu) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau, \mu), \mu) d\tau. \tag{3.12}$$

By the continuous differentiability of $f(t, x, \mu)$ with respect to x and μ, one obtains that the solution $x(t, \mu)$ is differentiable with respect to μ on some neighborhood of μ_0. That is,

$$\frac{\partial x(t, \mu)}{\partial \mu} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[\frac{\partial f}{\partial x}(\tau, x(\tau, \mu), \mu) \frac{\partial x(t, \mu)}{\partial \mu} + \frac{\partial f}{\partial \mu}(\tau, x(\tau, \mu), \mu) \right] d\tau. \tag{3.13}$$

By setting $x_\mu(t, \mu) = \frac{\partial x(t, \mu)}{\partial \mu}$, we can write the equation (3.13) as

$$x_\mu(t, \mu) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[\frac{\partial f}{\partial x}(\tau, x(\tau, \mu), \mu) x_\mu(t, \mu) + \frac{\partial f}{\partial \mu}(\tau, x(\tau, \mu), \mu) \right] d\tau.$$

At $\mu = \mu_0$, $x_\mu(t,\mu_0)$ depends only on some time t and describes the time evolution of the sensitivity of parameter μ. Taking Caputo type fractional derivative of both sides of (3.13), it yields that

$$ {}_0^C D_t^\alpha x_\mu(t,\mu) = \frac{\partial f}{\partial x}(t, x(t,\mu), \mu) x_\mu(t,\mu) + \frac{\partial f}{\partial \mu}(t, x(t,\mu), \mu), \qquad x_\mu(0,\mu) = 0. \tag{3.14}$$

Denote that $P(t,\mu) = \frac{\partial f(t,x(t,\mu),\mu)}{\partial x}$ and $Q(t,\mu) = \frac{\partial f(t,x(t,\mu),\mu)}{\partial \mu}$, as μ is sufficiently close to μ_0, $P(t,\mu)$ and $Q(t,\mu)$ are well-defined on $[0,b]$. Furthermore, the evolution of equation (3.14) is fully determined by knowledge of the vector $x_\mu(t,\mu)$ at a time $t=0$ in the usual sense, and does not depend on information of the infinite time interval $x_\mu(\tau,\mu)$ $(-\infty < \tau \le 0)$. When $\mu = \mu_0$, the equation (3.14) is described by the follows form

$$ {}_0^C D_t^\alpha S(t) = P(t,\mu_0) S(t) + Q(t,\mu_0), \qquad S(0) = 0, \tag{3.15}$$

where $S(t) = x_\mu(t,\mu_0)$ is called the fractional-order sensitivity function, and formula (3.15) is call the fraction-order sensitivity equation. Apparently, $S(t)$ is also well-defined on on a finite time interval $[0,b]$, which is the unique solution of the equation (3.15). The matrices $P(t,\mu_0)$ and $Q(t,\mu_0)$ are given as follows.

$$\begin{aligned} P(t,\mu_0) &= \frac{\partial f(t,x,\mu)}{\partial x}\bigg|_{x=x(t,\mu_0),\mu=\mu_0} \in \mathbb{R}^{n\times n}, \\ Q(t,\mu_0) &= \frac{\partial f(t,x,\mu)}{\partial \mu}\bigg|_{x=x(t,\mu_0),\mu=\mu_0} \in \mathbb{R}^{n\times p}. \end{aligned}$$

Remark 3.14. The sensitivity function and sensitivity equation of Caputo fractional derivative is an extension of sensitivity analysis of parameters from the integer-order to the fractional-order. The corresponding theorem has been found in [16].

In the next section, two simple examples are given to illustrate the solution process of the sensitivity function.

4 Illustrative examples

Example 4.1. Let us consider the following Duffing forced-oscillation equation with Caputo fractional derivative

$$\begin{cases} {}_0^C D_t^{0.9} x_1 = x_2, \\ {}_0^C D_t^{0.9} x_2 = -a x_2 - b x_1^3 + c\cos(wt), \end{cases} \tag{4.1}$$

where a, b, c and w are real positive parameters of equation. If the nominal values of parameters are chosen as $a_0 = 0.3$, $b_0 = 1$, $c_0 = 39$ and $w_0 = 1$, then the nominal equation is written as

$$\begin{cases} {}_0^C D_t^{0.9} x_1 = x_2, \\ {}_0^C D_t^{0.9} x_2 = -0.3 x_2 - x_1^3 + 39\cos(t). \end{cases} \tag{4.2}$$

Differentiating with respect to $x = (x_1, x_2)^T$ and $\mu = (a,b,c,w)^T$, respectively, we see that the Jacobian matrices are described by

$$\begin{aligned} \frac{\partial f}{\partial x} &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3bx_1^2 & -a \end{bmatrix}, \\ \frac{\partial f}{\partial \mu} &= \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} & \frac{\partial f}{\partial c} & \frac{\partial f}{\partial w} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -x_2 & -x_1^3 & \cos(wt) & -cw\sin(wt) \end{bmatrix}. \end{aligned}$$

Consequently, one gets

$$P(t,\mu_0)=\frac{\partial f}{\partial x}\bigg|_{\mu_0}=\begin{bmatrix}0 & 1\\ -3x_1^2 & -0.3\end{bmatrix},$$

$$Q(t,\mu_0)=\frac{\partial f}{\partial \mu}\bigg|_{\mu_0}=\begin{bmatrix}0 & 0 & 0 & 0\\ -x_2 & -x_1^3 & \cos(t) & -39\sin(t)\end{bmatrix}.$$

Let the sensitivity function

$$S(t)=\frac{\partial x}{\partial \mu}\bigg|_{\mu_0}=\begin{bmatrix}\frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} & \frac{\partial x_1}{\partial w}\\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} & \frac{\partial x_2}{\partial w}\end{bmatrix}\triangleq\begin{bmatrix}x_3 & x_5 & x_7 & x_9\\ x_4 & x_6 & x_8 & x_{10}\end{bmatrix},$$

then the sensitivity equation

$$\begin{aligned}{}_0^CD_t^{0.9}S(t)&=\begin{bmatrix}{}_0^CD_t^{0.9}x_3 & {}_0^CD_t^{0.9}x_5 & {}_0^CD_t^{0.9}x_7 & {}_0^CD_t^{0.9}x_9\\ {}_0^CD_t^{0.9}x_4 & {}_0^CD_t^{0.9}x_6 & {}_0^CD_t^{0.9}x_8 & {}_0^CD_t^{0.9}x_{10}\end{bmatrix}=P(t,\mu_0)S(t)+Q(t,\mu_0)\\ &=\begin{bmatrix}0 & 1\\ -3x_1^2 & -0.3\end{bmatrix}\begin{bmatrix}x_3 & x_5 & x_7 & x_9\\ x_4 & x_6 & x_8 & x_{10}\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ -x_2 & -x_1^3 & \cos(t) & -39\sin(t)\end{bmatrix}.\end{aligned}\tag{4.3}$$

Combining Eqs. (4.2) and (4.3), one has

$$\begin{cases}{}_0^CD_t^{0.9}x_1(t)=x_2(t), & x_1(0)=x_{10},\\ {}_0^CD_t^{0.9}x_2(t)=-0.3x_2(t)-x_1^3(t)+39\cos(t), & x_2(0)=x_{20},\\ {}_0^CD_t^{0.9}x_3(t)=x_4(t), & x_3(0)=0,\\ {}_0^CD_t^{0.9}x_4(t)=-3x_3(t)x_1^2(t)-0.3x_4(t)-x_2(t), & x_4(0)=0,\\ {}_0^CD_t^{0.9}x_5(t)=x_6(t), & x_5(0)=0,\\ {}_0^CD_t^{0.9}x_6(t)=-3x_5(t)x_1^2(t)-0.3x_6(t)-x_1^3(t), & x_6(0)=0,\\ {}_0^CD_t^{0.9}x_7(t)=x_8(t), & x_7(0)=0,\\ {}_0^CD_t^{0.9}x_8(t)=-3x_7(t)x_1^2(t)-0.3x_8(t)+\cos(t), & x_8(0)=0,\\ {}_0^CD_t^{0.9}x_9(t)=x_{10}(t), & x_9(0)=0,\\ {}_0^CD_t^{0.9}x_{10}(t)=-3x_9(t)x_1^2(t)-0.3x_{10}(t)-39\sin(t), & x_{10}(0)=0.\end{cases}$$

In what follows, the Adams–Bashforth–Moulton predictor–corrector algorithm [9] is employed to the numerical calculation of fractional-order equation with Caputo derivative. When the initial value $(x_{10},x_{20})=(0.1,1)$, Fig. 4.1 depicts the sensitivities of x_1 with respect to x_3, x_5, x_7 and x_9, that is the sensitivity of parameters (a,b,c,w) for x_1. Also, Fig. 4.2 shows the sensitivity of parameters (a,b,c,w) for x_2. From Figs. 4.1–4.2, we can see that the solution is more sensitive for parameter w than for parameters a, b and c.

Remark 4.2. It should be noted that if the fractional-order equation is autonomous (namely, $f(t,x(t),\mu)=f(x(t),\mu)$), then the fractional sensitivity equation will be also autonomous. The next example is given to illustrate this specific case.

Example 4.3. Let us consider Caputo type fractional-order Lorenz equation [12]

$$\begin{cases}{}_0^CD_t^{0.95}x_1=-ax_1+ax_2,\\ {}_0^CD_t^{0.95}x_2=bx_1-x_2-x_1x_3,\\ {}_0^CD_t^{0.95}x_3=x_1x_2-cx_3,\end{cases}$$

where $\mu = (a, b, c)^T$ are real positive parameters that represent the Prandtl number, the geometric factor and the Rayleigh number, respectively. When the nominal values of parameters are chosen as $a_0 = 10$, $b_0 = 28$ and $c_0 = \frac{8}{3}$, then the nominal equation is described by

$$\begin{cases} {}_0^C D_t^{0.95} x_1 = -10x_1 + 10x_2, \\ {}_0^C D_t^{0.95} x_2 = 28x_1 - x_2 - x_1x_3, \\ {}_0^C D_t^{0.95} x_3 = x_1x_2 - \frac{8}{3}x_3. \end{cases} \tag{4.4}$$

Similar to the procedure of Example 4.1, one obtains

$$P(\mu_0) = \frac{\partial f}{\partial x}\bigg|_{\mu_0} = \begin{bmatrix} -10 & 10 & 0 \\ 28-x_3 & -1 & -x_1 \\ x_2 & x_1 & -\frac{8}{3} \end{bmatrix},$$

$$Q(\mu_0) = \frac{\partial f}{\partial \mu}\bigg|_{\mu_0} = \begin{bmatrix} -x_1 + x_2 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix}.$$

Let the sensitivity function

$$S = \frac{\partial x}{\partial \mu}\bigg|_{\mu_0} = \begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \\ \frac{\partial x_3}{\partial a} & \frac{\partial x_3}{\partial b} & \frac{\partial x_3}{\partial c} \end{bmatrix} \triangleq \begin{bmatrix} x_4 & x_6 & x_8 \\ x_5 & x_7 & x_9 \\ x_{10} & x_{11} & x_{12} \end{bmatrix},$$

then the sensitivity equation

$$\begin{aligned} {}_0^C D_t^{0.95} S &= \begin{bmatrix} {}_0^C D_t^{0.95} x_4 & {}_0^C D_t^{0.95} x_6 & {}_0^C D_t^{0.95} x_8 \\ {}_0^C D_t^{0.95} x_5 & {}_0^C D_t^{0.95} x_7 & {}_0^C D_t^{0.95} x_9 \\ {}_0^C D_t^{0.95} x_{10} & {}_0^C D_t^{0.95} x_{11} & {}_0^C D_t^{0.95} x_{12} \end{bmatrix} = P(\mu_0)S + Q(\mu_0) \\ &= \begin{bmatrix} -10 & 10 & 0 \\ 28-x_3 & -1 & -x_1 \\ x_2 & x_1 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x_4 & x_6 & x_8 \\ x_5 & x_7 & x_9 \\ x_{10} & x_{11} & x_{12} \end{bmatrix} + \begin{bmatrix} -x_1 + x_2 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix}. \end{aligned} \tag{4.5}$$

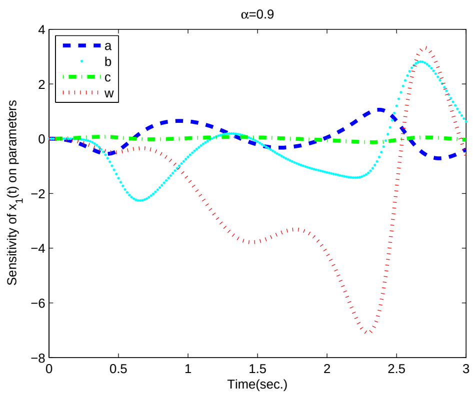

Figure 4.1: Evolution of the sensitivities of $x_1(t)$ on parameters (a, b, c, w).

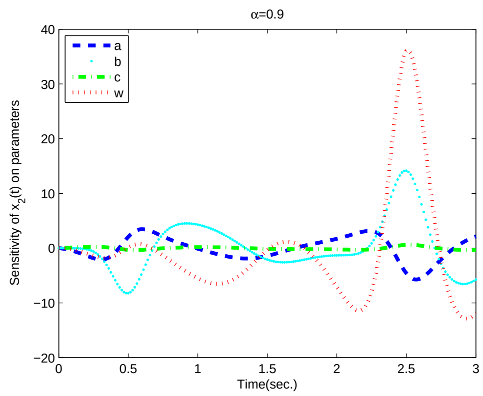

Figure 4.2: Evolution of the sensitivities of $x_2(t)$ on parameters (a, b, c, w).

Combining Eqs. (4.4) and (4.5), one has

$$
\begin{cases}
{}_0^C D_t^{0.95} x_1 = -10x_1 + 10x_2, & x_1(0) = x_{10},\\
{}_0^C D_t^{0.95} x_2 = 28x_1 - x_2 - x_1x_3, & x_2(0) = x_{20},\\
{}_0^C D_t^{0.95} x_3 = x_1x_2 - \frac{8}{3}x_3, & x_3(0) = x_{30},\\
{}_0^C D_t^{0.95} x_4 = -10x_4 + 10x_5 - x_1 + x_2, & x_4(0) = 0,\\
{}_0^C D_t^{0.95} x_5 = (28 - x_3)x_4 - x_5 - x_1x_{10}, & x_5(0) = 0,\\
{}_0^C D_t^{0.95} x_6 = -10x_6 + 10x_7, & x_6(0) = 0,\\
{}_0^C D_t^{0.95} x_7 = (28 - x_3)x_6 - x_7 - x_1x_{11} + x_1, & x_7(0) = 0,\\
{}_0^C D_t^{0.95} x_8 = -10x_8 + 10x_9, & x_8(0) = 0,\\
{}_0^C D_t^{0.95} x_9 = (28 - x_3)x_8 - x_9 - x_1x_{12}, & x_9(0) = 0,\\
{}_0^C D_t^{0.95} x_{10} = x_2x_4 + x_1x_5 - \frac{8}{3}x_{10}, & x_{10}(0) = 0,\\
{}_0^C D_t^{0.95} x_{11} = x_2x_6 + x_1x_7 - \frac{8}{3}x_{11}, & x_{11}(0) = 0,\\
{}_0^C D_t^{0.95} x_{12} = x_2x_8 + x_1x_9 - \frac{8}{3}x_{12} - x_3, & x_{12}(0) = 0.
\end{cases}
$$

When the initial value $(x_{10}, x_{20}, x_{30}) = (1, 1, 1)$, Fig. 4.3 shows the sensitivities of $x_1(t)$ with respect to $x_4(t)$, $x_6(t)$ and $x_8(t)$, that is the sensitivity of parameters (a, b, c) for $x_1(t)$. Also, Figs. 4.4–4.5 depict the sensitivity of parameters (a, b, c) for $x_2(t)$ and $x_3(t)$, respectively. From Figs. 4.3–4.5, we can see that the solution is more sensitive for parameter c than for parameters a and b. In addition, Fig. 4.6 displays a much closer relationship between the Rayleigh number c and the spatial temperature distribution in the fluid layer under gravity (i.e. the variable $x_3(t)$) than measures of the fluid velocity (i.e. the variables $x_1(t)$ and $x_2(t)$). Refs. [12, 15] show that the performance of the system is closely related to variations in the parameter c. Therefore, it is suggested that there should be described the effect of small parameter variations on solutions before the dynamical analysis of fractional Lorenz system.

5 Conclusion

This letter shows the effect of parameter variations on the performance of fractional differential equations and give the concept of fractional sensitivity functions and fractional sensitivity equations. At the same time, by employing Laplace transform and the inverse Laplace transform, some main results on fractional differential equations are proposed. Finally, two simple examples with numerical simulations are provided to show the validity and feasibility of the proposed theorem.

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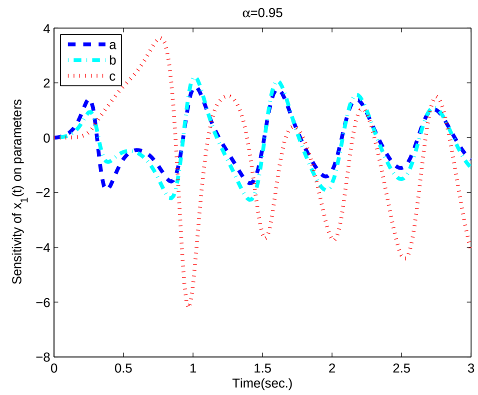

Figure 4.3: Evolution of the sensitivities of $x_1(t)$ on parameters (a, b, c).

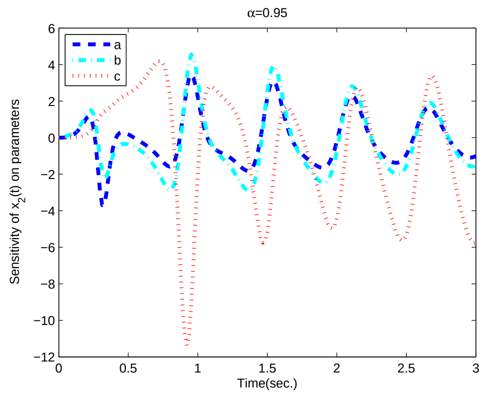

Figure 4.4: Evolution of the sensitivities of $x_2(t)$ on parameters (a, b, c).

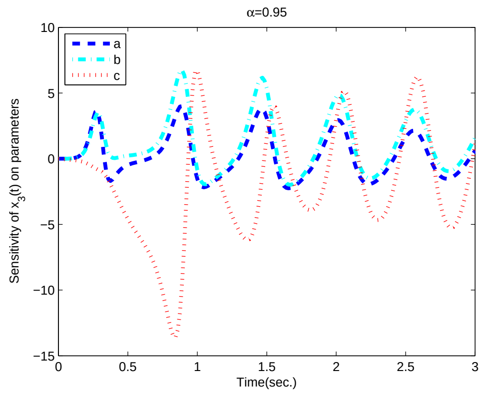

Figure 4.5: Evolution of the sensitivities of $x_3(t)$ on parameters (a, b, c).

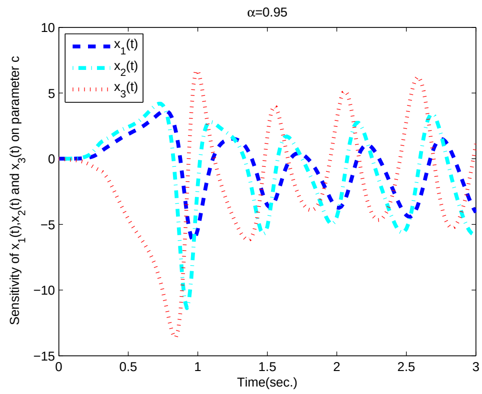

Figure 4.6: Evolution of the sensitivities of $x_1(t)$, $x_2(t)$ and $x_3(t)$ on parameter c.

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