



Fractional boundary value problems and Lyapunov-type inequalities with fractional integral boundary conditions

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Abstract. We discuss boundary value problems for Riemann–Liouville fractional differential equations with certain fractional integral boundary conditions. Such boundary conditions are different from the widely considered pointwise conditions in the sense that they allow solutions to have singularities, and different from other conditions given by integrals with a singular kernel since they arise from well-defined initial value problems. We derive Lyapunov-type inequalities for linear fractional differential equations and apply them to establish nonexistence, uniqueness, and existence-uniqueness of solutions for certain linear fractional boundary value problems. Parallel results are also obtained for sequential fractional differential equations. An example is given to show how computer programs and numerical algorithms can be used to verify the conditions and to apply the results.

Keywords: fractional differential equations, fractional integral boundary conditions, Lyapunov-type inequalities, boundary value problems, existence and uniqueness of solutions.

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1 Fractional integral boundary conditions

Boundary value problems (BVPs) for fractional differential equations are important in applications and have been studied extensively by many authors, see [3,6,10–13,16,19,20,22,27,29,34] and the references cited therein. A lot of work has been done on fractional BVPs consisting of a fractional differential equation in the form

$$(D_{a+}^{\alpha} x)(t) = f(t, x) \quad \text{on } (a, b) \quad (1.1)$$

with $\alpha > 0$, and a pointwise boundary condition (BC) at the end points; in particular, the Dirichlet BC

$$x(a) = x(b) = 0 \quad (1.2)$$

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for $1 < \alpha \leq 2$. Here with $\alpha > 0$ and $t > a$,

$$\left(I_{a^+}^\alpha x\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds \quad (1.3)$$

is the α th-order left-sided Riemann–Liouville fractional integral of $x(t)$ at a , and $(D_{a^+}^\alpha x)(t)$ denotes the α th-order left-sided Riemann–Liouville fractional derivative of $x(t)$ at a defined as

$$(D_{a^+}^\alpha x)(t) := \frac{d^n}{dt^n} (I_{a^+}^{n-\alpha} x)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \quad (1.4)$$

where $n = \lfloor \alpha \rfloor + 1$ with $\lfloor \alpha \rfloor$ the integer part of α and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function. In particular, when $\alpha = i \in \mathbb{N}_0$, then

$$\left(D_{a^+}^i u\right)(t) = u^{(i)}(t). \quad (1.5)$$

In the following, for the consistence of notations for BCs, we also denote $(D_{a^+}^{-\alpha} x)(t) := (I_{a^+}^\alpha x)(t)$ for $0 < \alpha < 1$.

We note that with Riemann–Liouville fractional derivative involved, any solution of Eq. (1.1) with a pointwise BC such as (1.2), if it exists, must be bounded on $[a, b]$. However, unlike integer-order differential equations, the majority of the solutions of Eq. (1.1) is unbounded at the left endpoint a no matter how good the right-hand function $f(t, x)$ is. This can be seen from [18, (2.1.39)] that every solution of Eq. (1.1) satisfies

$$x(t) = (I_{a^+}^\alpha D_{a^+}^\alpha x)(t) + \sum_{j=1}^n \frac{c_j}{\Gamma(\alpha-j+1)} (t-a)^{\alpha-j}, \quad (1.6)$$

with $c_j \in \mathbb{R}$ for $j = 1, \dots, n$, which shows that either $x(a) = 0$ or $x(t)$ is unbounded at a . Consequently, we should not expect that any BVP consisting Eq. (1.1) and a pointwise BC to have any solution unless the BC includes or implies the condition $x(a) = 0$. This is the reason why fractional BVPs have been studied mainly with the Dirichlet BC at a so far. More specifically, any pointwise BC including one of the following is ill-posed:

- (i) $x(a) = c$ for some $c \neq 0$,
- (ii) $x(a) + cx'(a) = 0$ for some $c \neq 0$,
- (iii) $x(a) + cx^{(i)}(b) = 0$ for some $c \neq 0, i = 0, 1$.

In fact, the BC in (i) violates (1.6); and the BCs in (ii) and (iii) are each equivalent to one of the two sets of conditions: $x(a) = x'(a) = 0$ and $x(a) = x^{(i)}(b) = 0$, and hence does not agree with the number requirement for well-posed BCs.

In this paper, we use fractional integral BCs to allow and “smoothen” the singularity of solutions at a . This idea is motivated by the initial conditions for Cauchy problems associated with Eq. (1.1) given in [18, (3.1.2)]:

$$\left(D_{a^+}^{\alpha-k} x\right)(a^+) = b_k, \quad b_k \in \mathbb{R}, \quad k = 1, 2, \dots, n, \quad (1.7)$$

where $n = \lfloor \alpha \rfloor + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$, and

$$\left(D_{a^+}^{\alpha-k} x\right)(a^+) := \lim_{t \rightarrow a^+} \left(D_{a^+}^{\alpha-k} x\right)(t)$$

except that

$$(D_{a^+}^{\alpha-n}x)(a^+) = \lim_{t \rightarrow a^+} (D_{a^+}^{\alpha-n}x)(t) = \lim_{t \rightarrow a^+} (I_{a^+}^{n-\alpha}x)(t) \quad \text{for } \alpha \notin \mathbb{N}.$$

Note that $\alpha - n = 0$ for $\alpha \in \mathbb{N}$. By (1.5)

$$(D_{a^+}^0x)(a^+) = \lim_{t \rightarrow a^+} (D_{a^+}^0x)(t) = x(a). \quad (1.8)$$

We notice that the existence and uniqueness of solutions have been established for the Cauchy problem (1.1), (1.7) with any $b_k \in \mathbb{R}$. From this point of view, a more reasonable BC should involve $(D_{a^+}^{\alpha-k}x)(a^+)$ rather than $x^{(k)}(a)$ for $k = 1, 2, \dots, n$. In particular, for Eq. (1.1) with $1 < \alpha \leq 2$, we may assign a homogeneous linear separated BC as

$$\begin{cases} c_{11} (D_{a^+}^{\alpha-2}x)(a^+) + c_{12} (D_{a^+}^{\alpha-2}x)(a^+) = 0, \\ c_{21} (D_{a^+}^{\alpha-2}x)(b) + c_{22} (D_{a^+}^{\alpha-2}x)(b) = 0; \end{cases} \quad (1.9)$$

and a coupled BC as

$$\begin{bmatrix} D_{a^+}^{\alpha-2}x \\ D_{a^+}^{\alpha-1}x \end{bmatrix} (b) = K \begin{bmatrix} D_{a^+}^{\alpha-2}x \\ D_{a^+}^{\alpha-1}x \end{bmatrix} (a^+); \quad (1.10)$$

where $c_{ij} \in \mathbb{R}$ and $K \in \mathbb{R}^{2 \times 2}$ such that $\det K \neq 0$. Nonhomogeneous BCs can be defined accordingly. Such BCs permit solutions unbounded at a and hence are more general than pointwise BCs. It is easy to see from (1.8) that when $\alpha = 2$, BCs (1.9) and (1.10) reduce to the two point BCs

$$\begin{cases} c_{11}x(a) + c_{12}x(a) = 0, \\ c_{21}x(b) + c_{22}x(b) = 0; \end{cases}$$

and

$$\begin{bmatrix} x \\ x' \end{bmatrix} (b) = K \begin{bmatrix} x \\ x' \end{bmatrix} (a);$$

respectively. Therefore, (1.9) and (1.10) are natural extensions of the self-adjoint BCs for second-order linear differential equations to fractional differential equations with $1 < \alpha \leq 2$.

We point out that the BCs considered in this paper are different from the general integral BCs with a singular kernel in the sense that they originate from the fractional initial conditions with which the existence-uniqueness results are derived. Problems with such BCs can be investigated in many approaches based on results on initial value problems. For instance, using the Fredholm alternative method to study the existence and uniqueness of boundary value problems, as shown in our Theorems 5.3 and 5.7. Such approaches are not allowed for general integral BCs.

In this paper, we consider the fractional BVP consisting of the linear equation

$$(D_{a^+}^\alpha x)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \quad (1.11)$$

and the BC

$$(D_{a^+}^{\alpha-2}x)(a^+) = (D_{a^+}^{\alpha-2}x)(b) = 0. \quad (1.12)$$

Lyapunov-type inequalities are derived and used to establish the existence and uniqueness for solutions of this BVP. Parallel results are also obtained for certain sequential fractional BVPs. Further discussions on higher order and nonlinear fractional BVPs will be given in forthcoming papers.

This paper is organized as follows: after this section, we briefly review the existing results on Lyapunov-type inequalities in Section 2. In Sections 3 and 4, we derive new Lyapunov-type inequalities for fractional differential equations and sequential fractional differential equations, respectively. Finally in Section 5, we apply the obtained Lyapunov-type inequalities to establish the existence and uniqueness for solutions of some fractional BVPs. We also give an example to show how computer programs and numerical algorithms can be used to verify the conditions and to apply the results.

2 Existing results on Lyapunov-type inequalities

For the second-order linear differential equation

$$x'' + q(t)x = 0 \quad \text{on } (a, b) \quad (2.1)$$

with $q \in L([a, b], \mathbb{R})$, the following result is known as the Lyapunov inequality, see [1, 21].

Theorem 2.1. *Assume Eq. (2.1) has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b |q(t)| dt > \frac{4}{b-a}. \quad (2.2)$$

It was first noticed by Wintner [30] and later by several other authors that inequality (2.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the positive part of $q(t)$, to become

$$\int_a^b q_+(t) dt > \frac{4}{b-a}. \quad (2.3)$$

Inequality (2.3) was further generalized to a more general form of second-order differential equations by Hartman [15, Chapter XI], and improved by Brown and Hinton [2] and Harris and Kong [14] later on.

Lyapunov-type inequalities have also been developed for higher order linear and half-linear differential equations by many authors. See [24–26, 31–33, 35] for the higher order linear case, and [4, 5] for half-linear case.

Recently, Ferreira obtained the following Lyapunov-type inequality for the Riemann–Liouville fractional-order differential equation (1.11) with the Dirichlet BC (1.2) in [8].

Theorem 2.2. *Assume Eq. (1.11) has a nontrivial solution $x(t)$ satisfying $x(a) = x(b) = 0$. Then*

$$\int_a^b |q(t)| dt > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}. \quad (2.4)$$

With a simple modification in the theorem, we can easily obtain a variation of Theorem 2.2.

Theorem 2.3. *Assume Eq. (1.11) has a nontrivial solution $x(t)$ satisfying $x(a) = x(b) = 0$. Then*

$$\int_a^b q_+(t) dt > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}.$$

These results are derived using the Green's function

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b \end{cases} \quad (2.5)$$

for BVP (1.11), (1.2) obtained in [8], which is an extension of the one given in [3] for the case that $a = 0$ and $b = 1$.

For more Lyapunov-type inequalities involving the Riemann–Liouville and Caputo fractional derivatives, we refer the reader to [7, 17, 23, 28].

3 Fractional Lyapunov-type inequalities

In this section, we let $-\infty < a < b < \infty$ and consider fractional differential equation

$$(D_{a^+}^\alpha x)(t) + q(t)x = 0 \quad \text{on } (a, b), \quad (3.1)$$

where $1 < \alpha \leq 2$ and $q \in L(a, b)$. To present our main results, we need the concept of γ -th right-sided Riemann–Liouville fractional derivative of a function $u(t)$ at b defined as

$$(D_{b^-}^\gamma u)(t) = \frac{-1}{\Gamma(n - \gamma)} \frac{d^n}{dt^n} \int_t^b (s - t)^{n - \gamma - 1} u(s) ds \quad \text{for } t < b, \quad (3.2)$$

where $\gamma \geq 0$ and $n = \lfloor \gamma \rfloor + 1$. In particular, when $0 \leq \gamma < 1$, (3.2) reduces to

$$(D_{b^-}^\gamma u)(t) = \frac{-1}{\Gamma(1 - \gamma)} \frac{d}{dt} \int_t^b (s - t)^{-\gamma} u(s) ds \quad \text{for } t < b. \quad (3.3)$$

More specifically, $(D_{b^-}^0 u)(t) = u(t)$. With the left-sided and right-sided fractional derivatives given in (1.4) and (3.3), we have the following fractional integration by parts formula, see [29, (2.64)]:

$$\int_a^b \phi(s) D_{a^+}^\gamma \psi(s) ds = \int_a^b \psi(s) D_{b^-}^\gamma \phi(s) ds \quad \text{for } 0 \leq \gamma < 1, \quad (3.4)$$

where $\phi \in L_p(a, b)$ and $\psi \in L_r(a, b)$ such that $p^{-1} + r^{-1} \leq 1 + \gamma$.

In the following we define

$$G(t, s) := \frac{1}{b - a} \begin{cases} (s - a)(b - t), & a \leq s \leq t \leq b, \\ (t - a)(b - s), & a \leq t \leq s \leq b; \end{cases} \quad (3.5)$$

and let $D_{b^-}^{2-\alpha}[G(t, s)q(s)]$ be the right-sided fractional derivative of $G(t, s)q(s)$ with respect to s and $[D_{b^-}^{2-\alpha}[G(t, s)q(s)]]_+$ be the positive part of $D_{b^-}^{2-\alpha}[G(t, s)q(s)]$. Now we present our main result on fractional Lyapunov-type inequalities.

Theorem 3.1. (a) Assume Eq. (3.1) has a nontrivial solution $x(t)$ satisfying

$$(D_{a^+}^{\alpha-2}x)(a^+) = (D_{a^+}^{\alpha-2}x)(b) = 0. \quad (3.6)$$

Then

$$\max_{t \in [a, b]} \left\{ \int_a^b |D_{b^-}^{2-\alpha}[G(t, s)q(s)]| ds \right\} > 1. \quad (3.7)$$

(b) Assume Eq. (3.1) has a solution $x(t)$ satisfying

$$(D_{a^+}^{\alpha-2}x)(a^+) = (D_{a^+}^{\alpha-2}x)(b) = 0 \quad \text{and} \quad (D_{a^+}^{\alpha-2}x)(t) \neq 0 \quad \text{on } (a, b). \quad (3.8)$$

Then

$$\max_{t \in [a, b]} \left\{ \int_a^b [D_{b^-}^{2-\alpha}[G(t, s)q(s)]]_+ ds \right\} > 1. \quad (3.9)$$

Proof. (a) Let $y(t) = (D_{a^+}^{\alpha-2}x)(t)$ for $a < t \leq b$ and $y(a) = (D_{a^+}^{\alpha-2}x)(a^+)$. Then $y(t)$ is continuous on $[a, b]$. Note that $x(t) = (D_{a^+}^{2-\alpha}y)(t)$. We claim that

$$(D_{a^+}^{\alpha-2}x)(t) = y''(t) \quad \text{for } 1 < \alpha \leq 2. \quad (3.10)$$

In fact, for $1 < \alpha < 2$, from (1.4) we have

$$(D_{a^+}^{\alpha}x)(t) = \frac{d^2}{dt^2} (I_{a^+}^{2-\alpha}x)(t) = \frac{d^2}{dt^2} (D_{a^+}^{\alpha-2}x)(t) = y''(t);$$

and (3.10) holds clearly when $\alpha = 2$ since $y(t) = x(t)$. Then it follows that BVP (3.1), (3.6) becomes the second-order linear BVP

$$-y'' = q(t)x, \quad y(a) = y(b) = 0. \quad (3.11)$$

Hence the solution $y(t)$ satisfies

$$y(t) = \int_a^b G(t,s)q(s)x(s)ds = \int_a^b G(t,s)q(s)D_{a^+}^{2-\alpha}y(s)ds, \quad (3.12)$$

where $G(t,s)$, given in (3.5), is the Green's function for BVP (3.11). For a fixed $t \in [a, b]$, applying (3.4) with $\phi(s) = G(t,s)q(s) \in L(a, b)$ and $\psi(s) = y(s) \in L_{\gamma}(a, b)$ for $\gamma = 2 - \alpha$, we obtain

$$y(t) = \int_a^b G(t,s)q(s)D_{a^+}^{2-\alpha}y(s)ds = \int_a^b y(s)D_{b^-}^{2-\alpha}[G(t,s)q(s)]ds. \quad (3.13)$$

By taking the absolute value on both sides we have

$$|y(t)| = \left| \int_a^b y(s)D_{b^-}^{2-\alpha}[G(t,s)q(s)]ds \right| \leq \int_a^b |y(s)| |D_{b^-}^{2-\alpha}[G(t,s)q(s)]| ds.$$

Let $m = \max_{t \in [a, b]} |y(t)|$. By taking the maximum of $|y(t)|$ on both sides we obtain

$$m \leq \max_{t \in [a, b]} \int_a^b |y(s)| |D_{b^-}^{2-\alpha}[G(t,s)q(s)]| ds. \quad (3.14)$$

If $|y(t)| < m$ a.e. on $[a, b]$, then

$$m < m \max_{t \in [a, b]} \int_a^b |D_{b^-}^{2-\alpha}[G(t,s)q(s)]| ds$$

which leads to (3.7). Otherwise, there exists $J = \cup_{i=1}^k [a_i, b_i] \subset [a, b]$ for $1 \leq k \leq \infty$ with $a_i < b_i$ such that $y(t) \equiv m$ on J and $y(t) < m$ a.e. on $[a, b] \setminus J$. Then for $t \in J$, $y''(t) = 0$ and

$$x(t) = (D_{a^+}^{2-\alpha}y)(t) = (D_{a^+}^{2-\alpha}m) = \frac{m(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} \neq 0.$$

From (3.11), $q(t) \equiv 0$ on J . This implies that for any $t \in [a, b]$, $D_{b^-}^{2-\alpha}[G(t,s)q(s)] = 0$ for $s \in J$. Hence it follows from (3.14) that

$$\begin{aligned} m &\leq \max_{t \in [a, b]} \int_{[a, b] \setminus J} |y(s)| |D_{b^-}^{3-\alpha}[G(t,s)q(s)]| ds \\ &< m \max_{t \in [a, b]} \int_{[a, b] \setminus J} |D_{b^-}^{3-\alpha}[G(t,s)q(s)]| ds \\ &= m \max_{t \in [a, b]} \int_a^b |D_{b^-}^{3-\alpha}[G(t,s)q(s)]| ds \end{aligned}$$

which also leads to (3.7).

(b) From the proof of Part (a) we see that (3.13) holds. By the assumption, $y(t) \neq 0$ on (a, b) . Without loss of generality, we assume that $y(t) > 0$ on (a, b) . Then it follows that

$$y(t) \leq \int_a^b y(s) \left[D_{b^-}^{2-\alpha} [G(t, s)q(s)] \right]_+ ds. \quad (3.15)$$

Now, a similar argument as in Part (a) leads to (3.9). \square

The corollary below is a special case of Theorem 3.1.

Corollary 3.2. Assume $D_{b^-}^{2-\alpha} [G(t, s)q(s)] \geq 0$ for $t, s \in [a, b]$ and Eq. (3.1) has a nontrivial solution $x(t)$ satisfying (3.6). Then

$$\int_a^b q_+(t) dt > \frac{\alpha^\alpha \Gamma(\alpha - 1)}{(\alpha - 1)^{\alpha-1} (b - a)^{\alpha-1}}. \quad (3.16)$$

Proof. By Theorem 3.1 we see that (3.7) holds. By the assumption and the definition of $D_{b^-}^{2-\alpha} [G(t, s)q(s)]$ given in (3.3) we have

$$\begin{aligned} \int_a^b |D_{b^-}^{2-\alpha} [G(t, s)q(s)]| ds &= \int_a^b D_{b^-}^{2-\alpha} [G(t, s)q(s)] ds \\ &= \frac{-1}{\Gamma(\alpha - 1)} \int_a^b \left(\int_s^b (\tau - s)^{\alpha-2} G(t, \tau) q(\tau) d\tau \right)' ds \\ &= \frac{1}{\Gamma(\alpha - 1)} \int_a^b (\tau - a)^{\alpha-2} G(t, \tau) q(\tau) d\tau. \end{aligned} \quad (3.17)$$

Hence (3.9) becomes

$$\max_{t \in [a, b]} \int_a^b (\tau - a)^{\alpha-2} G(t, \tau) q(\tau) d\tau > \Gamma(\alpha - 1).$$

Using the facts that $q(t) \leq q_+(t)$, $G(t, s) \geq 0$ on $[a, b] \times [a, b]$, and

$$\max_{t \in [a, b]} G(t, \tau) = G(\tau, \tau) = \frac{(\tau - a)(b - \tau)}{b - a}, \quad \tau \in [a, b],$$

we see that

$$\begin{aligned} \Gamma(\alpha - 1) &< \max_{t \in [a, b]} \int_a^b (\tau - a)^{\alpha-2} G(t, \tau) q_+(\tau) d\tau \\ &\leq \frac{1}{b - a} \int_a^b (\tau - a)^{\alpha-1} (b - \tau) q_+(\tau) d\tau. \end{aligned} \quad (3.18)$$

Denote $g(\tau) = (\tau - a)^{\alpha-1} (b - \tau)$. From the fact that $1 < \alpha \leq 2$, we see $g(\tau)$ is continuous on $[a, b]$, $g(a) = g(b) = 0$, and $g(\tau) > 0$ on (a, b) . Thus there exists a $c \in (a, b)$ such that $g(c) = \max_{\tau \in [a, b]} g(\tau)$. Now a simple calculation shows that $c = [(\alpha - 1)b + a] / \alpha$ and hence

$$g(\tau) \leq g(c) = \frac{(\alpha - 1)^{\alpha-1} (b - a)^\alpha}{\alpha^\alpha}. \quad (3.19)$$

Substituting (3.19) in (3.18) we see that (3.16) holds. \square

Remark 3.3. By (1.5) we have $(D_{a^+}^2 x)(t) = x''(t)$ and $(D_{a^+}^0 x)(t) = x(t)$. Hence for $\alpha = 2$, Eq. (3.1) with condition (3.8) becomes the second-order equation with pointwise condition

$$x'' + q(t)x = 0, \quad x(a) = x(b) = 0 \quad \text{and} \quad x(t) \neq 0 \quad \text{on} \quad (a, b). \quad (3.20)$$

Since $G(t, s) \geq 0$ on $[a, b] \times [a, b]$, it follows from Theorem 3.1, Part (b) with $\alpha = 2$ that

$$\begin{aligned} 1 &< \max_{t \in [a, b]} \left\{ \int_a^b [D_{b^-}^{2-\alpha} [G(t, s)q(s)]_+ ds \right\} \\ &= \max_{t \in [a, b]} \left\{ \int_a^b [G(t, s)q(s)]_+ ds \right\} \\ &= \max_{t \in [a, b]} \int_a^b G(t, s)q_+(s) ds. \end{aligned} \quad (3.21)$$

Note that

$$\max_{t \in [a, b]} G(t, s) = G(s, s) = \frac{(s-a)(b-s)}{b-a} \leq \frac{b-a}{4}.$$

Hence (3.21) reduces to

$$\int_a^b q_+(t) dt > \frac{4}{b-a},$$

which becomes the Lyapunov inequality for the second-order equation (2.1).

4 Sequential fractional Lyapunov-type inequalities

Here we let $-\infty < a < b < \infty$ and consider the sequential fractional differential equation

$$\left[\left(D_{a^+}^\beta (D_{a^+}^\alpha x) \right) \right] (t) + q(t)x = 0 \quad \text{on} \quad (a, b), \quad (4.1)$$

where $q \in L([a, b], \mathbb{R})$, and $0 < \alpha, \beta \leq 1$. In the following, we define

$$G(t, s) := \frac{1}{\Gamma(\beta+1)} \begin{cases} \frac{(t-a)^\beta(b-s)^\beta}{(b-a)^\beta} - (t-s)^\beta, & a \leq s \leq t \leq b, \\ \frac{(t-a)^\beta(b-s)^\beta}{(b-a)^\beta}, & a \leq t \leq s \leq b; \end{cases} \quad (4.2)$$

and let $D_{b^-}^{1-\alpha} [G(t, s)q(s)]$ and $[D_{b^-}^{1-\alpha} [G(t, s)q(s)]_+]$ be defined in the same way as in Section 3. Now we present Lyapunov-type inequalities for Eq. (4.1).

Theorem 4.1. (a) Assume Eq. (4.1) has a nontrivial solution $x(t)$ satisfying

$$\left(D_{a^+}^{\alpha-1} x \right) (a^+) = \left(D_{a^+}^{\alpha-1} x \right) (b) = 0. \quad (4.3)$$

Then

$$\max_{t \in [a, b]} \left\{ \int_a^b \left| D_{b^-}^{1-\alpha} [G(t, s)q(s)] \right| ds \right\} > 1. \quad (4.4)$$

(b) Assume Eq. (4.1) has a solution $x(t)$ satisfying

$$\left(D_{a^+}^{\alpha-1} x \right) (a^+) = \left(D_{a^+}^{\alpha-1} x \right) (b) = 0 \quad \text{and} \quad \left(D_{a^+}^{\alpha-1} x \right) (t) \neq 0 \quad \text{on} \quad (a, b). \quad (4.5)$$

Then

$$\max_{t \in [a, b]} \left\{ \int_a^b \left[D_{b^-}^{1-\alpha} [G(t, s)q(s)] \right]_+ ds \right\} > 1, \quad (4.6)$$

Proof. Let $y(t) = (D_{a^+}^{\alpha-1}x)(t)$ for $a < t \leq b$ and $y(a) = (D_{a^+}^{\alpha-1}x)(a^+)$. Then $y(t)$ is continuous on $[a, b]$. Note that $x(t) = (D_{a^+}^{1-\alpha}y)(t)$. As shown in the proof of Theorem 3.1, we have $(D_{a^+}^{\alpha}x)(t) = y'(t)$. It follows that BVP (4.1), (4.3) becomes the fractional BVP

$$-(D_{a^+}^{\beta}y')(t) = q(t)x, \quad y(a) = y(b) = 0. \quad (4.7)$$

We claim that $(D_{a^+}^{\beta+1}y)(t) = (D_{a^+}^{\beta}y')(t)$. In fact, from the relation [18, (2.1.28)] we have

$$(D_{a^+}^{\beta}y)(t) = \frac{1}{\Gamma(1-\beta)} \left[\frac{y(a)}{(t-a)^{\beta}} + \int_a^t \frac{y'(s)}{(t-s)^{\beta}} ds \right].$$

Using the fact that $y(a) = 0$ and differentiating both sides with respect to t we have

$$(D_{a^+}^{\beta+1}y)(t) = \frac{d}{dt}(D_{a^+}^{\beta}y)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t \frac{y'(s)}{(t-s)^{\beta}} ds = (D_{a^+}^{\beta}y')(t).$$

Thus BVP (4.7) becomes

$$-(D_{a^+}^{\beta+1}y)(t) = q(t)x, \quad y(a) = y(b) = 0. \quad (4.8)$$

Note that BVP (4.8) is in the form of BVP (1.11), (1.2) with α replaced by $\beta + 1$, and the Green's function $G(t, s)$ in (2.5) for BVP (1.11), (1.2) becomes the one in (4.2). Then the solution $y(t)$ satisfies

$$y(t) = \int_a^b G(t, s)q(s)x(s)ds = \int_a^b G(t, s)q(s)D_{a^+}^{1-\alpha}y(s)ds.$$

The rest of the proof is essentially the same as the proof of Theorem 3.1. We omit the details. \square

The following corollary is a special case of Theorem 4.1.

Corollary 4.2. Assume $D_{b^-}^{1-\alpha}[G(t, s)q(s)] \geq 0$ for $t, s \in [a, b]$ and $1 < \alpha + \beta \leq 2$. Suppose Eq. (4.1) has a nontrivial solution $x(t)$ satisfying (4.3). Then

$$\int_a^b q_+(t)dt > \frac{(\alpha + 2\beta - 1)^{\alpha+2\beta-1}\Gamma(\alpha)\Gamma(\beta+1)}{(\alpha + \beta - 1)^{\alpha+\beta-1}\beta^{\beta}(b-a)^{\alpha+\beta-1}}. \quad (4.9)$$

Proof. The proof is similar to that of Corollary 3.2. By Theorem 4.1 we see that (4.4) holds. From the assumption and the definition of $D_{b^-}^{1-\alpha}[G(t, s)q(s)]$ given in (3.3) we have

$$\begin{aligned} \int_a^b \left| D_{b^-}^{1-\alpha}[G(t, s)q(s)] \right| ds &= \int_a^b D_{b^-}^{1-\alpha}[G(t, s)q(s)] ds \\ &= \frac{-1}{\Gamma(\alpha)} \int_a^b \left(\int_s^b (\tau - s)^{\alpha-1} G(t, \tau) q(\tau) d\tau \right)' ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (\tau - a)^{\alpha-1} G(t, \tau) q(\tau) d\tau. \end{aligned} \quad (4.10)$$

Hence (4.4) becomes

$$\max_{t \in [a, b]} \int_a^b (\tau - a)^{\alpha-1} G(t, \tau) q(\tau) d\tau > \Gamma(\alpha).$$

Using the facts that $q(t) \leq q_+(t)$, $G(t, s) \geq 0$ on $[a, b] \times [a, b]$, and

$$\max_{t \in [a, b]} G(t, \tau) = G(\tau, \tau) = \frac{(\tau - a)^{\beta}(b - \tau)^{\beta}}{(b - a)^{\beta}\Gamma(\beta + 1)}, \quad \tau \in [a, b],$$

we see that

$$\begin{aligned}\Gamma(\alpha) &< \max_{t \in [a,b]} \int_a^b (\tau - a)^{\alpha-1} G(t, \tau) q_+(\tau) d\tau \\ &\leq \frac{1}{(b-a)^\beta \Gamma(\beta+1)} \int_a^b (\tau - a)^{\alpha+\beta-1} (b-\tau)^\beta q_+(\tau) d\tau.\end{aligned}\quad (4.11)$$

Denote $h(\tau) = (\tau - a)^{\alpha+\beta-1} (b - \tau)^\beta$. From the fact that $0 < \beta \leq 1$ and $1 < \alpha + \beta \leq 2$, we see $h(\tau)$ is continuous on $[a, b]$, $h(a) = h(b) = 0$ and $h(\tau) > 0$ on (a, b) . Thus there exists a $d \in (a, b)$ such that $h(d) = \max_{\tau \in [a,b]} h(\tau)$. Now a simple calculation shows that $d = [(\alpha + \beta - 1)b + \beta a] / (\alpha + 2\beta - 1)$ and hence

$$h(\tau) \leq h(d) = \frac{(\alpha + \beta - 1)^{\alpha+\beta-1} \beta^\beta}{(\alpha + 2\beta - 1)^{\alpha+2\beta-1}} (b-a)^{\alpha+2\beta-1}.\quad (4.12)$$

Substituting (4.12) in (4.11) we see that (4.9) holds. \square

Remark 4.3. By (1.5) we have $(D_{a^+}^2 x)(t) = x''(t)$ and $(D_{a^+}^0 x)(t) = x(t)$. Hence for $\alpha = \beta = 1$, Eq. (4.1) and condition (4.5) becomes (3.20). Letting $\beta = 1$ in (4.2), we see that $G(t, s)$ becomes the same as the one in (3.5). Since $G(t, s) \geq 0$ on $[a, b] \times [a, b]$, it follows from Theorem 4.1, Part (b) with $\alpha = 1$ that

$$\begin{aligned}1 &< \max_{t \in [a,b]} \left\{ \int_a^b \left[D_{b^-}^{1-\alpha} [G(t, s) q(s)] \right]_+ ds \right\} \\ &= \max_{t \in [a,b]} \left\{ \int_a^b \left[G(t, s) q(s) \right]_+ ds \right\} \\ &= \max_{t \in [a,b]} \int_a^b G(t, s) q_+(s) ds.\end{aligned}\quad (4.13)$$

Note that

$$\max_{t \in [a,b]} G(t, s) = G(s, s) = \frac{(s-a)(b-s)}{b-a} \leq \frac{b-a}{4}.$$

Hence (4.13) reduces to

$$\int_a^b q_+(t) dt > \frac{4}{b-a},$$

which becomes the Lyapunov inequality for the second-order equation (2.1).

5 Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type inequalities obtained in Sections 2 and 3 to study the nonexistence, uniqueness, and existence-uniqueness of solutions of related fractional-order linear BVPs. We first consider the BVP consisting of the equation

$$(D_{a^+}^\alpha x)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \quad (5.1)$$

and the BC

$$(D_{a^+}^{\alpha-2} x)(a^+) = (D_{a^+}^{\alpha-2} x)(b) = 0. \quad (5.2)$$

Definition 5.1. A solution $x(t)$ of Eq. (5.1) is said to be an I -positive solution if $(I_{a^+}^{n-\alpha} x)(t) > 0$ on (a, b) , where $n = \lfloor \alpha \rfloor + 1$.

The following result is on the nonexistence of solutions of BVP (5.1), (5.2).

Lemma 5.2. (a) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b |D_b^{2-\alpha} [G(t,s)q(s)]| ds \right\} \leq 1. \quad (5.3)$$

Then BVP (5.1), (5.2) has no nontrivial solution.

(b) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b [D_b^{2-\alpha} [G(t,s)q(s)]]_+ ds \right\} \leq 1. \quad (5.4)$$

Then BVP (5.1), (5.2) has no I -positive solution.

Proof. (a) Assume the contrary, i.e., BVP (5.1), (5.2) has a nontrivial solution $x(t)$. Then by Theorem 3.1 Part (a), (3.7) holds. This contradicts assumption (5.3).

(b) The proof is similar to Part (a) and hence is omitted. \square

Next we consider the fractional-order nonhomogeneous linear BVP consisting of the equation

$$(D_{a^+}^\alpha x)(t) + q(t)x = w(t) \quad \text{on } (a, b) \quad (5.5)$$

with $1 < \alpha \leq 2$ and $q, w \in L((a, b), \mathbb{R})$; and the BC

$$(D_{a^+}^{\alpha-2} x)(a^+) = k_1, \quad (D_{a^+}^{\alpha-2} x)(b) = k_2, \quad (5.6)$$

where $k_1, k_2 \in \mathbb{R}$. Based on Theorem 3.1, we obtain a criterion for BVP (5.5), (5.6) to have a unique solution and a relation among solutions if the problem has more than one solution.

Theorem 5.3. (a) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b |D_b^{2-\alpha} [G(t,s)q(s)]| ds \right\} \leq 1.$$

Then BVP (5.5), (5.6) has a unique solution on (a, b) for any $k_1, k_2 \in \mathbb{R}$.

(b) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b [D_b^{2-\alpha} [G(t,s)q(s)]]_+ ds \right\} \leq 1 < \max_{t \in [a,b]} \left\{ \int_a^b |D_b^{2-\alpha} [G(t,s)q(s)]| ds \right\}.$$

If BVP (5.5), (5.6) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (a, b)$ such that $(I_{a^+}^{2-\alpha} x_1)(c) = (I_{a^+}^{2-\alpha} x_2)(c)$.

Proof. (a) We first show that BVP (5.5), (5.6) has at most one solution for any $k_1, k_2 \in \mathbb{R}$. Assume the contrary, i.e., it has two solutions $x_1(t)$ and $x_2(t)$ in (a, b) . Let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of BVP (5.1), (5.2). By Lemma 5.2, Part (a), we have $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of the solution of BVP (5.5), (5.6).

Since BVP (3.1), (5.2) has only the zero solution, then by the Fredholm alternative theorem [9], we conclude that BVP (5.5), (5.6) has a unique solution.

(b) Let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the BVP (5.1), (5.2). By Lemma 5.2 Part (b), $x(t)$ is not an I -positive on $[a, b]$. Then there exists a $c \in (a, b)$ such that $(I_{a^+}^{2-\alpha} x)(c) = 0$, i.e., $(I_{a^+}^{2-\alpha} x_1)(c) = (I_{a^+}^{2-\alpha} x_2)(c)$. \square

With a similar argument, from Corollary 3.2 we obtain the result below.

Corollary 5.4. Assume $D_b^{2-\alpha}[G(t,s)q(s)] \geq 0$ in $[a,b]$ and

$$\int_a^b q_+(t)dt \leq \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}. \quad (5.7)$$

Then BVP (5.5), (5.6) has a unique solution on (a,b) for any $k_1, k_2 \in \mathbb{R}$.

Remark 5.5. We note from Section 1 that the BVP consisting of Eq. (5.5) and the pointwise BC

$$x(a) = k_1, \quad x(b) = k_2 \quad (5.8)$$

does not have a solution unless $k_1 = 0$. Even for the case with $k_1 = 0$, the existence and uniqueness of solutions of BVP (5.5), (5.8) cannot be established by the Fredholm alternative method. This is due to the fact that Eq. (5.5) with a pointwise initial condition may not have a unique solution.

For the case with $k_1 = k_2 = 0$ and $w(t) \equiv 0$, from Theorem 2.2, we can easily derive the following result: Assume

$$\int_a^b |q(t)|dt \leq \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}. \quad (5.9)$$

Then BVP (5.5), (5.8) has only the zero solution.

We observe that this result has been improved by Corollary 5.4 for $\alpha = 2$ since BVPs (5.5), (5.6) and (5.5), (5.8) become the same second-order homogeneous linear problem. When $1 < \alpha < 2$, we compare the two results by comparing the right-hand numbers of (5.7) and (5.9) (under the assumption that $D_b^{2-\alpha}[G(t,s)q(s)] \geq 0$ for BVP (5.5), (5.6)). We claim that

$$H(\alpha) := \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}} - \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1} > 0 \quad (5.10)$$

and $H(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 1^+$. In fact,

$$H(\alpha) = \frac{(\alpha-1)\Gamma(\alpha-1)}{(b-a)^{\alpha-1}} \left(\left(\frac{\alpha}{\alpha-1} \right)^\alpha - \left(4^{1-\frac{1}{\alpha}} \right)^\alpha \right).$$

Then (5.10) follows from the fact that $\alpha/(\alpha-1) > 4^{1-1/\alpha}$ for $1 < \alpha < 2$. This shows that condition (5.7) is weaker than condition (5.9), and much weaker when α is close to 1; which is reasonable since BC (5.6) allows the solution $x(t)$ to have a singularity at a , while BC (5.8) requires the solution to be bounded.

Now, we state the results for the sequential fractional BVPs which are parallel to Theorem 5.3 and Corollary 5.4. We omit the proofs since they are essentially in the same way. Consider the BVP consisting of the equation

$$\left[\left(D_{a^+}^\beta (D_{a^+}^\alpha x) \right) \right] (t) + q(t)x = 0, \quad 0 < \alpha, \beta \leq 1, \quad (5.11)$$

and the BC

$$\left(D_{a^+}^{\alpha-1} x \right) (a^+) = \left(D_{a^+}^{\alpha-1} x \right) (b) = 0. \quad (5.12)$$

The following result is on the nonexistence of solutions of BVP (5.11), (5.12).

Lemma 5.6. (a) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b \left| D_{b^-}^{1-\alpha} [G(t,s)q(s)] \right| ds \right\} \leq 1.$$

Then BVP (5.11), (5.12) has only the trivial solution.

(b) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b \left[D_{b^-}^{1-\alpha} [G(t,s)q(s)] \right]_+ ds \right\} \leq 1.$$

Then BVP (5.11), (5.12) has no I-positive solution.

Next we consider the sequential nonhomogeneous linear BVPs consisting of the equation

$$(D_{a^+}^\beta (D_{a^+}^\alpha x))(t) + q(t)x = w(t) \quad \text{on } (a,b), \quad (5.13)$$

where $0 < \alpha, \beta \leq 1$ and $q, w \in L((a,b), \mathbb{R})$, and the BC

$$(D_{a^+}^{\alpha-1} x)(a^+) = k_1, \quad (D_{a^+}^{\alpha-1} x)(b) = k_2, \quad (5.14)$$

where $k_1, k_2 \in \mathbb{R}$. Now we present a criterion for BVP (5.13), (5.14) to have a unique solution and a relation among the solutions if the problem has more than one solution.

Theorem 5.7. (a) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b \left| D_{b^-}^{1-\alpha} [G(t,s)q(s)] \right| ds \right\} \leq 1.$$

Then BVP (5.13), (5.14) have a unique solution on (a,b) for any $k_1, k_2 \in \mathbb{R}$.

(b) Assume

$$\max_{t \in [a,b]} \left\{ \int_a^b \left[D_{b^-}^{1-\alpha} [G(t,s)q(s)] \right]_+ ds \right\} \leq 1 < \max_{t \in [a,b]} \left\{ \int_a^b \left| D_{b^-}^{1-\alpha} [G(t,s)q(s)] \right| ds \right\}.$$

If BVP (5.13), (5.14) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (a,b)$ such that $(I_{a^+}^{1-\alpha} x_1)(c) = (I_{a^+}^{1-\alpha} x_2)(c)$.

As before, we have the following corollary from Corollary 4.2.

Corollary 5.8. Let $1 < \alpha + \beta \leq 2$. Assume $D_{b^-}^{1-\alpha} [G(t,s)q(s)] \geq 0$ in $[a,b]$ and

$$\int_a^b q_+(t) dt \leq \frac{(\alpha + 2\beta - 1)^{\alpha+2\beta-1} \Gamma(\alpha) \Gamma(\beta + 1)}{(\alpha + \beta - 1)^{\alpha+\beta-1} \beta^\beta (b-a)^{\alpha+\beta-1}}.$$

Then BVP (5.13), (5.14) have a unique solution on (a,b) for any $k_1, k_2 \in \mathbb{R}$.

Finally, we point out that the applications of the results in this paper involve evaluations of fractional derivatives of functions. However, conditions involving fractional derivatives and integrals are hard to check analytically, even with pointwise BCs. So computer programs and numerical algorithms are the main tools for applications. We refer the reader to [13] for numerical algorithms for computing fractional derivatives. Here, we give an example to illustrate the application of Theorem 5.3. A similar example for Theorem 5.7 can be easily elaborated and hence is left to the interested reader.

Example 5.9. We consider the BVP

$$(D_{0+}^{\alpha}x)(t) + k(\sin t)x = w(t), \quad (D_{0+}^{\alpha-2}x)(0^+) = k_1, \quad (D_{0+}^{\alpha-2}x)(2\pi) = k_2, \quad (5.15)$$

where $1 < \alpha \leq 2$, $w \in L((0, 2\pi), \mathbb{R})$, and $k, k_1, k_2 \in \mathbb{R}$. Using *Mathematica*, we sketch the graphs of the integrals $\int_0^{2\pi} |D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]| ds$ and $\int_0^{2\pi} [D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]]_+ ds$ as functions of t . From them we find that

$$\max_{t \in [0, 2\pi]} \int_0^{2\pi} |D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]| ds = 3.29$$

and

$$\max_{t \in [0, 2\pi]} \int_0^{2\pi} [D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]]_+ ds = 1.81,$$

see Figures 5.1 and 5.2 respectively. Hence, applying Theorem 5.3, we observe the following:

- (a) for $k \leq 0.3$, BVP (5.15) has a unique solution on $(0, 2\pi)$;
- (b) for $0.3 < k \leq 0.55$, if BVP (5.15) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (0, 2\pi)$ such that $(I_{0+}^{2-\alpha}x_1)(c) = (I_{0+}^{2-\alpha}x_2)(c)$.

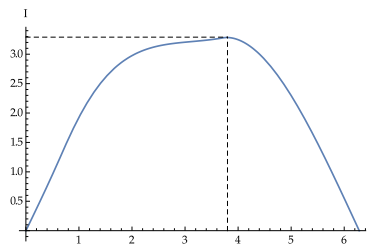


Figure 5.1: $\int_0^{2\pi} |D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]| ds$.

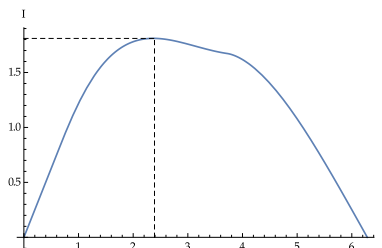


Figure 5.2: $\int_0^{2\pi} [D_{2\pi^-}^{2-\alpha}[G(t, s) \sin s]]_+ ds$

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