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Existence results for a nonlinear elliptic transmission problem of $p(x)$-Kirchhoff type

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Abstract. In this article, we establish the existence of weak solutions for a nonlinear transmission problem involving nonlocal coefficients of $p(x)$-Kirchhoff type in two different domains, which are connected by a nonlinear transmission condition at their interface. We get our results by means of the monotone operator theory and the (S_+) mapping theory; the weak formulation takes place in suitable variable exponent Sobolev spaces.

Keywords: nonlinear transmission problem, $p(x)$-Laplacian, monotone operator.

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1 Introduction

Let Ω, Ω_1, $\Omega_2 \subseteq \mathbb{R}^2$ be a bounded polygonal domains with their boundaries $\partial\Omega$, $\partial\Omega_1$, $\partial\Omega_2$ and closures $\overline{\Omega}$, $\overline{\Omega}_1$, $\overline{\Omega}_2$ satisfying the relations $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \phi$.
We denote

$$\Gamma_3 = \partial\Omega_1 \cap \partial\Omega_2, \qquad \Gamma_i = \partial\Omega_i \setminus \Gamma_3, \qquad i = 1,2. \qquad \text{(see Figure 1.1).}$$

We are concerned with the existence of solutions to the following nonlinear elliptic system

$$\begin{aligned}
&-M_i\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx\Big)\operatorname{div}(a(x,\nabla u_i)) = \operatorname{div}\vec{f} + h_i(x,u_i) \quad \text{in } \Omega_i,\ i=1,2.\\
&M_i\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx\Big)a(x,\nabla u_i).\vec{\nu} + k|u_i|^{\alpha(x)-2}u_i = \vec{f}.\,\vec{\nu} \quad \text{on } \Gamma_i,\ i=1,2.\\
&M_1\Big(\int_{\Omega_1} A(x,\nabla u_1)\,dx\Big)a(x,\nabla u_1).\vec{\nu}^1 = -M_2\Big(\int_{\Omega_2} A(x,\nabla u_2)\,dx\Big)a(x,\nabla u_2).\vec{\nu}^2\\
&\qquad = k|u_2-u_1|^{\alpha(x)-2}(u_2-u_1) + \vec{f}.\nu^1 \quad \text{on } \Gamma_3
\end{aligned} \tag{1.1}$$

where

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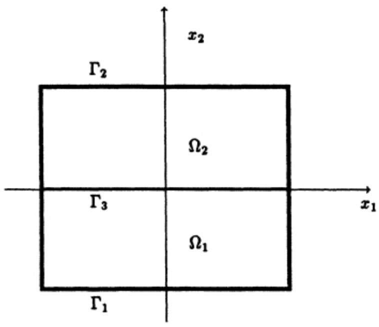

Figure 1.1: Domain Ω

(M0) $M_i : [0, +\infty[\longrightarrow [m_{0i}, +\infty[$, are non decreasing locally Lipschitz continuous functions,

(H0) $h_i : \Omega_i \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions and satisfy the growth condition $|h_i(x,t)| \leq c_{i0} + c_{i1}|t|^{\beta(x)-1}$, for any $x \in \overline{\Omega}$, $t \in \mathbb{R}$, $i = 1, 2$,

(A1) $a(x,\xi) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the continuous mapping $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$; there exist two positive constants $\Upsilon_1 \leq \Upsilon_2$ such that $\Upsilon_1 |\xi|^{p(x)} \leq a(x,\xi)\xi$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ and

(A2) $|a(x,\xi)| \leq \Upsilon_2 |\xi|^{p(x)-1}$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$,

(A3) $A(x,0) = 0$ for all $x \in \Omega$,

(A4) $A(x,\cdot)$ is strictly convex in $\mathbb{R}^N$,

m_{0i}, $i = 1,2$, k are positive numbers, p and α are continuous functions on $\overline{\Omega}$ satisfying appropriate conditions, $\vec{f} = (f_1, f_2)$ is a given vector field (determined from Maxwell's equations), $\vec{\nu} = (\nu_1, \nu_2)$ and $\vec{\nu}^i = (\nu_1^i, \nu_2^i)$ denote a unit outer normal to $\partial\Omega$ and to $\partial\Omega_i$, respectively; of course $\vec{\nu}^1 = -\vec{\nu}^2$ and $a(x,\nabla.).\vec{\nu}^1 = -\ a(x,\nabla.).\vec{\nu}^2$ on Γ_3, $\vec{\nu} = \vec{\nu}^i$ on Γ_i, $i = 1,2$. We confine ourselves to the case where $M_1 = M_2$ with $m_{01} = m_{02} = m_0$ for simplicity. Notice that the results of this work remain valid for $M_1 \neq M_2$.

The study of problems in differential equations and variational problems involving variable exponents has been an interesting topic in recent decades. The interest for such problems is based on the multiple possibilities to apply them. There are applications in nonlinear elasticity, theory of image restoration, electrorheological fluids and so on (see [1,13,46]). We refer the readers to [15,31,42] for an overview of this subject, to [11,19,20] for the p-Laplacian and [23–27,34] for the study of $p(x)$-Laplacian equations and the corresponding variational problems.

Transmission problems problems arise in several applications in physics and biology (see [36]). Some results are available for linear parabolic equations with linear and nonlinear conditions at interfaces, for biological models for the transfer of chemicals through semipermeable thin membranes (see [8,39,43]). There are cases where transmission conditions can allow to deal with models including chemical phenomena in materials with different porosity and diffusivity, and chemotaxis phenomena in regions with different substrate properties (see [30]).

Kirchhoff in 1883 [32] investigated an equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which is called the Kirchhoff equation, where $\rho,\ \rho_0,\ h,\ E,\ L$ are all constants, moreover, this equation contains a nonlocal coefficient

$$\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$$

which depends on the average

$$\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$$

hence the equation is no longer a pointwise identity, and therefore is often called nonlocal problem. Various equations of Kirchhoff type have been studied by many authors, especially after the work of Lions [35], where a functional analysis framework for the problem was proposed; see e.g. [3,9,29] for some interesting results and further references. In recent years, various Kirchhoff-type problems have been discussed in many papers. The Kirchhoff model is an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, which takes into account the changes in length of the string produced by transverse vibrations; while purely longitudinal motions of a viscoelastic bar of uniform cross section and its generalizations can be found in [12,33,40,41]. In particular, in a recent article, existence and multiplicity of nontrivial radial solutions are obtained via variational methods [34]. The study of nonlocal elliptic problem has already been extended to the case involving the p-Laplacian (for details, see [11,19,20]) and $p(x)$-Laplacian (see [14,17,22,29]).

More recently, Cabanillas L. et al. [7], have dealt with the $p(x)$-Kirchhoff type equation

$$-M\left(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\,dx\right)\left[\operatorname{div}(a(x,\nabla u)) - |u|^{p(x)-2}u\right] = f(x,u)|u|_{s(x)}^{t(x)} \quad \text{in } \Omega,$$
$$u = \text{constant} \quad \text{on } \partial\Omega,$$
$$\int_{\partial\Omega} a(x,\nabla u).\nu d\Gamma = 0.$$

by topological methods. Our work is motivated by the ones of Feistauer et al. [28] and Cecik et al. [10].

The aim of this article is to study the existence of a solution to the problem (1.1) in the Sobolev spaces with variable exponents; we use the well-known theorem named as Browder–Minty theorem and the degree theory of (S_+) type mappings to attack it.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. Section 3 is devoted to the proof of our general existence results.

2 Preliminaries

To discuss problem (1.1), we need some theory on $W^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space (for details, see [26]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \qquad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \left\{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\right\}$$

with the norm

$$|u|_{p(x),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x),\Omega} = |u|_{p(x),\Omega} + |\nabla u|_{p(x),\Omega}.$$

Proposition 2.1 ([26]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([26]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\, dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) for $u \neq 0$, $|u|_{p(x),\Omega} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x),\Omega} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;

(3) if $|u|_{p(x),\Omega} > 1$, then $|u|^{p^-}_{p(x),\Omega} \le \rho(u) \le |u|^{p^+}_{p(x),\Omega}$;

(4) if $|u|_{p(x),\Omega} < 1$, then $|u|^{p^+}_{p(x),\Omega} \le \rho(u) \le |u|^{p^-}_{p(x),\Omega}$;

(5) $\lim_{k\to+\infty} |u_k|_{p(x),\Omega} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;

(6) $\lim_{k\to+\infty} |u_k|_{p(x),\Omega} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3 ([23,26]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([23,24]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the Hölder-type inequality*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)}. \tag{2.1}$$

Proposition 2.5 ([21]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^\partial(x)$ $(q(x) < p^\partial(x))$ for $x \in \partial\Omega$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$, where*

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.6 ([26]). *If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies*

$$|h(x,t)| \le a(x) + b|t|^{p_1(x)/p_2(x)}, \quad \text{for any } x \in \overline{\Omega},\ t \in \mathbb{R},$$

where $p_1(x), p_2(x) \in C_+(\Omega), a(x) \in L^{p_2(x)}(\Omega), a(x) \ge 0$, and $b \ge 0$ is a constant, then the Nemytsky operator $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_h(u))(x) = h(x, u(x))$ is a continuous and bounded operator.

In the sequel we shall assume that $\vec{f} \in \left[L^{p'(x)}(\Omega)\right]^2$.

Let us define the Banach space $E = W^{1,p(x)}(\Omega_1) \times W^{1,p(x)}(\Omega_2)$ equipped with the norm

$$\|u\|_E = \|u_1\|_{1,p(x),\Omega_1} + \|u_2\|_{1,p(x),\Omega_2}, \qquad \forall u = (u_1,u_2) \in E$$

where $\|u_i\|_{1,p(x),\Omega_i}$ is the norm of u_i in $W^{1,p(x)}(\Omega_i)$, $i = 1,2$. By $|u|_E$ we denote the seminorm in E

$$|u|_E = |\nabla u_1|_{p(x),\Omega_1} + |\nabla u_2|_{p(x),\Omega_2}.$$

It is obvious that

$$|\nabla u_i|_{p(x),\Omega_i} \le |u|_E \le \|u\|_E, \qquad \forall u = (u_1,u_2) \in E.$$

Remark 2.7. From the assumptions on A, arguing as in [37], we get after some computations that

$$\begin{aligned}
\frac{\Upsilon_1}{p^+} \min\left\{|\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i}\right\} &\le \Upsilon_1 \int_{\Omega_i} \frac{1}{p(x)}|\nabla u_i|^{p(x)}\,dx \le \int_{\Omega_i} A(x,\nabla u_i)\,dx \\
&\le \Upsilon_2 \int_{\Omega_i} \frac{1}{p(x)}|\nabla u_i|^{p(x)}\,dx \\
&\le \frac{\Upsilon_2}{p^-} \max\left\{|\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i}\right\}.
\end{aligned}$$

3 Existence of solutions

In this section, we shall state and prove the main result of the paper. For simplicity, we use c, c_i, $i = 1,2,\ldots$ to denote the general positive constants (the exact value may change from line to line).

Let us define the forms

$$\begin{aligned}
b(u,v) &= \sum_{i=1}^{2} M\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx\Big) \int_{\Omega_i} a(x,\nabla u_i)\nabla v_i\,dx, \\
c(u,v) &= k\sum_{i=1}^{2}\int_{\Gamma_i} |u_i|^{\alpha(x)-2}u_i v_i\,dS, \qquad \alpha(x) > 2, \\
d(u,v) &= k\int_{\Gamma_3} |u_2 - u_1|^{\alpha(x)-2}(u_2-u_1)(v_2-v_1)\,dS, \\
l(u,v) &= -\sum_{i=1}^{2}\int_{\Omega_i} h(x,u_i)v_i\,dx, \qquad 1 < \beta(x) < p^*(x), \\
L(v) &= \sum_{i=1}^{2}\int_{\Omega_i} \vec{f}.\nabla v_i\,dx, \\
g(u,v) &= b(u,v) + c(u,v) + d(u,v) + l(u,v), \\
&\quad u = (u_1,u_2),\ v = (v_1,v_2) \in E.
\end{aligned}$$

We say that $u = (u_1, u_2) \in E$ is a weak solution of problem (1.1) if

$$g(u,v) = L(v), \quad \text{for all } v \in E.$$

Here we recall how the theory of monotone operators is used to prove existence of solutions to (1.1) . For this, it will be useful to consider the differential operator as a mapping from E into its dual space, i.e.,

$$G : E \to E', \qquad \langle G(u), v\rangle = g(u,v)$$

provided of course that for fixed u this indeed defines a bounded linear functional on E.

The following lemma states the rather obvious relation between the operator G and the differential equation (1.1)

Lemma 3.1. *For each $u \in E$ the forms $g(u,\cdot)$, $b(u,\cdot)$, $c(u,\cdot)$, $d(u,\cdot)$, $l(u,\cdot)$ and L are linear and continuous on E .*

Proof. The boundedness of the forms is an easy consequence of Hölder's inequality, Remark 2.7, Propositions 2.2–2.5 and monotonicity of M. Indeed,

$$\begin{aligned}|b(u,v)| &\le c\sum_{i=1}^{2} M\left(\frac{\Upsilon_2}{p^-}\int_{\Omega_i}|\nabla u_i|^{p(x)}\,dx\right)\int_{\Omega_i}|a(x,\nabla u_i)||\nabla v_i|\,dx\\ &\le c\sum_{i=1}^{2} M\left(\frac{\Upsilon_2}{p^-}|u|_E^{\gamma+1}\right)\max\left\{|\nabla u_i|^{p^+-1}_{p(x),\Omega_i}, |\nabla u_i|^{p^--1}_{p(x),\Omega_i}\right\}|\nabla v_i|_{p(x),\Omega_i}\\ &\le cM\left(\frac{1}{p^-}\|u\|_E^{\gamma+1}\right)\|u\|_E^{\gamma}\|v\|_E\end{aligned}$$

where

$$\gamma = \begin{cases} p^+-1 & \text{if } |\nabla u_i|_{p(x),\Omega_i} > 1\\ p^--1 & \text{if } |\nabla u_i|_{p(x),\Omega_i} < 1,\end{cases}$$

$$\begin{aligned}|c(u,v)| &\le k\sum_{i=1}^{2}\int_{\Gamma_i}|u_i|^{\alpha(x)-1}|v_i|\,dx \le kc\sum_{i=1}^{2}\left||u_i|^{\alpha(x)-1}\right|_{\alpha'(x),\Gamma_i}|v_i|_{\alpha(x),\Gamma_i}\\ &\le kc\sum_{i=1}^{2}|u_i|^{\theta}_{\alpha(x),\Gamma_i}|v_i|_{\alpha(x),\Gamma_i} \le kc\|u\|_E^{\theta}\|v\|_E,\end{aligned}$$

where

$$\theta = \begin{cases} \alpha^+-1 & \text{if } |\nabla u_i|_{\alpha(x),\Gamma_i} > 1\\ \alpha^--1 & \text{if } |\nabla u_i|_{\alpha(x),\Gamma_i} < 1,\end{cases}$$

and

$$\begin{aligned}|d(u,v)| &\le k\int_{\Gamma_3}|u_1-u_2|^{\alpha(x)-1}|v_1-v_2|\,dx \le kc\left||u_1-u_2|^{\alpha(x)-1}\right|_{\alpha'(x),\Gamma_3}|v_1-v_2|_{\alpha(x),\Gamma_3}\\ &\le kc^{\theta}|u_1-u_2|^{\theta}_{\alpha(x),\Gamma_3}|v_1-v_2|_{\alpha(x),\Gamma_3} \le kc'^{\theta}\|u\|_E^{\theta}\|v\|_E,\end{aligned}$$

where

$$\theta = \begin{cases} \alpha^+-1 & \text{if } |u_1-u_2|_{\alpha(x),\Gamma_3} > 1,\\ \alpha^--1 & \text{if } |u_1-u_2|_{\alpha(x),\Gamma_3} < 1.\end{cases}$$

and

$$\begin{aligned}|l(u,v)| &\le \sum_{i=1}^{2}\int_{\Omega_i}|h(x,u_i)||v_i|\,dx \le c\sum_{i=1}^{2}\left(\int_{\Omega_i}|u_i|^{\beta(x)-1}|v_i|\,dx + \int_{\Omega_i}|v_i|\,dx\right)\\ &\le c\sum_{i=1}^{2}\left(|u_i|^{\delta}_{\beta(x),\Omega_i}|v_i|_{\beta(x),\Omega_i} + |v_i|_{p(x),\Omega_i}\right) \le kc(\|u\|_E^{\delta}+1)\|v\|_E,\end{aligned}$$

where

$$\delta = \begin{cases} \beta^+ - 1 & \text{if } |u_i|_{\beta(x),\Omega_i} > 1, \\ \beta^- - 1 & \text{if } |u_i|_{\beta(x),\Omega_i} < 1. \end{cases}$$

□

By Lemma 3.1 we can define the mapping $G : E \to E'$ and the functional $\varphi \in E'$ by the identities

$$\langle G(u), v\rangle = g(u,v)$$
$$\langle \varphi, v\rangle = L(v)$$

for each $u, v \in E$.
Now, it is clear that solving (1.1) is the same as finding $u \in E$ such that

$$G(u) = \varphi.$$

Theorem 3.2. *Assume that $(M0)$, $(H0)$ and $(A1)$–$(A4)$ hold. In addition, suppose that*
$(H1)$ $h(x,0) = 0$ and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a decreasing function with respect to the second variable, i.e.

$$h(x,s_1) \leq h(x,s_2) \qquad \text{for a.e. } x \in \Omega \text{ and } s_1, s_2 \in \mathbb{R},\ s_1 \geq s_2.$$

If $p^+ < \alpha^-$, problem (1.1) has precisely one weak solution .

For the proof of our theorem we need to establish some lemmas.

Lemma 3.3. *Let $r, s \geq 1$, $\beta > 1$. Then there exists a positive constant $c_2 = c_2(r,s)$ such that*

$$|u|_E^\beta + \frac{\|u\|_E^\beta}{\min\{\|u\|_E^r, \|u\|_E^s\}}\left(\sum_{i=1}^{2} |u_i|^r_{\alpha(x),\Gamma_i} + |u_1 - u_2|^s_{\alpha(x),\Gamma_3}\right) \geq c_2\|u\|_E^\beta. \tag{3.1}$$

Proof. Firstly, we prove that there exists $c_2 > 0$ such that

$$|u|_E^\beta + \sum_{i=1}^{2} |u_i|^r_{\alpha(x),\Gamma_i} + |u_1 - u_2|^s_{\alpha(x),\Gamma_3} \geq c_2 \tag{3.2}$$

for all $u = (u_1, u_2) \in E$ with $\|u\|_E = 1$. Let us assume that (3.2) is not valid. Then there exists a sequence $\{u^\nu\} \subset E$ such that

$$\begin{aligned} &a) \quad \|u^\nu\|_E = 1 \\ &b) \quad u^\nu \rightharpoonup u = (u_1, u_2) \text{ weakly in } E, \\ &c) \quad |u^\nu|_E^\beta + \sum_{i=1}^{2} |u_i^\nu|^r_{\alpha(x),\Gamma_i} + |u_1^\nu - u_2^\nu|^s_{\alpha(x),\Gamma_3} \leq \frac{1}{\nu}. \end{aligned}$$

From Proposition 2.5 and $b)$ it follows that

$$u^\nu \to u = (u_1, u_2) \text{ strongly in } L^{\alpha(x)}(\Gamma_1) \times L^{\alpha(x)}(\Gamma_2). \tag{3.3}$$

Using (3.3) , the weak lower semicontinuity of the seminorm $|u|_E$ and $c)$ we get

$$|u|_E^\beta + \sum_{i=1}^{2} |u_i|^r_{\alpha(x),\Gamma_i} + |u_1 - u_2|^s_{\alpha(x),\Gamma_3} = 0.$$

Then, $u_i = k_i \equiv$ constant for $i = 1,2$. So $u_i \mid_{\Gamma_i} = k_i$. As $|u_i|_{\alpha(x),\Gamma_i} = 0$ we have $k_i = 0$ for $i = 1,2$, therefore $u = 0$. This is a contradiction to $a)$.
Finally to prove (3.1) let $u \in E, u \neq 0$ and $w = \frac{u}{\|u\|_E}$. From (3.2) we have

$$\frac{|u|_E^{\beta}}{\|u^{\nu}\|_E^{\beta}} + \frac{1}{\min\{\|u\|_E^r, \|u\|_E^s\}} \left(\sum_{i=1}^{2} |u_i|^r_{\alpha(x),\Gamma_i} + |u_1 - u_2|^s_{\alpha(x),\Gamma_3} \right) \geq c_2.$$

Multiplying this inequality by $\|u^{\nu}\|_E^{\beta}$ the assertion (3.1) follows. □

Lemma 3.4. *There exists a constant $c_3 > 0$ such that for any $u \in E$ with $\|u\|_E \geq 1$*

$$g(u,u) \geq c_3 \|u\|_E^{p^-}. \tag{3.4}$$

Proof. For any $u = (u_1, u_2) \in E$ we have

$$\begin{aligned} g(u,u) \geq & \sum_{i=1}^{2} M\Big(\int_{\Omega_i} A(x, \nabla u_i)\, dx \Big) \int_{\Omega_i} a(x, \nabla u_i) \nabla u_i \, dx \\ & + k \sum_{i=1}^{2} \int_{\Gamma_i} |u_i|^{\alpha(x)}\, dS + k \int_{\Gamma_3} |u_2 - u_1|^{\alpha(x)}\, dS - \sum_{i=1}^{2} \int_{\Omega_i} h(x, u_i) u_i \, dx \\ \geq & \, m_0 \sum_{i=1}^{2} \min\left\{ |\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i} \right\} + k \sum_{i=1}^{2} \min\left\{ |u_i|^{\alpha^-}_{p(x),\Gamma_i}, |u_i|^{\alpha^+}_{p(x),\Gamma_i} \right\} \\ & + k \min\left\{ |u_1 - u_2|^{\alpha^-}_{\alpha(x),\Gamma_3}, |u_1 - u_2|^{\alpha^+}_{\alpha(x),\Gamma_3} \right\}. \end{aligned} \tag{3.5}$$

Now, if

$$\begin{aligned} \min\left\{ |\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i} \right\} &= |\nabla u_i|^{p^-}_{p(x),\Omega_i} \\ \min\left\{ |u_i|^{\alpha^-}_{p(x),\Gamma_i}, |u_i|^{\alpha^+}_{p(x),\Gamma_i} \right\} &= |u_i|^{\alpha^-}_{p(x),\Gamma_i} \\ \min\left\{ |u_1 - u_2|^{\alpha^-}_{\alpha(x),\Gamma_3}, |u_1 - u_2|^{\alpha^+}_{\alpha(x),\Gamma_3} \right\} &= |u_1 - u_2|^{\alpha^-}_{\alpha(x),\Gamma_3}, \end{aligned} \tag{3.6}$$

using inequalities (3.5) and (3.6), it follows that

$$g(u,u) \geq m_0 c_4 |u|_E^{p^-} + k \sum_{i=1}^{2} |u_i|^{\alpha^-}_{p(x),\Gamma_i} + k |u_1 - u_2|^{\alpha^-}_{\alpha(x),\Gamma_3}.$$

Provided that $\|u\|_E > 1$, putting $\beta = p^-,\ r = s = \alpha^-$ in (3.1) , and noting that $\|u\|_E^{p^- - \alpha^-} \leq 1$ we obtain

$$g(u,u) \geq m_0 c_4 |u|_E^{p^-} + k \sum_{i=1}^{2} |u_i|^{\alpha^-}_{p(x),\Gamma_i} + k |u_1 - u_2|^{\alpha^-}_{\alpha(x),\Gamma_3} \geq c_{2'} \|u\|_E^{p^-}$$

for some $c_{2'} > 0$. For other cases, the proofs are similar and we omit them here. So we have

$$g(u,u) \geq c_3 \min\left\{ \|u\|_E^{p^-}, \|u\|_E^{p^+} \right\} = c_3 \|u\|_E^{p^-}.$$

This ends the proof of Lemma 3.4. □

The proof of the next Lemma is done by adapting some arguments employed in the proof of Theorem 2.1 i) in [18].

Lemma 3.5. *The form a is strictly monotone:*

$$g(u, u-v) - g(v, u-v) \geq 0, \quad \textit{for all } u, v \in E, u \neq v.$$

Proof. Denote $\rho_{1,p(x)}(u_i) = \int_{\Omega_i} \frac{1}{p(x)} |\nabla u_i|^{p(x)} dx$, for all $u_i \in W^{1,p(x)}(\Omega_i),\ i = 1, 2$.

So, for any $u_i, v_i \in W^{1,p(x)}(\Omega_i)$ with $u_i \neq v_i$ we may assume, without loss of generality , that $\rho_{1,p(x)}(u_i) > \rho_{1,p(x)}(v_i)$. By virtue of monotonicity of M we have

$$M\left(\Upsilon_2 \int_{\Omega_i} \frac{1}{p(x)} |\nabla u_i|^{p(x)} dx\right) \geq M\left(\Upsilon_1 \int_{\Omega_i} \frac{1}{p(x)} |\nabla v_i|^{p(x)} dx\right)$$

Noting that $a(x, \cdot)$ is monotone by assumption $(A4)$, and following a similar procedure to that used in [18] , we get

$$\begin{aligned}
&g(u, u-v) - g(v, u-v) \\
&\quad \geq \frac{m_0}{2} \sum_{i=1}^{2} \int_{\Omega_i} [a(x, \nabla u_i) - a(x, \nabla v_i)]\,(\nabla u_i - \nabla v_i)\,dx \\
&\qquad + \frac{k}{2} \sum_{i=1}^{2} \int_{\Gamma_i} (|u_i|^{\alpha(x)-2} - |v_i|^{\alpha(x)-2})(|u_i|^2 - |v_i|^2)\,dS \\
&\qquad + \frac{k}{2} \int_{\Gamma_3} (|u_1 - u_2|^{\alpha(x)-2} - |v_1 - v_2|^{\alpha(x)-2})(|u_1 - u_2|^2 - |v_1 - v_2|^2)\,dS \\
&\qquad - \sum_{i=1}^{2} \int_{\Omega_i} (h(x, u_i) - h(x, v_i)(u_i - v_i)\,dx \geq 0,
\end{aligned} \tag{3.7}$$

i.e. g is monotone.

If $g(u, u-v) - g(v, u-v) = 0$ then all four terms in the right-hand side of (3.7) are equal to zero. Hence, $u_i = v_i = k_i =$ const. a.e. in Ω_i and $u_i = v_i$ a.e. on $\Gamma_i,\ i = 1, 2$. Therefore, $k_i = 0$ and $u = v$ a.e. □

Lemma 3.6. *There exists a constant $c_6 > 0$ such that*

$$\begin{aligned}
&|g(u, v) - g(w, v)| \\
&\quad \leq c_6 \Big[\Big(\max\{\|u\|_E^{p^+}, \|u\|_E^{p^-}, \|u\|_E^{\alpha^+-2}, \|u\|_E^{\alpha^--2}\} \\
&\qquad\qquad + \max\{\|w\|_E^{p^+}, \|w\|_E^{p^-}, \|w\|_E^{\alpha^+-2}, \|w\|_E^{\alpha^--2}\} \Big) \|u - w\|_E \\
&\qquad\quad + \sum_{i=1}^{2} \Big(|N_h(u_i) - N_h(w_i)|_{\frac{\beta(x)}{\beta(x)-1}, \Omega_i} \\
&\qquad\qquad + M\left(\tfrac{\Upsilon_2}{p^-} \max\{|\nabla u_i|_{p(x),\Omega_i}^{p^-}, |\nabla u_i|_{p(x),\Omega_i}^{p^+}\}\right) \\
&\qquad\qquad \times |a(x, \nabla u_i) - a(x, \nabla w_i)|_{p'(x),\Omega_i} \Big) \Big] \|v\|_E
\end{aligned} \tag{3.8}$$

Proof. By definition of the form g, for all $u, v, w \in E$, we have

$$\begin{aligned} & g(u,v) - g(w,v) \\ &= \sum_{i=1}^{2} \Big\{ \Big[M\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx \Big) - M\Big(\int_{\Omega_i} A(x,\nabla w_i)\,dx \Big) \Big] \int_{\Omega_i} a(x,\nabla u_i)\nabla v_i\,dx \\ &\qquad + M\Big(\int_{\Omega_i} A(x,\nabla w_i)\,dx \Big) \int_{\Omega_i} (a(x,\nabla u_i) - a(x,\nabla w_i))\,\nabla v_i\,dx \Big\} \\ &\quad + k\sum_{i=1}^{2} \int_{\Gamma_i} (|u_i|^{\alpha(x)-2}u_i - |w_i|^{\alpha(x)-2}w_i)v_i\,dx \\ &\quad + k\int_{\Gamma_3} \Big(|u_2-u_1|^{\alpha(x)-2}(u_2-u_1) - |w_2-w_1|^{\alpha(x)-2}(w_2-w_1) \Big)\,(v_2-v_1)\,dx \\ &\quad - \sum_{i=1}^{2} \int_{\Omega_i} (h(x,u_i) - h(x,w_i))\,v_i\,dx \end{aligned} \tag{3.9}$$

Now in virtue of the Lipschitz condition satisfied by M and the elementary inequalities

$$\begin{aligned} &a)\ \big||z|^{\alpha-2}z - |y|^{\alpha-2}y\big| \le C|z-y|^{\alpha-1} \quad \text{for all} \quad y,z\in\mathbb{R}^n, \quad \text{if} \quad 1<\alpha\le 2 \\ &b)\ \big||z|^{\alpha-2}z - |y|^{\alpha-2}y\big| \le C|z-y|(|z|+|y|)^{\alpha-2} \quad \text{for all} \quad y,z\in\mathbb{R}^n, \quad \text{if} \quad 2\le\alpha<\infty, \end{aligned}$$

the equality (3.9) reduces to

$$\begin{aligned} & |g(u,v) - g(w,v)| \\ &\le c\sum_{i=1}^{2} \Big[\int_{\Omega_i} L_{M_i} \frac{\big||\nabla u_i|^{p(x)} - |\nabla w_i|^{p(x)}\big|}{p(x)}\,dx \int_{\Omega_i} |\nabla u_i|^{p(x)-1}|\nabla v_i|\,dx \\ &\qquad + M\Big(\tfrac{\Upsilon_2}{p^-} \max\Big\{ |\nabla u_i|^{p^-}_{p(x),\Omega_i}, |\nabla u_i|^{p^+}_{p(x),\Omega_i} \Big\} \Big) |a(x,\nabla u_i) - a(x,\nabla w_i)|_{p'(x),\Omega_i} |\nabla v_i|_{p(x),\Omega_i} \Big] \\ &\quad + kc_\alpha \sum_{i=1}^{2} \int_{\Gamma_i} |u_i - w_i|(|u_i|^{\alpha(x)-2} + |w_i|^{\alpha(x)-2})|v_i|\,dx \\ &\quad + kc_\alpha \int_{\Gamma_3} |(u_2-u_1) - (w_2-w_1)| \Big(|u_2-u_1|^{\alpha(x)-2} + |w_2-w_1|^{\alpha(x)-2} \Big) |v_2 - v_1|\,dx \\ &\quad + c_7 \sum_{i=1}^{2} |N_h(u_i) - N_h(w_i)|_{\beta'(x),\Omega_i} |v_i|_{\beta(x),\Omega_i}, \end{aligned}$$

where $L_{M_i} > 0$, the Lipschitz constant of M, depends on $\max\{\|u\|_E, \|w\|_E\}$. Therefore, by Propositions 2.2–2.6, after some calculations, we arrive at the estimate (3.8). □

Proof of Theorem 3.2. First, we note that using hypothesis $(A2)$ and Proposition 2.2 in [24] we get

$$|a(x,\nabla u_i) - a(x,\nabla w_i)|_{p'(x),\Omega_i} \to 0 \qquad \text{if } u \to v \text{ in E}$$

From Lemmas 3.4–3.6 the operator a is bounded, coercive, strictly monotone and continuous (hence hemicontinuous) in E. Therefore, by the Browder–Minty theorem [4, Theorem 7.3.2], problem (1.1) admits a unique weak solution. □

Next, we use the degree theory of (S_+) type mappings to prove the second result of this paper.

Let us recall the definition of the mapping of type (S_+). Let X be a Banach space and $D \subset X$ an open set. "$u_\nu \rightharpoonup u$" and "$u_\nu \to u$" denote respectively the weak convergence and the strong convergence in X.

A mapping $A : \overline{D} \to X^*$ is said to be of type (S_+) if for any sequence $\{u_\nu\} \subset \overline{D}$ for which $u_\nu \rightharpoonup u$ in X and $\limsup_{\nu\to\infty}\langle A(u_\nu), u_\nu - u\rangle \le 0$, $u_\nu \to u$ in X. For (S_+) mapping theory, including the degree theory and the surjection theorem, we refer the reader to [6,44,45].

Theorem 3.7. *Assume that* $(M0)$ *and* $(H0)$ *hold. If* $p^+ < \alpha^-$ *and* $\beta^+ < p^-$, *problem* (1.1) *has a weak solution .*

Proof. By a simple adaptation of Theorem 2.1 ii) in [18] to our problem, we can prove that the mapping

$$\langle B_0(u_i), v_i\rangle = M\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx\Big)\int_{\Omega_i} a(x,\nabla u_i)\nabla v_i\,dx$$

for all $u_i, v_i \in W^{1,p(x)}(\Omega_i)$ is an operator of type (S_+).

Define the mappings B, C, D and $S : E \to E^*$ respectively by

$$\begin{aligned}
\langle B(u), v\rangle &= \sum_{i=1}^{2} M\Big(\int_{\Omega_i} A(x,\nabla u_i)\,dx\Big)\int_{\Omega_i} a(x,\nabla u_i)\nabla v_i\,dx,\\
\langle C(u), v\rangle &= k\sum_{i=1}^{2}\int_{\Gamma_i}|u_i|^{\alpha(x)-2}u_i v_i\,dS, \qquad 2<\alpha(x)<p^{\partial}(x),\\
\langle D(u), v\rangle &= k\int_{\Gamma_3}|u_2-u_1|^{\alpha(x)-2}(u_2-u_1)(v_2-v_1)\,dS,\\
\langle L(u), v\rangle &= -\sum_{i=1}^{2}\int_{\Omega_i} h(x,u_i)v_i\,dx, \qquad 1<\beta(x)<p^*(x)\\
\langle N_h(u_i), v_i\rangle &= \int_{\Omega_i} h(x,u_i)v_i\,dx,\\
T(v) &= \sum_{i=1}^{2}\int_{\Omega_i}\vec{f}.\nabla v_i\,dx,\\
u &= (u_1,u_2),\ v=(v_1,v_2)\in E.
\end{aligned}$$

Then $B(u) = \sum_{i=1}^{2} B_0(u_i)$, $L(u) = -\sum_{i=1}^{2} N_h(u_i)$ and $G(u) = B(u) + C(u) + D(u) + L(u)$. It is clear that $u \in E$ is a solution of (1.1) if an only if $G(u) = T$.

From the above analysis, it is obvious that $B : E \to E'$ is continuous , bounded and of type (S_+). Moreover, using the compacity of embeddings $W^{1,p(x)}(\Omega_i) \hookrightarrow L^{\alpha(x)}(\Gamma_i)$ and $W^{1,p(x)}(\Omega_i) \hookrightarrow L^{\beta(x)}(\Omega_i)$ we deduce that the operators C, D, L are compact (cf. e.g. [5,26]). Noting that the sum of an (S_+) type mapping and a compact mapping is of type (S_+) , it follows that the mapping $G = B + C + D + L$ is continuous, bounded, and of type (S_+).

Then, proceeding similarly as in the proof of Lemma 3.4, for $\|u\|_E$ large enough we have that

$$\begin{aligned}
\langle G(u), u\rangle &\ge m_0 c_4 |u|_E^{p^-} + k\sum_{i=1}^{2}|u_i|^{\alpha^-}_{p(x),\Gamma_i} + k|u_1-u_2|^{\alpha^-}_{\alpha(x),\Gamma_3} - \sum_{i=1}^{2}\int_{\Omega_i} h(x,u_i)u_i\,dx\\
&\ge c_{2'}\|u\|_E^{p^-} - c_8\left(\|u\|_E + \|u\|_E^{\beta^+}\right) \ge c_9\|u\|_E^{p^-} > 0
\end{aligned}$$

By the topological degree theory for (S_+) type mappings, for $R > 0$ large enough, we have

$$\deg(G, B(0,R), 0) = 1$$

Therefore the equation $G(u) = 0$ has at least one solution $u \in B(0, R)$. Furthermore

$$\lim_{\|u\|_E \to +\infty} \frac{\langle G(u), u \rangle}{\|u\|_E} \geq c_9 \|u\|_E^{p^- - 1} = +\infty$$

So, the mapping G is coercive, and hence, by the surjection theorem for the pseudomonotone mappings (see [45, Theorem 27.A]), the mapping G is surjective. □

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