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Interval oscillation criteria for nonlinear impulsive differential equations with variable delay

Xiaoliang Zhou ✉ [1] and **Wu-Sheng Wang**[2]

[1]Department of Mathematics, Lingnan Normal University, Zhanjiang, Guangdong 524048, PR China
[2]Department of Mathematics, Hechi University, Yizhou, Guangxi 546300, PR China

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Abstract. In this paper, the interval qualitative properties of a class of second order nonlinear differential equations are studied. For the hypothesis of delay being variable $\tau(t)$, an "interval delay function" is introduced to estimate the ratio of functions $x(t-\tau(t))$ and $x(t)$ on each considered interval, then Riccati transformation and H functions are applied to obtain interval oscillation criteria. The known results gained by Huang and Feng [*Comput. Math. Appl.* **59**(2010), 18–30] under the assumption of constant delay τ are developed. Moreover, examples are also given to illustrate the effectiveness and non-emptiness of our results.

Keywords: interval oscillation, impulsive, variable delay, interval delay function.

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1 Introduction

In recent years, interval oscillation of impulsive differential equations was arousing the interest of many researchers, see, for example, [2,5,6,8–10]. In [5], Liu and Xu considered the forced super-linear impulsive ordinary differential equation

$$\begin{aligned} &(r(t)x'(t))' + p(t)|x(t)|^{\alpha-1}x(t) = f(t), \qquad t \geq t_0,\ t \neq \tau_k,\ k = 1,2,\dots \\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \qquad k = 1,2,\dots \end{aligned} \tag{1.1}$$

where $\alpha > 1$ and obtained some interval criteria which extend results of Nasr [7] and Wong [13]. Making use of a Picone-type identity, Özbekler and Zafer [8,9] studied super-half-linear impulsive equations of the form

$$\begin{aligned} &(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)\varphi_\beta(x) = f(t), \qquad t \neq \theta_i, \\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\varphi_\beta(x) = f_i, \qquad t = \theta_i,\ i \in \mathbb{N}. \end{aligned} \tag{1.2}$$

✉ Corresponding author. Email: zxlmath@163.com

where $\beta > \alpha > 0$ and $\varphi_\gamma(s) = |s|^{\gamma-1}s$. The interval oscillation problem for mixed nonlinear impulsive differential equations of the form

$$\begin{aligned} &(r(t)\Phi_\alpha(x'(t)))' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t)) = f(t), \qquad t \neq \tau_k, \\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \qquad k = 1, 2, \dots, \end{aligned} \tag{1.3}$$

where $\Phi_\gamma(s) = |s|^{\gamma-1}s$ and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, has been discussed in [2,10].

However, almost all of interval oscillation results for the impulsive equations in the existing literature were established only for the case of "without delay", in other words, for the case of "with delay" the study on the interval oscillation is very scarce. To the best of our knowledge, Huang and Feng [4] gave the first research in this subject recently. They considered the second order impulsive differential equations with constant delay of the form

$$\begin{aligned} &x''(t) + p(t)g(x(t-\tau)) = f(t), \qquad t \geq t_0,\ t \neq \tau_k, \\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \qquad k = 1, 2, \dots, \end{aligned} \tag{1.4}$$

and established some interval oscillation criteria which developed some known results for the equations without delay in [1,5,13].

Later, by idea of [4], Guo *et al.* [3] studied the delay case of mixed nonlinear impulsive differential equations (1.3) as follows

$$\begin{aligned} &(r(t)\Phi_\alpha(x'(t)))' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = f(t), \qquad t \neq \tau_k, \\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \qquad k = 1, 2, \dots \end{aligned} \tag{1.5}$$

where $\Phi_\gamma(s) = |s|^{\gamma-1}s$ and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$. They corrected some errors in proof of [4] (cf. Remark 2.4 in [3]) and obtained some results which developed some known results of [2,6,10]. In 2014, Zhou *et al.* [14] investigated interval qualitative properties of a class of nonlinear impulsive differential equations under three factors – impulse, damping and delay of the form

$$\begin{aligned} &[r(t)\varphi_\gamma(x'(t))]' + q_0(t)\varphi_\gamma(x'(t)) + p_0(t)\varphi_\gamma(x(t)) + p(t)g(x(t-\sigma)) = f(t), \qquad t \neq \tau_k, \\ &x(t^+) = a_k x(t), \qquad x'(t^+) = b_k x'(t), \qquad t = \tau_k,\ k = 1, 2, \dots \end{aligned} \tag{1.6}$$

where $\varphi_\gamma(s) = |s|^{\gamma-1}s$ and γ is positive.

As the comment above, the delay considered in [3,4,14] is constant. It is natural to ask if it is possible to research the interval oscillation of the impulsive equations with variable delay. In fact, when there is no impulse, the interval oscillation of differential equations with variable delay of the form

$$x''(t) + q(t)|x(\tau(t))|^\gamma \operatorname{sgn} x(\tau(t)) = f(t) \tag{1.7}$$

in the linear ($\gamma = 1$) and the superlinear ($\gamma > 1$) cases has been studied by Sun [12]. Some techniques to estimate the unknown function $x(\tau(t))/x(t)$ on each considered interval were used in [12], which inspires us to consider more complex problem.

In this paper, motivated mainly by [4,12], we consider the following second order nonlinear impulsive differential equations with variable delay

$$\begin{aligned} &x''(t) + p(t)g(x(t-\tau(t))) = f(t), \qquad t \geq t_0, \ t \neq \theta_k, \\ &x(t^+) = a_k x(t), \ x'(t^+) = b_k x'(t), \qquad t = \theta_k, \ k = 1,2,\dots \end{aligned} \tag{1.8}$$

where $\{\theta_k\}$ denotes the impulsive moments sequence with $0 \leq t_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \cdots$ and $\lim_{k\to\infty} \theta_k = \infty$. By introducing an "interval delay function" and discussing its zero points on intervals of impulse moments, we estimate the function of ratio of $x(t-\tau(t))$ and $x(t)$ on each considered interval, then making use of Riccati transformation and H functions (introduced first by Philos [11]), we establish some interval oscillation criteria, which generalize or improve the results of [4]. Moreover, we also give two examples to illustrate the effectiveness and non-emptiness of the results.

2 Main results

We first introduce some definitions and assumptions.

Let $I \subset \mathbb{R}$ be an interval, a functional space $PLC(I,\mathbb{R})$ is defined as follows:

$PLC(I,\mathbb{R}) := \{y : I \to \mathbb{R} \mid y$ is continuous on $I \setminus \{t_i\}$ and at each $t_i, y(t_i^+)$ and $y(t_i^-)$ exist, and the left continuity of y is assumed, i.e. $y(t_i^-) = y(t_i), \ i \in \mathbb{N}\}$.

Throughout the paper, we always assume that the following conditions hold:

(A_1) $p(t), f(t) \in PLC([t_0,\infty),\mathbb{R})$; $g \in C(\mathbb{R},\mathbb{R})$, $xg(x) > 0$ and there exists a positive constant η such that $\frac{g(x)}{x} \geq \eta$ for all $x \in \mathbb{R} \setminus \{0\}$;

(A_2) $\{a_k\}$, $\{b_k\}$ are real-valued sequences satisfying $b_k \geq a_k > 0$, $k = 1,2,\dots$;

(A_3) $\tau(t) \in C([t_0,\infty))$ and there exists a nonnegative constant τ such that $0 \leq \tau(t) \leq \tau$ for all $t \geq t_0$ and $\theta_{k+1} - \theta_k > \tau$ for all $k = 1,2,\dots$

Let $k(s) = \max\{i : t_0 < \theta_i < s\}$. For the discussion of impulse moments of $x(t)$ and $x(t-\tau(t))$ on two intervals $[c_j, d_j]$ $(j = 1,2)$, we need to consider the following possible cases for $k(c_j) < k(d_j)$

(S_1) $\theta_{k(c_j)} + \tau < c_j$ and $\theta_{k(d_j)} + \tau < d_j$;

(S_2) $\theta_{k(c_j)} + \tau < c_j$ and $\theta_{k(d_j)} + \tau > d_j$;

(S_3) $\theta_{k(c_j)} + \tau > c_j$ and $\theta_{k(d_j)} + \tau < d_j$;

(S_4) $\theta_{k(c_j)} + \tau > c_j$ and $\theta_{k(d_j)} + \tau > d_j$,

and the possible cases for $k(c_j) = k(d_j)$

$(\bar{S}_1)$ $\theta_{k(c_j)} + \tau < c_j$;

$(\bar{S}_2)$ $c_j < \theta_{k(c_j)} + \tau < d_j$;

$(\bar{S}_3)$ $\theta_{k(c_j)} + \tau > d_j$.

In order to save space, throughout the paper, we study (1.8) under the case of combination of (S_1) with $(\bar{S}_1)$ only. The discussions for other cases are similar and omitted.

We define a function (called "interval delay function"):

$$D_k(t) = t - \theta_k - \tau(t), \quad t \in [\theta_k, \theta_{k+1}], \quad k = 1, 2, \ldots$$

The following condition is always assumed

(A_4) $D_k(t)$ has at most one zero point on $(\theta_k, \theta_{k+1}]$ for any $k = 1, 2, \ldots$

Remark 2.1. The situation for zero points of $D_k(t)$ on $(\theta_k, \theta_{k+1}]$ may be very complicated. That is to say, the number of zero points of $D_k(t)$ on $(\theta_k, \theta_{k+1}]$ may be arbitrary. Assumption (A_4) is just a simple situation for $D_k(t)$. The research of other complex situations will be left to the reader.

Remark 2.2. In (A_4), the assumption that $D_k(t)$ has at most one zero point on $(\theta_k, \theta_{k+1}]$ may be divided into three cases as follows.

(A_{4-1}) There is one zero point $t_k \in (\theta_k, \theta_{k+1}]$ such that $D_k(t_k) = 0$, $D_k(t) < 0$ for $t \in (\theta_k, t_k)$ and $D_k(t) > 0$ for $t \in (t_k, \theta_{k+1}]$. See Figure 1; or

(A_{4-2}) There is one zero point $t_k \in (\theta_k, \theta_{k+1}]$ such that $D_k(t_k) = 0$, $D_k(t) > 0$ for $t \in (\theta_k, t_k) \cup (t_k, \theta_{k+1}]$. See Figure 2; or

(A_{4-3}) There is not any zero point such that $D_k(t) = 0$. This case must lead to $D_k(t) > 0$ for all $t \in (\theta_k, \theta_{k+1}]$. See Figure 3.

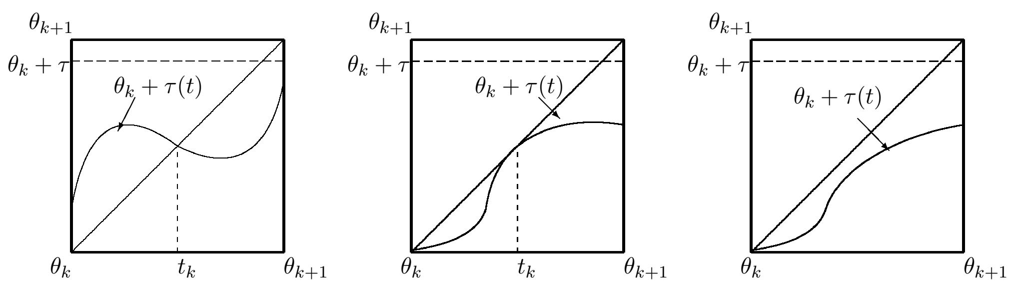

Figure 1 Figure 2 Figure 3

In Remark 2.2, the case (A_{4-1}) is more complex to consider than other two cases for the estimation of $\frac{x(t-\tau(t))}{x(t)}$. We study (1.8) under the assumption (A_{4-1}) only throughout the paper. The discussions for cases (A_{4-2}) and (A_{4-3}) are similar and omitted.

Lemma 2.3. *Assume that for any $T \geq t_0$, there exist $c_1, d_1 \notin \{\theta_k\}$, such that $T < c_1 - \tau < c_1 < d_1$ and*

$$\begin{cases} p(t) \geq 0, & t \in [c_1 - \tau, d_1] \backslash \{\theta_k\}; \\ f(t) \leq 0, & t \in [c_1 - \tau, d_1] \backslash \{\theta_k\}. \end{cases} \tag{2.1}$$

If $x(t)$ is a positive solution of (1.8), then there exist the following estimations of $\frac{x(t-\tau(t))}{x(t)}$:

(I) when $k(c_1) < k(d_1)$, $t_i \in (\theta_i, \theta_{i+1}]$ for $i = k(c_1) + 1, \ldots, k(d_1) - 1$,

(a) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_i-\tau(t)}{t-\theta_i}, \quad t \in (t_i, \theta_{i+1}]$;

(b) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_i}{b_i(t+\tau(t)-\theta_i)}$, $\quad t \in (\theta_i, t_i)$;

(c) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_{k(c_1)}-\tau(t)}{t-\theta_{k(c_1)}}$, $\quad t \in [c_1, \theta_{k(c_1)+1}]$;

(d) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_{k(d_1)}-\tau(t)}{t-\theta_{k(d_1)}}$, $\quad t \in (t_{k(d_1)}, d_1]$;

(e) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_{k(d_1)}}{b_{k(d_1)}(t+\tau(t)-\theta_{k(d_1)})}$, $\quad t \in (\theta_{k(d_1)}, t_{k(d_1)})$;

(II) *when* $k(c_1) = k(d_1)$,

(f) $\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_{k(c_1)}-\tau(t)}{t-\theta_{k(c_1)}}$, $\quad t \in [c_1, d_1]$.

Proof. From (1.8), (2.1) and (A_1), we obtain, for $t \in [c_1, d_1]\backslash\{\theta_k\}$, that

$$x''(t) = f(t) - p(t)g(x(t-\tau(t))) \le 0.$$

Hence $x'(t)$ is nonincreasing on the interval $[c_1, d_1]\backslash\{\theta_k\}$. Next, we give the proof of cases (a) and (b) only. For other cases, the proof is similar and will be omitted.

Case (a). If $t_i < t \le \theta_{i+1}$, then $(t-\tau(t), t) \subset (\theta_i, \theta_{i+1}]$. Thus there is no impulsive moment in $(t-\tau(t), t)$. For any $s \in (t-\tau(t), t)$, we have

$$x(s) - x(\theta_i^+) = x'(\xi_1)(s-\theta_i), \qquad \xi_1 \in (\theta_i, s).$$

Since $x(\theta_i^+) > 0$ and $x'(s)$ is nonincreasing on (θ_i, θ_{i+1}), we have

$$x(s) > x'(\xi_1)(s-\theta_i) \ge x'(s)(s-\theta_i). \tag{2.2}$$

Thus

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\theta_i}. \tag{2.3}$$

Integrating both sides of above inequality from $t-\tau(t)$ to t, we obtain

$$\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_i-\tau(t)}{t-\theta_i}, \qquad t \in (t_i, \theta_{i+1}]. \tag{2.4}$$

Case (b). If $\theta_i < t < t_i$, then $\theta_i - \tau < t - \tau(t) < \theta_i < t$. There is an impulsive moment θ_i in $(t-\tau(t), t)$. For any $t \in (\theta_i, t_i)$, we have

$$x(t) - x(\theta_i^+) = x'(\xi_2)(t-\theta_i), \qquad \xi_2 \in (\theta_i, t).$$

Using the impulsive condition of (1.8) and the monotone properties of $x'(t)$, we get

$$x(t) - a_i x(\theta_i) \le x'(\theta_i^+)(t-\theta_i) = b_i x'(\theta_i)(t-\theta_i).$$

Since $x(\theta_i) > 0$, we have

$$\frac{x(t)}{x(\theta_i)} - a_i \le b_i \frac{x'(\theta_i)}{x(\theta_i)}(t-\theta_i). \tag{2.5}$$

In addition,

$$x(\theta_i) > x(\theta_i) - x(\theta_i - \tau(t)) = x'(\xi_3)\tau(t), \qquad \xi_3 \in (\theta_i - \tau(t), \theta_i). \tag{2.6}$$

Similarly to the analysis of (2.2) and (2.3), we have

$$\frac{x'(\theta_i)}{x(\theta_i)} < \frac{1}{\tau(t)}. \tag{2.7}$$

From (2.5) and (2.7), we get

$$\frac{x(t)}{x(\theta_i)} < a_i + \frac{b_i}{\tau(t)}(t-\theta_i).$$

In view of (A_2), we have

$$\frac{x(\theta_i)}{x(t)} > \frac{\tau(t)}{\tau(t)a_i + b_i(t-\theta_i)} \geq \frac{\tau(t)}{b_i(t+\tau(t)-\theta_i)} > 0. \tag{2.8}$$

On the other hand, using similar analysis of (2.2) and (2.3), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\theta_i+\tau(t)}, \qquad s \in (\theta_i - \tau(t), \theta_i). \tag{2.9}$$

Integrating (2.9) from $t-\tau(t)$ $(> (\theta_i - \tau(t))$ to θ_i where $t \in (\theta_i, t_i)$, we have

$$\frac{x(t-\tau(t))}{x(\theta_i)} > \frac{t-\theta_i}{\tau(t)} \geq 0. \tag{2.10}$$

From (2.8) and (2.10), we obtain

$$\frac{x(t-\tau(t))}{x(t)} > \frac{t-\theta_i}{b_i(t+\tau(t)-\theta_i)}, \qquad t \in (\theta_i, t_i). \tag{2.11}$$

□

Lemma 2.4. *Assume that for any $T \geq t_0$, there exist $c_2, d_2 \notin \{\theta_k\}$, such that $T < c_2 - \tau < c_2 < d_2$ and*

$$\begin{cases} p(t) \geq 0, & t \in [c_2 - \tau, d_2] \backslash \{\theta_k\}; \\ f(t) \geq 0, & t \in [c_2 - \tau, d_2] \backslash \{\theta_k\}. \end{cases} \tag{2.12}$$

If $x(t)$ is a negative solution of (1.8), then estimations (a)–(f) in Lemma 2.3 are correct with the replacement of $[c_1, d_1]$ by $[c_2, d_2]$.

The proof of Lemma 2.4 is similar to that of Lemma 2.3 and will be omitted.

Lemma 2.5. *Assume that for any $T \geq t_0$ there exist $c_1, d_1 \notin \{\theta_k\}$ such that $T < c_1 - \tau < c_1 < d_1$ and (2.1) holds. Let $x(t)$ be a positive solution of (1.8) and $u(t)$ be defined by*

$$u(t) := \frac{x'(t)}{x(t)}, \quad \text{for } t \in [c_1, d_1]. \tag{2.13}$$

If $k(c_1) < k(d_1)$ and θ_i, $i = k(c_1)+1, \ldots, k(d_1)$, are impulsive moments in $[c_1, d_1]$, then, there are the following estimations of $u(t)$:

(g) $u(\theta_i) \leq \frac{1}{\theta_i - \theta_{i-1}}$, for $\theta_i \in [c_1, d_1]$, $i = k(c_1)+2, \ldots, k(d_1)$;

(h) $u(\theta_{k(c_1)+1}) \leq \frac{1}{\theta_{k(c_1)+1} - c_1}$, for $\theta_{k(c_1)+1} \in [c_1, d_1]$.

Proof. For $t \in (\theta_{i-1}, \theta_i] \subset [c_1, d_1]$, $i = k(c_1)+2, \dots, k(d_1)$, we have

$$x(t) - x(\theta_{i-1}) = x'(\xi)(t - \theta_{i-1}), \qquad \xi \in (\theta_{i-1}, t).$$

From the proof of Lemma 2.3, we know that $x'(t)$ is nonincreasing. In view of $x(\theta_{i-1}) > 0$, we obtain

$$x(t) > x'(\xi)(t - \theta_{i-1}) \geq x'(t)(t - \theta_{i-1}).$$

Then

$$\frac{x'(t)}{x(t)} < \frac{1}{t - \theta_{i-1}}.$$

Let $t \to \tau_i^-$, it follows

$$u(\theta_i) = \frac{x'(\theta_i)}{x(\theta_i)} \leq \frac{1}{\theta_i - \theta_{i-1}}, \qquad i = k(c_1)+2, \dots, k(d_1). \tag{2.14}$$

Using similar analysis on $(c_1, \theta_{k(c_1)+1}]$, we can get

$$u(\theta_{k(c_1)+1}) \leq \frac{1}{\theta_{k(c_1)+1} - c_1}. \tag{2.15}$$

□

Lemma 2.6. *Assume that for any $T \geq t_0$, there exist $c_2, d_2 \notin \{\theta_k\}$ such that $T < c_2 - \tau < c_2 < d_2$ and (2.12) holds. Let $x(t)$ be a negative solution of (1.8) and $u(t)$ be defined by*

$$u(t) := \frac{x'(t)}{x(t)}, \quad \text{for } t \in [c_2, d_2], \tag{2.16}$$

If $k(c_2) < k(d_2)$ and θ_i, $i = k(c_2)+1, \dots, k(d_2)$, are impulsive moments in $[c_2, d_2]$, then, the estimations (g), (h) in Lemma 2.5 are correct with the replacement of $[c_1, d_1]$ by $[c_2, d_2]$.

The proof of Lemma 2.6 is similar to that of Lemma 2.5 and will be omitted.

We introduce a space $\Omega(c, d)$ as follows

$$\Omega(c, d) := \{w \in C^1[c, d] : w(t) \not\equiv 0,\ w(c) = w(d) = 0\}.$$

In order to save a space, we define

$$\int_{[c_j, d_j]} := \int_{c_j}^{\theta_{k(c_j)+1}} + \sum_{i=k(c_j)+1}^{k(d_j)-1} \Big(\int_{\theta_i}^{t_i} + \int_{t_i}^{\theta_{i+1}} \Big) + \int_{\theta_{k(d_j)}}^{t_{k(d_j)}} + \int_{t_{k(d_j)}}^{d_j}, \quad \text{for } j = 1, 2.$$

Lemma 2.7. *Assume that for any $T \geq t_0$, there exist $c_1, d_1 \notin \{\theta_k\}$ such that $T < c_1 - \tau < c_1 < d_1$ and (2.1) hold. Let $x(t)$ be a positive solution of (1.8) and $u(t)$ be defined by (2.13). If $w_1(t) \in \Omega(c_1, d_1)$, then for $k(c_1) < k(d_1)$,*

$$\eta \int_{[c_1, d_1]} \frac{x(t - \tau(t))}{x(t)} p(t) w_1^2(t) dt - \int_{c_1}^{d_1} w_1'^2(t) dt \leq \sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i - a_i}{a_i} w_1^2(\theta_i) u(\theta_i) \tag{2.17}$$

and for $k(c_1) = k(d_1)$,

$$\eta \int_{c_1}^{d_1} p(t) w_1^2(t) \frac{x(t - \tau(t))}{x(t)} dt - \int_{c_1}^{d_1} w_1'^2(t) dt \leq 0. \tag{2.18}$$

Proof. Differentiating $u(t)$ and in view of (1.8) and condition (A_1), we obtain, for $t \neq \theta_k$,

$$\begin{aligned} u'(t) &= \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2 = \frac{f(t) - p(t)g(x(t-\tau(t)))}{x(t)} - u^2(t) \\ &\leq -\eta p(t)\frac{x(t-\tau(t))}{x(t)} - u^2(t). \end{aligned} \tag{2.19}$$

If $k(c_1) < k(d_1)$, we assume impulsive moments in $[c_1, d_1]$ are $\theta_{k(c_1)+1}, \theta_{k(c_1)+2}, \dots, \theta_{k(d_1)}$ and zero points of $D_i(t)$ in intervals (θ_i, θ_{i+1}) are t_i, $i = k(c_1)+1, \dots, k(d_1)$. Choosing a $w_1(t) \in \Omega(c_1, d_1)$, multiplying both sides of (2.19) by $w_1^2(t)$ and then integrating it from c_1 to d_1, we obtain

$$\int_{[c_1,d_1]} u'(t)w_1^2(t)dt \leq -\eta \int_{[c_1,d_1]} \frac{x(t-\tau(t))}{x(t)} p(t)w_1^2(t)dt - \int_{[c_1,d_1]} u^2(t)w_1^2(t)dt.$$

Using the integration by parts formula on the left side of above inequality and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\begin{aligned} \sum_{i=k(c_1)+1}^{k(d_1)} w_1^2(\theta_i)[u(\theta_i) - u(\theta_i^+)] \leq &-\eta \int_{[c_1,d_1]} \frac{x(t-\tau(t))}{x(t)} p(t)w_1^2(t)dt \\ &+ \int_{[c_1,d_1]} V(w_1(t), u(t))dt, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} V(w_1(t), u(t)) &= 2w_1'(t)w_1(t)u(t) - w_1^2(t)u^2(t) \\ &= w_1'^2(t) - (w_1(t)u(t) - w_1'(t))^2 \leq w_1'^2(t). \end{aligned} \tag{2.21}$$

Meanwhile, for $t = \theta_k$, $k = 1, 2, \dots,$ we have

$$u(\theta_k^+) = \frac{b_k}{a_k} u(\theta_k).$$

Hence

$$\sum_{i=k(c_1)+1}^{k(d_1)} w_1^2(\theta_i)[u(\theta_i) - u(\theta_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i} w_1^2(\theta_i)u(\theta_i). \tag{2.22}$$

Therefore, from (2.20)–(2.22), we can get (2.17).

If $k(c_1) = k(d_1)$, there is no impulsive moment in $[c_1, d_1]$. Multiplying both sides of (2.19) by $w_1^2(t)$ and integrating it from c_1 to d_1, we obtain

$$\int_{c_1}^{d_1} u'(t)w_1^2(t)dt \leq \int_{c_1}^{d_1} u(t)^2 w_1^2(t)dt - \eta \int_{c_1}^{d_1} p(t)w_1^2(t)\frac{x(t-\tau(t))}{x(t)}dt.$$

Using the integration by parts on the left-hand side and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\int_{c_1}^{d_1} V(w_1(t), u(t))dt - \eta \int_{c_1}^{d_1} p(t)w_1^2(t)\frac{x(t-\tau(t))}{x(t)}dt \geq 0,$$

where $V(w_1(t), u(t))$ is defined by (2.21). Thus

$$\int_{c_1}^{d_1} w_1'^2(t)dt - \eta \int_{c_1}^{d_1} p(t)w_1^2(t)\frac{x(t-\tau(t))}{x(t)}dt \geq 0. \tag{2.23}$$

This is (2.18). Therefore we complete the proof. □

Lemma 2.8. *Assume that for any $T \geq t_0$, there exist $c_2, d_2 \notin \{\theta_k\}$ such that $T < c_2 - \tau < c_2 < d_2$ and (2.1) holds. Let $x(t)$ be a negative solution of (1.8) and $u(t)$ be defined by (2.16). If $w_2(t) \in \Omega(c_2, d_2)$, then (2.17) and (2.18) hold with the replacements of $[c_1, d_1]$ and w_1 by $[c_2, d_2]$ and w_2 respectively.*

The proof of Lemma 2.8 is similar to that of Lemma 2.7 and will be omitted.

For convenience in the expression below, we use the following notations.

For two constants $c, d \notin \{\theta_k\}$ with $c < d$, $k(c) < k(d)$, a function $\varphi \in C([c,d], \mathbb{R})$ and a function $\phi \in PLC([c,d], \mathbb{R})$, we define a functional $Q : C([c,d], \mathbb{R}) \to \mathbb{R}$ by

$$Q_c^d[\varphi] := \varphi(\theta_{k(c)+1}) \frac{b_{k(c)+1} - a_{k(c)+1}}{a_{k(c)+1}(\theta_{k(c)+1} - c)} + \sum_{i=k(c)+2}^{k(d)} \varphi(\theta_i) \frac{b_i - a_i}{a_i(\theta_i - \theta_{i-1})}, \tag{2.24}$$

where $\sum_s^t = 0$ if $s > t$, and a functional $I : PLC([c,d], \mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned} I_c^d[\phi] := {} & \int_c^{\theta_{k(c)+1}} \phi(t) \frac{t - \theta_{k(c)} - \tau(t)}{t - \theta_{k(c)}} dt + \sum_{i=k(c)+1}^{k(d)-1} \left[\int_{\theta_i}^{t_i} \phi(t) \frac{t - \theta_i}{b_i(t - \theta_i + \tau(t))} dt \right. \\ & \left. + \int_{t_i}^{\theta_{i+1}} \phi(t) \frac{t - \theta_i - \tau(t)}{t - \theta_i} dt \right] + \int_{\theta_{k(d)}}^{t_{k(d)}} \phi(t) \frac{t - \theta_{k(d)}}{b_{k(d)}(t - \theta_{k(d)} + \tau(t))} dt \\ & + \int_{t_{k(d)}}^{d} \phi(t) \frac{t - \tau(t) - \theta_{k(d)}}{t - \theta_{k(d)}} dt. \end{aligned} \tag{2.25}$$

where t_i are zero points of $D_i(t)$ $(i = k(c)+1, \ldots, k(d))$ on (c,d).

Theorem 2.9. *Assume that for any $T \geq t_0$, there exist $c_j, d_j \notin \{\theta_k\}, j = 1, 2$, such that $T < c_1 - \tau < c_1 < d_1 \leq c_2 - \tau < c_2 < d_2$, (2.1)) and (2.12) hold. If there exist $w_j(t) \in \Omega(c_j, d_j)$ such that, for $k(c_j) < k(d_j)$, $j = 1, 2$,*

$$\eta I_{c_j}^{d_j}[p(t) w_j^2(t)] - \int_{c_j}^{d_j} w_j'^2(t) dt > Q_{c_j}^{d_j}[w_j^2], \tag{2.26}$$

and for $k(c_j) = k(d_j)$, $j = 1, 2$,

$$\eta \int_{c_j}^{d_j} p(t) w_j^2(t) \frac{t - \theta_{k(c_j)} - \tau(t)}{t - \theta_{k(c_j)}} dt - \int_{c_j}^{d_j} w_j'^2(t) dt > 0, \tag{2.27}$$

then (1.8) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.8). If $x(t)$ is positive solution, we choose the interval $[c_1 - \tau, d_1]$ to consider.

When the case $k(c_1) < k(d_1)$ holds, from Lemmas 2.3, 2.5 and 2.7 we easily obtain

$$\eta I_{c_1}^{d_1}[p(t) w_1^2(t)] - \int_{c_1}^{d_1} w_1'^2(t) dt \leq Q_{c_1}^{d_1}[w_1^2], \tag{2.28}$$

which contradicts condition (2.26) for $j = 1$.

When the cases $k(c_1) = k(d_1)$ holds, from Lemma 2.3 and Lemma 2.5 we easily obtain

$$\eta \int_{c_1}^{d_1} p(t) w_1^2(t) \frac{t - \theta_{k(c_1)} - \tau(t)}{t - \theta_{k(c_1)}} dt - \int_{c_1}^{d_1} w_1'^2(t) dt \leq 0, \tag{2.29}$$

which contradicts condition (2.27).

If $x(t)$ is negative solution, we choose the interval $[c_2-\tau, d_2]$ to consider. From Lemmas 2.4, 2.6 and 2.8 we easily obtain contradictions to (2.26) and (2.27) for $j=2$.

Therefore we complete the proof. □

Remark 2.10. When $\tau(t)=\tau$, (1.8) reduces to the (1.4) studied by Huang and Feng [4]. Therefore Theorem 2.9 develops Theorem 2.1 in [4].

In formula (2.25), t_i and $t_{k(d_j)}$ are zero points of $D_i(t)$ and $D_{k(d_j)}(t)$ respectively. In general, it is not easy to solve these zero points from $D_i(t)=0$ and $D_{k(d_j)}(t)=0$. This makes the calculation of (2.25) more difficult. To overcome this difficulty it is need to avoid these points being upper limit (or lower limit) of integrals. So, we give the following theorem.

Theorem 2.11. *Assume that for any $T\geq t_0$, there exist $c_j, d_j \notin \{\theta_k\}$, $j=1,2$, such that $T<c_1-\tau<c_1<d_1\leq c_2-\tau<c_2<d_2$, (2.1) and (2.12) hold. If there exist $w_j(t)\in\Omega(c_j,d_j)$ such that, for $k(c_j)<k(d_j)$, $j=1,2$,*

$$\begin{aligned}
&\eta\int_{c_j}^{\theta_{k(c_j)+1}} p(t)w_j^2(t)\frac{t-\theta_{k(c_j)}-\tau(t)}{t-\theta_{k(c_j)}}dt+\eta\sum_{i=k(c_j)+1}^{k(d_j)-1}\int_{\theta_i}^{\theta_{i+1}} p(t)w_j^2(t)\frac{t-\theta_i-\tau(t)}{t}dt\\
&\quad+\eta\int_{\theta_{k(d_j)}}^{d_j} p(t)w_j^2(t)\frac{t-\theta_{k(d_j)}-\tau(t)}{t}dt-\int_{c_j}^{d_j} w_j'^2(t)dt>Q_{c_j}^{d_j}[w_j^2],
\end{aligned}\tag{2.30}$$

and for $k(c_j)=k(d_j)$, $j=1,2$,

$$\eta\int_{c_j}^{d_j} p(t)w_j^2(t)\frac{t-\theta_{k(c_j)}-\tau(t)}{t-\theta_{k(c_j)}}dt-\int_{c_j}^{d_j} w_j'^2(t)dt>0,\tag{2.31}$$

then (1.8) is oscillatory.

Proof. The proof is similar to that of Theorem 2.9, only the estimations of $\frac{x(t-\tau(t))}{x(t)}$ on $(t_i,\theta_{i+1}]$, (θ_i,t_i), $(t_{k(d_j)},d_j)$ and $(\theta_{k(d_j)},t_{k(d_j)})$ need to be modified.

From (2.4), we have

$$\frac{x(t-\tau(t))}{x(t)}>\frac{t-\theta_i-\tau(t)}{t-\theta_i}>\frac{t-\theta_i-\tau(t)}{t},\qquad t\in(t_i,\theta_{i+1}].\tag{2.32}$$

From (2.11), we have

$$\frac{x(t-\tau(t))}{x(t)}>\frac{t-\theta_i}{b_i(t+\tau(t)-\theta_i)}>0>\frac{t-\theta_i-\tau(t)}{t},\qquad t\in(\theta_i,t_i).\tag{2.33}$$

Similarly, from (2.17) and (2.18), we have

$$\frac{x(t-\tau(t))}{x(t)}>\frac{t-\theta_{k(d_j)}-\tau(t)}{t},\qquad t\in(t_{k(d_j)},d_j]\cup(\theta_{k(d_j)},t_{k(d_j)}).\tag{2.34}$$

Therefore we complete the proof. □

In the following we will establish Kamenev-type interval oscillation criteria for (1.8) by the ideas of Philos [11].

Let $D=\{(t,s): t_0\leq s\leq t\}$, $H_1,H_2\in C^1(D,\mathbb{R})$, then a pair of functions H_1,H_2 is said to belong to a function set H, defined by $(H_1,H_2)\in H$, if there exist $h_1,h_2\in L_{\mathrm{loc}}(D,\mathbb{R})$ satisfying the following conditions:

(A_5) $H_1(t,t) = H_2(t,t) = 0,\ H_1(t,s) > 0,\ H_2(t,s) > 0$ for $t > s$;

(A_6) $\frac{\partial}{\partial t} H_1(t,s) = h_1(t,s)H_1(t,s),\ \ \frac{\partial}{\partial s} H_2(t,s) = h_2(t,s)H_2(t,s)$.

We assume there exist $c_j, d_j, \delta_j \notin \{\theta_k, k = 1,2,\dots\}$ $(j = 1,2)$ which satisfy $T < c_1 - \tau < c_1 < \delta_1 < d_1 < c_2 - \tau < c_2 < \delta_2 < d_2$ for any $T \geq t_0$. Noticing whether there are impulsive moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$ or not, we should consider the following four cases

(S_5) $k(c_j) < k(\delta_j) < k(d_j)$; (S_6) $k(c_j) = k(\delta_j) < k(d_j)$;

(S_7) $k(c_j) < k(\delta_j) = k(d_j)$; (S_8) $k(c_j) = k(\delta_j) = k(d_j)$.

Moreover, in the discussion of the impulse moments of $x(t - \tau(t))$, it is necessary to consider the following three cases:

$(\bar{S}_5)$ $t_{k(\delta_j)} < \delta_j$; $(\bar{S}_6)$ $t_{k(\delta_j)} > \delta_j$; $(\bar{S}_7)$ $t_{k(\delta_j)} = \delta_j$.

In the following theorems, we only consider the case of combination of (S_5) with $(\bar{S}_5)$. For the other combinations, similar conclusions can be given and their details will be omitted here.

For convenience in the expression below, we define, for $j = 1,2$, that

$$\Pi_{c_j}^{\delta_j}[H_1(t,c_j)] := \eta I_{c_j}^{\delta_j}[p(t)H_1(t,c_j)] - \frac{1}{4}\int_{c_j}^{\delta_j} H_1(t,c_j)h_1^2(t,c_j)dt,$$
$$\Pi_{\delta_j}^{d_j}[H_2(d_j,t)] := \eta I_{\delta_j}^{d_j}[p(t)H_2(d_j,t)] - \frac{1}{4}\int_{\delta_j}^{d_j} H_2(d_j,t)h_2^2(d_j,t)dt.$$

Theorem 2.12. *Assume that for any $T \geq t_0$, there exist $c_j, d_j, \delta_j \notin \{\theta_k\}, j = 1,2$, such that $T < c_1 - \tau < c_1 < d_1 \leq c_2 - \tau < c_2 < d_2$ and (2.1) and (2.12) hold. If there exists a pair of $(H_1, H_2) \in H$ such that, for $j = 1,2$,*

$$\frac{\Pi_{c_j}^{\delta_j}[H_1(t,c_j)]}{H_1(\delta_j,c_j)} + \frac{\Pi_{\delta_j}^{d_j}[H_2(d_j,t)]}{H_2(d_j,\delta_j)} > \frac{Q_{c_j}^{\delta_j}[H_1(\cdot,c_j)]}{H_1(\delta_j,c_j)} + \frac{Q_{\delta_j}^{d_j}[H_2(d_j,\cdot)]}{H_2(d_j,\delta_j)}, \tag{2.35}$$

then (1.8) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.8). If $x(t)$ is positive solution, we choose the interval $[c_1 - \tau, d_1]$ to consider.

Defining function $u(t)$ as in (2.13) and using the same proof as in Lemma 2.5, we can get (2.19). Multiplying both sides of (2.19) by $H_1(t,c_1)$ and integrating it from c_1 to δ_1, we have

$$\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt \leq -\int_{c_1}^{\delta_1} H_1(t,c_1)u^2(t)dt - \eta\int_{c_1}^{\delta_1} p(t)H_1(t,c_1)\frac{x(t-\tau(t))}{x(t)}dt. \tag{2.36}$$

Noticing impulsive moments $\theta_{k(c_1)+1}, \theta_{k(c_1)+2}, \dots, \theta_{k(\delta_1)}$ in $[c_1, \delta_1]$ and using the integration by parts on the left-hand side of above inequality, we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt &= \left(\int_{c_1}^{\theta_{k(c_1)+1}} + \int_{\theta_{k(c_1)+1}}^{\theta_{k(c_1)+2}} + \cdots + \int_{\theta_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)du(t)\\ &= \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{a_i - b_i}{a_i} H_1(\theta_i,c_1)u(\theta_i) + H_1(\delta_1,c_1)u(\delta_1)\\ &\quad - \int_{c_1}^{\delta_1} H_1(t,c_1)h_1(t,c_1)u(t)dt.\end{aligned} \tag{2.37}$$

Substituting (2.37) into (2.36), we have

$$\eta \int_{c_1}^{\delta_1} p(t)H_1(t,c_1)\frac{x(t-\tau(t))}{x(t)}dt \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i-a_i}{a_i}H_1(\theta_i,c_1)u(\theta_i) - H_1(\delta_1,c_1)u(\delta_1) + \int_{c_1}^{\delta_1} H_1(t,c_1)\Big(h_1(t,c_1)u(t)-u^2(t)\Big)dt. \tag{2.38}$$

Since

$$h_1(t,c_1)u(t)-u^2(t) \leq \frac{1}{4}h_1^2(t,c_1),$$

then we have

$$\eta \int_{c_1}^{\delta_1} p(t)H_1(t,c_1)\frac{x(t-\tau(t))}{x(t)}dt \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i-a_i}{a_i}H_1(\theta_i,c_1)u(\theta_i) - H_1(\delta_1,c_1)u(\delta_1) + \frac{1}{4}\int_{c_1}^{\delta_1} H_1(t,c_1)h_1^2(t,c_1)dt. \tag{2.39}$$

On the other hand, using same method as in Lemma 2.3, we can get estimations of the function $\frac{x(t-\sigma)}{x(t)}$ in several sub-intervals of $[c_1,\delta_1]$ and then we obtain

$$\begin{aligned}
\int_{c_1}^{\delta_1} p(t)H_1(t,c_1)\frac{x(t-\tau(t))}{x(t)}dt &> \int_{c_1}^{\theta_{k(c_1)+1}} p(t)H_1(t,c_1)\frac{t-\theta_{k(c_1)}-\tau(t)}{t-\theta_{k(c_1)}}dt \\
&+ \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left\{\int_{\theta_i}^{t_i} p(t)H_1(t,c_1)\frac{t-\theta_i}{b_i(t+\tau(t)-\theta_i)}dt + \int_{t_i}^{\theta_{i+1}} p(t)H_1(t,c_1)\frac{t-\theta_i-\tau(t)}{t-\theta_i}dt\right\} \\
&+ \int_{\theta_{k(\delta_1)}}^{t_{k(\delta_1)}} p(t)H_1(t,c_1)\frac{t-\theta_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau(t)-\theta_{k(\delta_1)})}dt + \int_{t_{k(\delta_1)}}^{\delta_1} p(t)H_1(t,c_1)\frac{t-\theta_{k(\delta_1)}-\tau(t)}{t-\theta_{k(\delta_1)}}dt \\
&=: I_{c_1}^{\delta_1}[p(t)H_1(t,c_1)].
\end{aligned} \tag{2.40}$$

From (2.39) and (2.40), we have

$$\Pi_{c_1}^{\delta_1}[H_1(t,c_1)] \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i-a_i}{a_i}H_1(\theta_i,c_1)u(\theta_i) - H_1(\delta_1,c_1)u(\delta_1). \tag{2.41}$$

Multiplying both sides of (2.19) by $H_2(d_1,t)$ and using similar process to the above, we obtain

$$\Pi_{\delta_1}^{d_1}[H_2(d_1,t)] \leq \sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i-a_i}{a_i}H_2(d_1,\theta_i)u(\theta_i) + H_2(d_1,\delta_1)u(\delta_1). \tag{2.42}$$

Dividing (2.41) and (2.42) by $H_1(\delta_1,c_1)$ and $H_2(d_1,\delta_1)$ respectively and adding them, we get

$$\frac{\Pi_{c_1}^{\delta_1}[H_1(t,c_1)]}{H_1(\delta_1,c_1)} + \frac{\Pi_{\delta_1}^{d_1}[H_2(d_1,t)]}{H_2(d_1,\delta_1)} \leq \frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i-a_i}{a_i}H_1(\theta_i,c_1)u(\theta_i) + \frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i-a_i}{a_i}H_2(d_1,\theta_i)u(\theta_i). \tag{2.43}$$

Meanwhile, by using same way as in Lemma 2.5, we get the estimations of $u(\theta_i)$

$$u(\theta_{k(c_1)+1}) \le \frac{1}{\theta_{k(c_1)+1} - c_1}, \qquad u(\theta_i) \le \frac{1}{\theta_i - \theta_{i-1}}, \qquad i = k(c_1)+2, \dots, k(\delta_1);$$
$$u(\theta_{k(\delta_1)+1}) \le \frac{1}{\theta_{k(\delta_1)+1} - c_1}, \qquad u(\theta_i) \le \frac{1}{\theta_i - \theta_{i-1}}, \qquad i = k(\delta_1)+2, \dots, k(d_1).$$

Then we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i - a_i}{a_i} H_1(\theta_i, c_1) u(\theta_i) \le Q_{c_1}^{\delta_1}[H_1(\cdot, c_1)], \tag{2.44}$$

and

$$\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i - a_i}{a_i} H_2(d_1, \theta_i) u(\theta_i) \le Q_{\delta_1}^{d_1}[H_2(d_1, \cdot)]. \tag{2.45}$$

From (2.43), (2.44) and (2.45), we can obtain a contradiction to the condition (2.35) for $j = 1$.

If $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.8). The proof is similar and will be omitted. Therefore we complete the proof. □

Remark 2.13. Theorem 2.12 develops Theorem 2.5 in [4].

Define

$$\begin{aligned}\widetilde{\Pi}_{c_j}^{\delta_j}[H_1(t, c_j)] := &\int_{c_j}^{\theta_{k(c_j)+1}} \widetilde{H}_1(t, c_j) \frac{t - \theta_{k(c_j)} - \tau(t)}{t - \theta_{k(c_j)}} dt + \sum_{i=k(c_j)+1}^{k(\delta_j)-1} \int_{\theta_i}^{\theta_{i+1}} \widetilde{H}_1(t, c_j) \frac{t - \theta_i - \tau(t)}{t} dt \\ &+ \int_{\theta_{k(\delta_j)}}^{\delta_j} \widetilde{H}_1(t, c_j) \frac{t - \theta_{k(\delta_j)} - \tau(t)}{t} dt - \frac{1}{4} \int_{c_j}^{\delta_j} H_1(t, c_j) h_1^2(t, c_j) dt\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Pi}_{\delta_j}^{d_j}[H_2(d_j, t)] := &\int_{\delta_j}^{\theta_{k(\delta_j)+1}} \widetilde{H}_2(d_j, t) \frac{t - \theta_{k(\delta_j)} - \tau(t)}{t - \theta_{k(\delta_j)}} dt + \sum_{i=k(\delta_j)+1}^{k(d_j)-1} \int_{\theta_i}^{\theta_{i+1}} \widetilde{H}_2(d_j, t) \frac{t - \theta_i - \tau(t)}{t} dt \\ &+ \int_{\theta_{k(d_j)}}^{d_j} \widetilde{H}_2(d_j, t) \frac{t - \theta_{k(d_j)} - \tau(t)}{t} dt - \frac{1}{4} \int_{\delta_j}^{d_j} H_2(d_j, t) h_2^2(d_j, t) dt,\end{aligned}$$

where

$$\widetilde{H}_1(t, c_j) = \eta p(t) H_1(t, c_j), \qquad j = 1, 2,$$
$$\widetilde{H}_2(d_j, t) = \eta p(t) H_2(d_j, t), \qquad j = 1, 2.$$

Similar to the proof way of Theorem 2.11 and Theorem 2.12, we get the following theorem in which the zero points of $D_i(t) = 0$ and $D_{k(d_j)}(t) = 0$ are avoided to be upper limit (or lower limit) of integrals.

Theorem 2.14. *Assume that for any $T \ge t_0$, there exist $c_j, d_j, \delta_j \notin \{\theta_k\}, j = 1, 2$, such that $T < c_1 - \sigma < c_1 < d_1 \le c_2 - \sigma < c_2 < d_2$ and (2.4) and (2.12) hold. If there exists a pair of $(H_1, H_2) \in H$ such that*

$$\frac{\widetilde{\Pi}_{c_j}^{\delta_j}[H_1(t, c_j)]}{H_1(\delta_j, c_j)} + \frac{\widetilde{\Pi}_{\delta_j}^{d_j}[H_2(d_j, t)]}{H_2(d_j, \delta_j)} > \frac{Q_{c_j}^{\delta_j}[H_1(\cdot, c_j)]}{H_1(\delta_j, c_j)} + \frac{Q_{\delta_j}^{d_j}[H_2(d_j, \cdot)]}{H_2(d_j, \delta_j)}, \qquad j = 1, 2, \tag{2.46}$$

then (1.8) is oscillatory.

3 Examples

In this section, we give two examples to illustrate the effectiveness and non-emptiness of our results.

Example 3.1. Consider the following impulsive differential equation with variable delay

$$\begin{aligned} & x''(t) + \eta x\Big(t - \frac{\pi}{12}\sin^2 t\Big) = -\sin(2t), \qquad t \neq \theta_{n,i}, \\ & x(t^+) = a_{n,i}x(t), \qquad x'(t^+) = b_{n,i}x'(t), \qquad t = \theta_{n,i}, \end{aligned} \tag{3.1}$$

where $\theta_{n,i} = 2n\pi + \frac{2\pi}{9} + (i-1)\frac{\pi}{3}$, $i = 1, 2$, $n \in \mathbb{N}$ and η is positive constant.

For any $T > 0$, we can choose large n_0 such that $T < c_1 = 2n\pi + \frac{\pi}{6}$, $d_1 = 2n\pi + \frac{\pi}{3}$, $c_2 = 2n\pi + \frac{\pi}{2}$, $d_2 = 2n\pi + \frac{2\pi}{3}$, for $n = n_0, n_0+1, \dots$. There are impulsive moments $\theta_{n,1}$ in $[c_1, d_1]$ and $\theta_{n,2}$ in $[c_2, d_2]$. The variable delay $\tau(t) = \frac{\pi}{12}\sin^2 t$ satisfies $0 \leq \tau(t) \leq \tau = \frac{\pi}{12}$. From $\theta_{n,2} - \theta_{n,1} = \frac{\pi}{3} > \frac{\pi}{12}$ and $\theta_{n+1,1} - \theta_{n,2} = \frac{5\pi}{3} > \frac{\pi}{12}$ for all $n > n_0$, we know that condition $\theta_{k+1} - \theta_k > \tau$ is satisfied. Let $D_k(t) = t - \theta_{n,i} - \tau(t)$, then $D'_k(t) = 1 - \frac{\pi}{6}\sin t\cos t > 0$ for all t and there exist zero points t_i of $D_k(t)$ in $(\theta_{n,i}, d_i)$. Moreover, we also see the conditions (S_1), (2.1) and (2.12) are satisfied.

For $t \in [c_1, d_1]$, letting $w_1(t) = \sin 6t$ and by simple calculation, we get $t_{n,1} \in (\theta_{n,1}, d_1)$ and $t_{n,1} \approx 0.873$ and the left side of (2.26)

$$\begin{aligned} \eta\bigg\{ & \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \sin^2(6t)\frac{t + \frac{26\pi}{18} - \frac{\pi}{12}\sin^2 t}{t + \frac{26\pi}{18}}dt + \int_{\frac{2\pi}{9}}^{t_{n,1}} \sin^2(6t)\frac{t - \frac{2\pi}{9}}{b_{n,1}(t - \frac{2\pi}{9} + \frac{\pi}{12}\sin^2 t)}dt \\ & + \int_{t_{n,1}}^{\frac{\pi}{3}} \sin^2(6t)\frac{t - \frac{2\pi}{9} - \frac{\pi}{12}\sin^2 t}{t - \frac{2\pi}{9}}dt\bigg\} - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(36\cos^2(6t)dt \approx \left(0.086 + \frac{0.052}{b_{n,1}}\right)\eta - 9.424. \end{aligned}$$

On the other hand, the right side of (2.26) is

$$Q_{c_1}^{d_1}[w_1^2] = \frac{27(b_{n,1} - a_{n,1})}{2\pi a_{n,1}}.$$

Thus condition (2.26) is satisfied if

$$(0.086 + \frac{0.052}{b_{n,1}})\eta - 9.424 > \frac{27(b_{n,1} - a_{n,1})}{2\pi a_{n,1}}. \tag{3.2}$$

For $t \in [c_2, d_2]$, letting $w_2(t) = \sin 6t$ and using similar calculation to the above, we also show that the condition (2.26) is satisfied if

$$\left(0.071 + \frac{0.032}{b_{n,2}}\right)\eta - 9.424 > \frac{27(b_{n,2} - a_{n,2})}{2\pi a_{n,2}}. \tag{3.3}$$

Hence, by Theorem 2.9, Eq. (3.1) is oscillatory if

$$\begin{cases} \left(0.086 + \dfrac{0.052}{b_{n,1}}\right)\eta - 9.424 > \dfrac{27(b_{n,1} - a_{n,1})}{2\pi a_{n,1}}, \\ \left(0.071 + \dfrac{0.032}{b_{n,2}}\right)\eta - 9.424 > \dfrac{27(b_{n,2} - a_{n,2})}{2\pi a_{n,2}}. \end{cases} \tag{3.4}$$

Particularly, if $a_{n,i} = b_{n,i}$, for $n \in \mathbb{N}$, $i = 1, 2$, condition (3.4) becomes

$$\begin{cases} \left(0.086 + \dfrac{0.052}{b_{n,1}}\right) \eta > 9.424, \\ \left(0.071 + \dfrac{0.032}{b_{n,2}}\right) \eta > 9.424. \end{cases}$$

Example 3.2. Consider the following impulsive differential equation with variable delay

$$\begin{aligned} & x''(t) + p(t)x\left(t - \frac{1}{3\pi}\sin^2(\pi t)\right) = f(t), \qquad t \neq \theta_{n,i}, \\ & x(t^+) = a_{n,i}x(t), \qquad x'(t^+) = b_{n,i}x'(t), \qquad t = \theta_{n,i}, \end{aligned} \tag{3.5}$$

where $\theta_{n,i} : \theta_{n,1} = 8n + \frac{3}{2}$, $\theta_{n,2} = 8n + \frac{5}{2}$, $\theta_{n,3} = 8n + \frac{11}{2}$, $\theta_{n,4} = 8n + \frac{13}{2}$, $n \in \mathbb{N}$. Let

$$p(t) = \begin{cases} (t - 8n)^3, & t \in [8n, 8n + 3], \\ 27(8n + 4 - t), & t \in [8n + 3, 8n + 4], \\ (t - 8n - 4)^3, & t \in [8n + 4, 8n + 7], \\ 27(8n + 8 - t), & t \in [8n + 7, 8n + 8], \end{cases}$$

and

$$f(t) = \begin{cases} (t - 8n)^3(t - 8n - 4)^3, & t \in [8n, 8n + 4], \\ (t - 8n - 4)^3(8n + 8 - t)^3, & t \in [8n + 4, 8n + 8]. \end{cases}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 8n$ and let $[c_1, d_1] = [8n + 1, 8n + 3]$, $[c_2, d_2] = [8n + 5, 8n + 7]$, $\delta_1 = 8n + 2$ and $\delta_2 = 8n + 6$. It is easy to see that condition (2.1) and (2.12) are satisfied. Moreover, we easily see that there exist zero points t_i $(i = 1, 2, 3, 4)$ of $D_i(t) = t - \theta_{n,i} - \frac{1}{3\pi}\sin^2(\pi t)$ in $(\theta_{n,1}, \delta_1)$, $(\theta_{n,2}, d_1)$, $(\theta_{n,3}, \delta_2)$ and $(\theta_{n,4}, d_2)$ respectively. Therefore, the assumptions (S_1), $(\bar{S}_1)$, (A_{4-1}), (S_5) and $(\bar{S}_5)$ are satisfied.

For $j = 1$, letting $H_1(t, s) = H_2(t, s) = (t - s)^3$, we have $h_1(t, s) = -h_2(t, s) = \frac{3}{t-s}$. By simple calculation, we get

$$\begin{aligned} \frac{\Pi_{c_1}^{\delta_1}[H_1(t, c_1)]}{H_1(\delta_1, c_1)} = & \int_{8n+1}^{8n+\frac{3}{2}} (t - 8n)^3(t - 8n - 1)^3 \frac{t - 8n + \frac{3}{2} - \frac{1}{3\pi}\sin^2(\pi t)}{t - 8n + \frac{3}{2}} dt \\ & + \int_{8n+\frac{3}{2}}^{t_1} (t - 8n)^3(t - 8n - 1)^3 \frac{t - 8n - \frac{3}{2}}{b_{n,1}(t - 8n - \frac{3}{2} + \frac{1}{3\pi}\sin^2(\pi t))} dt \\ & + \int_{t_1}^{8n+2} (t - 8n)^3(t - 8n - 1)^3 \frac{t - 8n - \frac{3}{2} - \frac{1}{3\pi}\sin^2(\pi t)}{t - 8n - \frac{3}{2}} dt \\ & - \frac{1}{4}\int_{8n+1}^{8n+2} (t - 8n - 1)^3 \frac{3^2}{(t - 8n - 1)^2} dt \\ \approx & \frac{0.045}{b_{n,1}} - 0.731, \end{aligned}$$

where $t_1 \approx 8n + 1.596$ is a zero point of $D_k(t) = t - 8n - \frac{3}{2} - \frac{1}{3\pi}\sin^2(\pi t)$ in $(8n + \frac{3}{2}, 8n + 2)$

and

$$\begin{aligned}
\frac{\Pi_{\delta_1}^{d_1}[H_2(d_1,t)]}{H_2(d_1,\delta_1,)} &= \int_{8n+2}^{8n+\frac{5}{2}} (8n+3-t)^3(t-8n)^3 \frac{t-8n-\frac{3}{2}-\frac{1}{3\pi}\sin^2(\pi t)}{t-8n-\frac{3}{2}} dt \\
&\quad + \int_{8n+\frac{5}{2}}^{t_2} (t-8n)^3(8n+3-t)^3 \frac{t-8n-\frac{5}{2}}{b_{n,2}(t-8n-\frac{5}{2}+\frac{1}{3\pi}\sin^2(\pi t))} dt \\
&\quad + \int_{t_2}^{8n+3} (t-8n)^3(8n+3-t)^3 \frac{t-8n-\frac{5}{2}-\frac{1}{3\pi}\sin^2(\pi t)}{t-8n-\frac{5}{2}} dt \\
&\quad - \frac{1}{4}\int_{8n+2}^{8n+3} (8n+3-t)^3 \frac{3^2}{(8n+3-t)^2} dt \\
&\approx \frac{3.296}{b_{n,2}} + 5.735,
\end{aligned}$$

where $t_2 \approx 8n+2.593$ is a zero point of $D_k(t) = t-8n-\frac{5}{2}-\frac{1}{3\pi}\sin^2(\pi t)$ in $(8n+\frac{5}{2}, 8n+3)$ Then the left side of the inequality (2.35) is

$$\frac{\Pi_{c_1}^{\delta_1}[H_1(t,c_1)]}{H_1(\delta_1,c_1)} + \frac{\Pi_{\delta_1}^{d_1}[H_2(d_1,t)]}{H_2(d_1,\delta_1,)} \approx \frac{0.045}{b_{n,1}} + \frac{3.296}{b_{n,2}} + 5.004.$$

Because $\theta_{k(c_1)+1} = \theta_{k(\delta_1)} = \theta_{n,1} = 8n+\frac{3}{2} \in (c_1,\delta_1)$ and $\theta_{k(\delta_1)+1} = \theta_{k(d_1)} = \theta_{n,2} = 8n+\frac{5}{2} \in (\delta_1, d_1)$, the right side of the inequality (2.35) is

$$\frac{Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)]}{H_1(\delta_1,c_1)} + \frac{Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]}{H_2(d_1,\delta_1)} = \frac{b_{n,1}-a_{n,1}}{4a_{n,1}} + \frac{b_{n,2}-a_{n,2}}{4a_{n,2}}.$$

Thus (2.35) is satisfied for $j=1$ if

$$\frac{0.045}{b_{n,1}} + \frac{3.296}{b_{n,2}} + 5.004 > \frac{b_{n,1}-a_{n,1}}{4a_{n,1}} + \frac{b_{n,2}-a_{n,2}}{4a_{n,2}}. \tag{3.6}$$

For $j=2$, using similar argument as above we get that $t_3 \approx 1.596$, $t_4 \approx 2.593$ and the left side of inequality (2.35) is

$$\begin{aligned}
&\frac{\Pi_{c_2}^{\delta_2}[H_1(t,c_2)]}{H_1(\delta_2,c_2)} + \frac{\Pi_{\delta_2}^{d_2}[H_2(d_2,t)]}{H_2(d_2,\delta_2)} \\
&= \int_1^{\frac{3}{2}} \frac{u^3(u-1)^3(u+\frac{3}{2}-\frac{1}{3\pi}\sin^2(\pi u))}{u+\frac{3}{2}} du + \int_{\frac{3}{2}}^{t_3} \frac{u^3(u-1)^3(u-\frac{3}{2})}{b_{n,3}(u-\frac{3}{2}+\frac{1}{3\pi}\sin^2(\pi u))} du \\
&\quad + \int_{t_3}^{2} \frac{u^3(u-1)^3(u-\frac{3}{2}-\frac{1}{3\pi}\sin^2(\pi u))}{u-\frac{3}{2}} du + \int_2^{\frac{5}{2}} \frac{u^3(3-u)^3(u-\frac{3}{2}-\frac{1}{3\pi}\sin^2(\pi u))}{u-\frac{3}{2}} du \\
&\quad + \int_{\frac{5}{2}}^{t_4} \frac{u^3(3-u)^3(u-\frac{5}{2})}{b_{n,4}(u-\frac{5}{2}+\frac{1}{3\pi}\sin^2(\pi u))} du + \int_{t_4}^{3} \frac{u^3(3-u)^3(u-\frac{5}{2}-\frac{1}{3\pi}\sin^2(\pi u))}{u-\frac{5}{2}} du \\
&\quad -\frac{1}{4}\int_1^2 (u-1)^3 \frac{3^2}{(u-1)^2} du - \frac{1}{4}\int_2^3 (3-u)^3 \frac{3^2}{(3-u)^2} du \\
&\approx \frac{0.090}{b_{n,3}} + \frac{3.897}{b_{n,4}} + 3.558
\end{aligned}$$

and the right side of the inequality (2.35) is

$$\frac{Q_{c_2}^{\delta_2}[H_1(\cdot,c_2)]}{H_2(\delta_2,c_2)} + \frac{Q_{\delta_2}^{d_2}[H_2(d_2,\cdot)]}{H_2(d_2,\delta_2)} = \frac{b_{n,3}-a_{n,3}}{4a_{n,3}} + \frac{b_{n,4}-a_{n,4}}{4a_{n,4}}.$$

Therefor (2.35) is satisfied for $j = 2$ if

$$\frac{0.090}{b_{n,3}} + \frac{3.897}{b_{n,4}} + 3.558 > \frac{b_{n,3} - a_{n,3}}{4a_{n,3}} + \frac{b_{n,4} - a_{n,4}}{4a_{n,4}}. \tag{3.7}$$

Hence, by Theorem 2.12, we know (3.5) is oscillatory if (3.6) and (3.7) hold.

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