Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **34**, 1–22; doi: 10.14232/ejqtde.2016.1.34 http://www.math.u-szeged.hu/ejqtde/

Differential inclusions of arbitrary fractional order with anti-periodic conditions in Banach spaces

JinRong Wang[1], **Ahmed G. Ibrahim**[2] and **Michal Fečkan**✉ [3, 4]

[1]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
[2]Department of Mathematics, Faculty of Science, King Faisal University, Al-Ahasa 31982, Saudi Arabia
[3]Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia
[4]Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

Received 30 December 2015, appeared 2 June 2016
Communicated by Paul Eloe

Abstract. In this paper, we establish various existence results of solutions for fractional differential equations and inclusions of arbitrary order $q \in (m-1, m)$, where m is an arbitrary natural number greater than or equal to two, in infinite dimensional Banach spaces, and involving the Caputo derivative in the generalized sense (via the Liouville–Riemann sense). We study the existence of solutions under both convexity and nonconvexity conditions on the multivalued side. Some examples of fractional differential inclusions on lattices are given to illustrate the obtained abstract results.

Keywords: fractional differential inclusions, anti-periodic solutions, Caputo derivative in the generalized sense, measure of noncompactness, fractional lattice inclusions.

2010 Mathematics Subject Classification: 26A33, 34K20, 34K45.

1 Introduction.

During the past two decades, fractional differential equations and fractional differential inclusions have gained considerable importance due to their applications in various fields, such as physics, mechanics and engineering. For some of these applications, one can see [18, 23] and the references therein. For some recent developments on initial-value problems for differential equations and inclusions of fractional order, we refer the reader to the references [2, 17, 30, 34, 38–45].

Some applied problems in physics require fractional differential equations and inclusions with boundary conditions. Recently, many authors have been studied differential inclusions with various boundary conditions. Some of these works have been done in finite dimensional spaces and of positive integer order, for example, Ibrahim et al. [25] and Gomaa [19, 20].

Several results have studied fractional differential equations and inclusions with various boundary value conditions in finite dimensional spaces. We refer, for example, to Agarwal

✉ Corresponding author. Email: Michal.Feckan@fmph.uniba.sk

et al. [1] where conditions are established for the existence of solutions for various classes of initial and boundary value problems for fractional differential equations and inclusions involving the Caputo derivative. Next, Ouahab [34] studied a fractional differential inclusion with Dirichlet boundary conditions under both convexity and nonconvexity conditions on the multi-valued right-hand side and Ntouyas et al. [33] discussed the existence of solutions for fractional differential inclusions with three-point integral boundary conditions involving convex and non-convex multivalued maps.

For some recent works on boundary value problems for fractional differential equations and inclusions in infinite dimensional spaces, we refer to Benchohra et al. [7] where the existence of solutions are established for nonlinear fractional differential inclusions with two point boundary conditions.

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical problems and have received a considerable attention. Examples include anti-periodic trigonometric polynomials in the study of interpolation problems, anti-periodic wavelets, anti-periodic conditions in physics, and so forth (for details, see [3]).

Some recent works on anti-periodic boundary value problems of fractional order q where $m-1<q<m$ and $m=2,3,4,5$ can be found in [1,3–5,11,12,26].

In this paper, we establish various existence results of solutions for fractional differential equations and inclusions of arbitrary order $q\in(m-1,m)$, where m is a natural number greater than or equal to two, in infinite dimensional Banach spaces, and involving the Caputo derivative in the generalized sense (via the Riemann–Liouville sense). More precisely, let $J=[0,T]$, $T>0$, E be a real separable Banach space with a norm $\|\cdot\|$. We study the following fractional differential equations and inclusions with anti-periodic conditions:

$$\begin{cases} {}^cD^q_g x(t)=f(t,x(t)) & \text{a.e. on } J,\\ x^{(k)}(0)=-x^{(k)}(T), & k=0,1,2,\dots,m-1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD^q_g x(t)\in F(t,x(t)) & \text{for a.e. } t\in J,\\ x^{(k)}(0)=-x^{(k)}(T), & k=0,1,2,\dots,m-1, \end{cases} \tag{1.2}$$

respectively, where ${}^cD^q_g x(t)$ is the generalized Caputo derivative which is defined via the Riemann–Liouville fractional derivative of order q with the lower limit zero for the function x at the point t, $f:J\times E\to E$ and $F:J\times E\to 2^E$ is a multifunction.

We would like to point out that Agarwal et al. [1] considered the problems (1.1) and (1.2) when $m=4$ and the dimension of E is finite. Thereafter, Ahmad et al. [3] considered the problem (1.1) when $m=2$, Ahmad [4] proved the existence of solutions for (1.1) when $m=3$ and the dimension of E is finite, Cernea [12] proved existence theorems of solutions for (1.2) when $m=3$ and the dimension of E is finite, and Ibrahim [26] established various existence achievements in infinite dimensional Banach spaces for (1.1) and (1.2) when $m=3$. Alsaedi et al. [5] considered (1.1) in finite dimensional spaces in the case when $m=5$. As a consequence, the obtained results in [1,3–5,11,26] are particular cases of our derived results. We must mention that Kaslik and Sivasundaram [28] gave non-existence of periodic solutions of fractional order differential equations in the interval $[0,\infty)$ by using the Mellin transform approach. However, we consider fractional differential equations and inclusions with anti-periodic conditions on $[0,T]$ in this paper, not on the interval $[0,\infty)$. In other words, we try to find solutions to fractional differential equations and inclusions with anti-periodic conditions on finite time intervals, not to find periodic solutions on the infinite interval. So our problem is much different from [28].

The present paper is organized as follows. In Section 2, we collect some background material and basic results about fractional calculus. Hence, we prove auxiliary lemmas which will be used later. In Section 3, we give existence results for (1.1). In Section 4, we prove various existence results for (1.2). We consider the case when the values of F are convex as well as nonconvex. In Section 5, we apply our abstract results to fractional differential inclusions on lattices continuing on our previous work [45].

The proofs rely on the methods and results for boundary value fractional differential inclusions, the properties of noncompact measure and fixed point techniques.

2 Preliminaries and notations

Let $C(J,E)$ be the space of E-valued continuous functions on J with the norm $\|x\|_{C(J,E)} = \max\{\|x(t)\|,\ t \in J\}$, $AC^n(J,E)$ be the space of E-valued functions f on J, which have continuous derivatives up to the order $n-1$ on J such that f^{n-1} is absolute continuous on J, $L^1(J,E)$ be the space of all E-valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\|dt$, $P_b(E) = \{B \subseteq E : B$ is nonempty and bounded$\}$, $P_{cl}(E) = \{B \subseteq E : B$ is nonempty and closed$\}$, $P_k(E) = \{B \subseteq E : B$ is nonempty and compact$\}$, $P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact$\}$, $P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex$\}$, $\mathrm{conv}(B)$ (respectively, $\overline{\mathrm{conv}}(B)$) be the convex hull (respectively, convex and closed hull in E) of a subset B.

Let $G : J \to 2^E$ be a multifunction. By S^1_G we denote the set of integrable selections of G, i.e., $S^1_G = \{f \in L^1(J,E) : f(t) \in G(t)$ a.e.$\}$. This set may be empty. For $P_{cl}(E)$-valued measurable multifunction, S^1_G is nonempty and bounded in $L^1(J,E)$ if and only if $t \to \sup\{\|x\| :\ x \in G(t)\} \in L^1(J,\mathbb{R}^+)$ (such a multifunction is said to be integrably bounded) (see [22, Theorem 3.2]). Note that $S^1_G \subseteq L^1(J,E)$ is closed if the values of G are closed and it is convex if and only if for almost all $t \in J$, $G(t)$ is convex set in E.

Definition 2.1. Let X and Y be two topological spaces. A multifunction $G : X \to P(Y)$ is said to be upper semicontinuous, if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$.

For more information about multifunctions, see, [6, 10, 24, 27]. Now, let us recall the following definitions and facts about the integration and differentiation of fractional order.

Definition 2.2. [29, p. 69] The Riemann–Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f \in L^p(J,E)$, $p \in [1,\infty)$ is defined as follows:

$$I^q f(t) = (g_q * f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, \qquad t \in J,$$

where the integration is in the sense of Bochner, Γ is the Euler gamma function, $g_q(t) = \frac{t^{q-1}}{\Gamma(q)}$, for $t > 0$, $g_q(t) = 0$, for $t \leq 0$ and $*$ denotes the convolution of functions. For $q = 0$, we set $I^0 f(t) = f(t)$.

It is known [29] that $I^q I^\beta f(t) = I^{q+\beta} f(t)$, $\beta,\ q \geq 0$. Moreover, by applying Young's inequality, it follows that

$$\|I^q f\|_{L^p(J,E)} = \|g_q * f\|_{L^p(J,E)} \leq \|g_q\|_{L^1(J,\mathbb{R})} \|f\|_{L^p(J,E)} = g_{q+1}(T) \|f\|_{L^p(J,E)}.$$

Then, I^q maps $L^p(J,E)$ to $L^p(J,E)$. Let $\lceil q \rceil$ be the least integer greater than or equal to the number q. The set of natural numbers is denoted by $\mathbb{N}$.

Definition 2.3. [29, p. 70] Let $q > 0$, $m = \lceil q \rceil$ and $f \in L^1(J,E)$ be such that $g_{m-q} * f \in W^{m,1}(J,E)$. The Riemann–Liouville fractional derivative of order q for f is defined by

$$D^q f(J) = \frac{d^m}{dt^m} I^{m-q} f(t) = \frac{d^m}{dt^m} (g_{m-q} * f)(t),$$

where

$$W^{m,1}(J,E) = \left\{ f(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + I^m \varphi(t),\ t \in J : \varphi \in L^1(J,E),\ c_k \in E \right\}.$$

Note in the above definition that $\varphi = f^{(m)}$ and $c_k = f^{(k)}(0)$, $k = 0,1,\ldots,m-1$. In the following lemma, we mention some elementary properties for the Riemann–Liouville fractional integrals and derivatives.

Lemma 2.4. *Let $q > 0$ and $m = \lceil q \rceil$.*

(i) If $f \in L^1(J,E)$ then $g_{m-q} * (I^q f) \in W^{m,1}(J,E)$ and $D^q I^q f(t) = f(t)$ a.e.

(ii) If $\gamma > q$ and $f \in L^1(J,E)$, then $D^q I^\gamma f(t) = I^{\gamma-q} f(t)$ a.e. In particular, if $\gamma > k$, $k \in \mathbb{N}$, then $D^k I^\gamma f(t) = I^{\gamma-k} f(t)$ a.e.

(iii) If $p > \frac{1}{q}$ and $f \in L^p(J,E)$ then $I^q f$ is continuous on J.

Proof. Proofs of these properties are exactly as in the scalar case [36, Chapter 1]. □

Definition 2.5. [29, p. 91] Let $q > 0$ and $m = \lceil q \rceil$. The Caputo derivative in the generalized sense (via Riemann–Liouville fractional derivative) of order q for a given function f is defined by

$${}^cD_g^q f(t) = D^q \left[f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k \right]$$

provided that the right side is well defined.

Remark 2.6. Let $q > 0$, $m = \lceil q \rceil$ and $f : J \to \mathbb{R}$, $f(t) = t^n$, $n = 0,1,2,\ldots,(m-1)$. Then

$${}^cD_g^q t^n = D^q \left[t^n - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k \right] = D^q[t^n - t^n] = 0.$$

Remark 2.7. [29, Theorem 2.1] Let E be a reflexive Banach space, $m \in \mathbb{N}$ and $q \in (m-1,m)$. If $f \in AC^m(J,E)$, then ${}^cD_g^q f(t)$ exists a.e. and

$${}^cD_g^\alpha f(t) = I^{m-q}\left(f^{(m)}(t)\right) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(m)}(s) ds \quad \text{for a.e. } t \in J.$$

Moreover, if $f \in C^{(m)}(J,E)$, then this equation is valid for all $t \in J$.

To proceed, we state the following lemma as a simple consequence of Lemma 2.4 and formula [36, (3.13)].

Lemma 2.8. *Let $m \in \mathbb{N}$, $q \in (m-1, m)$ and $f \in L^{\frac{1}{\gamma}}(J, E)$, where $\gamma \in (0, q-(m-1))$. Then*

(i) If $\sigma > \gamma$ then the function $t \to I^{\sigma} f(t)$ is continuous on J.

(ii) For any $t \in T$ and $k \in \{0, 1, 2, \ldots, m-1\}$, the function $(I^q f)^{(k)}(t)$ is continuous satisfying $(I^q f)^{(k)}(0) = 0$. Moreover,

$$^cD_g^q \left(I^q f(t)\right) = D^q \left(I^q f(t)\right) \quad \text{for a.e. } t \in J,$$

and hence, $^cD_g^q(I^q f(t)) = f(t)$ a.e. on J.

The following lemma is essential to derive existence results of solutions for (1.1) and (1.2).

Lemma 2.9. *Let $m \in \mathbb{N}$ and $q \in (m-1, m)$. If $z \in L^{\frac{1}{\gamma}}(J, E), \gamma \in (0, q-(m-1))$ and $x : J \to E$ is given by*

$$x(t) = I^q z(t) - \sum_{k=0}^{m-1} b_{k+1} t^k, \tag{2.1}$$

where

$$b_m = \frac{1}{2(m-1)!} I^{q-(m-1)} z(T), \tag{2.2}$$

$$b_1 = \frac{1}{2} \left[I^q z(T) - \sum_{k=1}^{m-1} b_{k+1} T^k \right], \tag{2.3}$$

and for $2 \leq n \leq m-1$,

$$b_n = \frac{1}{2(n-1)!} \left[I^{q-(n-1)} z(T) - \sum_{k=n}^{m-1} \frac{k! b_{k+1}}{(k-(n-1))!} T^{k-(n-1)} \right]. \tag{2.4}$$

Then, $x^{(m-1)}$ is continuous on J, $^cD_g^q x(t)$ exists a.e. for $t \in J$ and

$$\begin{cases} ^cD_g^q x(t) = z(t) & \text{for a.e. } t \in J, \\ x^{(k)}(0) = -x^{(k)}(T), & k = 0, 1, 2, \ldots, m-1. \end{cases} \tag{2.5}$$

Proof. First we note, since for any $k = 0, \ldots, m-1$, it holds $q - k \geq q - (m-1) > \gamma$, by Lemma 2.8, the functions $I^{q-k} z$, $D^k(I^q z)$ are continuous on J. Hence b_{k+1} in (2.2), (2.3) and (2.4) are well-defined, and $x^{(k)}$ are continuous on J for all $k = 0, \ldots, m-1$.

Now, in view of (ii) of Lemma 2.8, $D_g^q z(t)$ exists for a.e. $t \in J$ and $^cD_g^q z(t) = D^q z(t)$ a.e. Thus, for a.e. $t \in J$

$$^cD_g^q x(t) = {}^c D_g^q (I^q z(t)) - {}^c D_g^q \left(\sum_{k=0}^{m-1} b_{k+1} t^k \right) = z(t) - {}^c D_g^q \left(\sum_{k=0}^{m-1} b_{k+1} t^k \right) = z(t),$$

by Remark 2.6. So (2.1) is a general solution of (2.5). Next, in view of (2.1), condition $x(0) = -x(T)$ gives

$$0 = x(0) + x(T) = \left[I^q z(T) - \sum_{k=1}^{m-1} b_{k+1} T^k \right] - 2b_1,$$

which implies (2.3). Furthermore, by differentiating both sides of (2.1), we get from (ii) of Lemma 2.4 for $1 \leq n \leq m-2$

$$x^{(n)}(t) = I^{\alpha-n} z(t) - (n!) b_{n+1} - \sum_{k=n+1}^{m-1} \frac{k! b_{k+1}}{(k-n))!} t^{k-n}, \qquad t \in J.$$

So conditions $x^{(n)}(0) = -x^{(n)}(T)$ give

$$0 = x^{(n)}(0) + x^{(n)}(T) = I^{q-n}z(T) - 2(n!)b_{n+1} - \sum_{k=n+1}^{m-1} \frac{k!b_{k+1}}{(k-n))!}T^{k-n},$$

which implies (2.4). Similarly,

$$x^{(m-1)}(t) = I^{q-(m-1)}z(t) - (m-1)!b_m, \qquad t \in J. \tag{2.6}$$

Then, condition $x^{(m-1)}(0) = -x^{(m-1)}(T)$ gives

$$0 = x^{(m-1)}(0) + x^{(m-1)}(T) = I^{q-(m-1)}z(T) - 2(m-1)!b_m,$$

which implies (2.2). The proof is finished. □

Now, by using Lemma 2.9 to establish formulae of solutions for (2.5), when $m = 2,3,4,5$, we get known results mentioned in Introduction. We list them one by one as follows.

(i) for $m = 2$, we get (see [3])

$$x(t) = I^q z(t) - \frac{1}{2}I^q z(T) + \frac{T-2t}{4}I^{q-1}z(T); \tag{2.7}$$

(ii) for $m = 3$, we get (see [4,12,26])

$$x(t) = I^q z(t) - \frac{1}{2}I^q z(T) + \frac{T-2t}{4}I^{q-1}z(T) + \frac{t(T-t)}{4}I^{q-2}z(T); \tag{2.8}$$

(iii) for $m = 4$, we get (see [1])

$$\begin{aligned} x(t) = I^q z(t) - \frac{1}{2}I^q z(T) + \frac{T-2t}{4}I^{q-1}z(T) + \frac{t(T-t)}{4}I^{q-2}z(T) \\ + \frac{(6t^2T - 4t^3 - T^3)}{48}I^{q-3}z(T); \end{aligned} \tag{2.9}$$

(iv) for $m = 5$, we get (see [5])

$$\begin{aligned} x(t) = I^q z(t) - \frac{1}{2}I^q z(T) + \frac{T-2t}{4}I^{q-1}z(T) + \frac{t(T-t)}{4}I^{q-2}z(T) \\ + \frac{(6t^2T - 4t^3 - T^3)}{48}I^{q-3}z(T) + \frac{(2t^3T - t^4 - tT^3)}{48}I^{q-4}z(T). \end{aligned} \tag{2.10}$$

Next, for $m = 6$, we get

$$\begin{aligned} x(t) = I^q z(t) - \frac{1}{2}I^q z(T) + \frac{T-2t}{4}I^{q-1}z(T) + \frac{t(T-t)}{4}I^{q-2}z(T) \\ + \frac{(6t^2T - 4t^3 - T^3)}{48}I^{q-3}z(T) + \frac{(2t^3T - t^4 - tT^3)}{48}I^{q-4}z(T) \\ + \left(\frac{T^5}{4(5!)} - \frac{t^2T^3}{4(4!)} + \frac{t^4T}{4(4!)} - \frac{t^5}{2(5!)}\right)I^{q-5}z(T). \end{aligned} \tag{2.11}$$

From the above examples (2.7)–(2.10) and (2.11), it is worthwhile to observe that we expect that (2.1) has a form

$$x(t) = I^q z(t) - \sum_{j=0}^{m-1} \theta_j(t) I^{q-j}z(T) \tag{2.12}$$

for polynomials $\theta_j(t)$ of degree j. Certainly by above arguments, such $x(t)$ satisfies ${}^cD^q x(t) = z(t)$ a.e. on J. The anti-periodic conditions of (2.5) imply

$$I^{q-k}z(T) = \sum_{j=k}^{m-1} \left(\theta_j^{(k)}(0) + \theta_j^{(k)}(T)\right) I^{q-j}z(T), \qquad k = 0,1,2,\dots,m-1 \tag{2.13}$$

where we use $\theta_j^{(k)}(t) = 0$ on J for $k > j$. Since z is arbitrarily, we set

$$\begin{aligned} \theta_k^{(k)}(t) &= \frac{1}{2}, \\ \theta_j^{(k)}(0) + \theta_j^{(k)}(T) &= 0, \qquad k = 0,1,2,\dots,j-1. \end{aligned} \tag{2.14}$$

It is easy to see that (2.14) determine uniquely $\theta_j(t)$. Since the solution of (2.5) is unique, we see that (2.12) with (2.14) give the solution (2.1), which coincides with the above computations (2.11). Moreover we see that $\theta_j(t)$ are really independent of m. Furthermore, set

$$\eta_j(s) = \frac{\theta_j(sT)}{T^j}, \qquad s \in [0,1], \quad j = 0,1,2,\dots,m-1.$$

Then

$$\begin{aligned} \eta_k^{(k)}(s) &= \frac{1}{2}, \\ \eta_j^{(k)}(0) + \eta_j^{(k)}(1) &= 0, \qquad k = 0,1,2,\dots,j-1. \end{aligned} \tag{2.15}$$

Again, (2.15) determine uniquely $\eta_j(t)$. But now $\eta_j(t)$ are independent also of T, so we have

$$\eta_j(s) = \frac{1}{2}\sum_{k=1}^{j+1} \frac{\gamma_k^{(j+1)}}{(k-1)!} s^{k-1}, \qquad \gamma_k^{(j)} \in \mathbb{R}, \ j,k = 1,2,3,\dots,m. \tag{2.16}$$

Then

$$\theta_j(t) = \frac{1}{2}\sum_{k=1}^{j+1} \frac{\gamma_k^{(j+1)} T^{j+1-k}}{(k-1)!} t^{k-1}, \qquad j = 0,1,2,\dots,m-1.$$

Hence

$$\begin{aligned} x(t) &= I^q z(t) - \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=1}^{j+1} \frac{\gamma_k^{(j+1)} T^{j+1-k}}{(k-1)!} t^{k-1} I^{q-j}z(T) \\ &= I^q z(t) - \frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1} \frac{\gamma_k^{(j+1)} T^{j+1-k}}{(k-1)!} I^{q-j}z(T) t^{k-1}. \end{aligned} \tag{2.17}$$

By comparing (2.1) with (2.17), we obtain

$$b_k = \sum_{j=k-1}^{m-1} \frac{\gamma_k^{(j+1)} T^{j+1-k}}{2(k-1)!} I^{q-j}z(T), \qquad k = 1,2,3,\dots,m.$$

Consequently, $\gamma_k^{(j)}$ is the coefficients of $\frac{T^{j-k}I^{q-(j-1)}z(T)}{2(k-1)!}$ in b_k for any $j,k = 1,2,\dots,m$ with $k \le j$.

Next, by inserting (2.16) into (2.15), after elementary calculations, we derive

$$\gamma_r^{(r)} = 1, \quad r = 1,2,\dots,m, \quad \gamma_{r-k}^{(r)} = -\sum_{n=1}^{k} \frac{\gamma_{r-(k-n)}^{(r)}}{2(n)!}, \quad k = 1,2,\dots,r-1. \tag{2.18}$$

Using (2.18), we can derive step by step, for instance

$$\begin{aligned}
&\gamma_{r-1}^{(r)} = -\frac{1}{2}, \quad r \geq 2, \quad \gamma_{r-2}^{(r)} = 0, \quad r \geq 3, \quad \gamma_{r-3}^{(r)} = \frac{1}{24}, \quad r \geq 4 \\
&\gamma_{r-4}^{(r)} = 0, \quad r \geq 5, \quad \gamma_{r-5}^{(r)} = -\frac{1}{2(5!)}, \quad r \geq 6, \quad \gamma_{r-6}^{(r)} = \frac{3}{2(6!)}, \quad r \geq 7.
\end{aligned} \tag{2.19}$$

Furthermore, let us set

$$\tilde{\eta}_j(s) = \int_0^s \eta_{j-1}(z)dz - \frac{1}{2}\int_0^1 \eta_{j-1}(z)dz, \qquad j = 1,2,3,\ldots,m-1.$$

We can easily check

$$\begin{aligned}
\tilde{\eta}_k^{(k)}(s) &= \frac{1}{2}, \\
\tilde{\eta}_j^{(k)}(0) + \tilde{\eta}_j^{(k)}(1) &= 0, \qquad k = 0,1,2,\ldots,j-1.
\end{aligned}$$

So by uniqueness, $\tilde{\eta}_j = \eta_j$, i.e., we get

$$\eta_j(s) = \int_0^s \eta_{j-1}(z)dz - \frac{1}{2}\int_0^1 \eta_{j-1}(z)dz, \qquad j = 1,2,3,\ldots,m-1. \tag{2.20}$$

On the other hand, (2.16) implies

$$\gamma_k^{(j)} = \eta_{j-1}^{(k-1)}(0).$$

Hence for any $r,k \in \mathbb{N}$, $r > k$, by (2.20), we obtain

$$\gamma_{r-k}^{(r)} = \eta_{r-1}^{(r-k-1)}(0) = \eta_k(0) = \gamma_1^{(k+1)},$$

which justifies (2.19).

Summarizing, we arrive at the following result.

Corollary 2.10. *Let $m \in \mathbb{N}$ and $m-1 < q < m$. If $z \in L^{\frac{1}{\sigma}}(J,E)$, $\sigma \in (0, q-(m-1))$, then the function x given by (2.17) and $\gamma_{r-k}^{(r)}$, $r = 1,2,3,\ldots,m$, $k = 0,1,2,\ldots,r-1$ determined by (2.18), is a solution of (2.5).*

Remark 2.11. One can verify that the expression (2.17) coincides with the known relations (2.7)–(2.10) and (2.11) for $m = 2,3,4,5,6$, respectively.

Remark 2.12. Let $m \in \mathbb{N}$ and $m-1 < q < m$. If $z \in L^{\frac{1}{\sigma}}(J,E)$, $\sigma \in (0, q-(m-1))$ then $z \in L^1(J,E)$ and the function x given by (2.17) satisfies the inequality

$$\begin{aligned}
\|x\|_{C(J,E)} \leq{}& \frac{3T^{q-1}}{2\Gamma(q)}\|z\|_{L^1(J,E)} + \frac{T^{q-1}}{2}\sum_{j=1}^{m-2}\frac{1}{\Gamma(q-j)}\sum_{k=1}^{j+1}\frac{|\gamma_k^{(j+1)}|}{(k-1)!}\|z\|_{L^1(J,E)} \\
&+ \frac{T^{q-\sigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\sum_{k=1}^{m}\frac{|\gamma_k^{(m)}|}{(k-1)!}\|z\|_{L^{\frac{1}{\sigma}}(J,E)},
\end{aligned} \tag{2.21}$$

where by Hölder's inequality

$$\|I^{q-(m-1)}z(T)\| \leq \frac{T^{q-m+1-\sigma}}{\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\|z\|_{L^{\frac{1}{\sigma}}(J,E)}.$$

Our main tools are the Schaefer fixed point theorem for single-valued mappings and the O'Regan–Precup fixed point theorem [35, Theorem 3.1] for multivalued mappings, which is a generalization of the Mönch fixed point theorem.

Lemma 2.13. *Let $Z : E \to E$ be continuous and map every bounded subset into relatively compact subset. If the set $E(Z) = \{x \in E : x = \lambda Z(x),\ \lambda \in [0,1)\}$ is bounded, then Z has a fixed point.*

Lemma 2.14. *Let D be a closed convex subset of E, and $N : D \to P_c(D)$. Assume the graph of N is closed, N maps compact sets into relatively compact sets and that, for some $x_0 \in U$, one has*

$$Z \subseteq D,\ \ Z = \operatorname{conv}(\{x_0\} \cup T(Z)),\ \overline{Z} = \overline{C}\ \text{ with } C \subseteq Z \text{ countable} \Longrightarrow Z \text{ is relatively compact.} \tag{2.22}$$

Then T has a fixed point.

For more about fixed point theorems see [16]. Finally, we give the concept of solutions for (1.1) and (1.2).

Definition 2.15. Let $f : J \times E \to E$ be a function. A function $x \in C^{(m-1)}(J, E)$ is called a solution for (1.1), if ${}^cD_g^q x(t)$ exists a.e. and

$$\begin{cases} {}^cD_g^q x(t) = f(t, x(t)) & \text{a.e. on } J = [0, T], \\ x^{(k)}(0) = -x^{(k)}(T), & k = 0, 1, 2, \dots, m-1. \end{cases}$$

Definition 2.16. Let $F : J \times E \to 2^E$ be a multifunction. A function $x \in C^{(m-1)}(J, E)$ is called a solution for (1.2), if ${}^cD_g^q x(t)$ exists a.e. and

$$\begin{cases} {}^cD_g^q x(t) = z(t) & \text{a.e. on } J = [0, T], \\ x^{(k)}(0) = -x^{(k)}(T), & k = 0, 1, 2, \dots, m-1. \end{cases}$$

where $z \in L^1(J, E)$ with $z(t) \in F(t, x(t))$ for a.e. $t \in J$.

Of course, f and F in the above definitions are specified below with appropriate properties.

3 Existence results for (1.1)

In the following, we give the first existence result for (1.1).

Theorem 3.1. *Let $m \in \mathbb{N}$, $m \geq 2$ and $q \in (m-1, m)$. Let $f : J \times E \to E$ be a function such that the following conditions are satisfied.*

(H_1) *f is continuous.*

(H_2) *There exists a function $\varphi \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, $\sigma \in (0, q-(m-1))$ such that for any $x \in E$*

$$\|f(t, x)\| \leq \varphi(t)(1 + \|x\|) \quad \text{for a.e. } t \in J.$$

(H_3) *For a.e. $s \in J$, the function $f(s, \cdot)$ maps any bounded subset into relatively compact subset in E.*

Then, the problem (1.1) has a solution provided that $\delta < 1$ for

$$\begin{aligned}\delta = &\left[\frac{3T^{q-1}}{2\Gamma(q)} + \frac{T^{q-1}}{2}\sum_{j=1}^{m-2}\frac{1}{\Gamma(q-j)}\sum_{k=1}^{j+1}\frac{|\gamma_k^{(j+1)}|}{(k-1)!}\right]\|\varphi\|_{L^1(J,\mathbb{R}^+)} \\ &+ \frac{T^{q-\sigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\sum_{k=1}^{m}\frac{|\gamma_k^{(m)}|}{(k-1)!}\|\varphi\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.\end{aligned} \tag{3.1}$$

Proof. Let us consider the operator $N : C(J,E) \to C(J,E)$ defined by

$$(N(x))(t) = I^q z(t) - \frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{\gamma_k^{(j+1)}T^{j+1-k}}{(k-1)!}I^{q-j}z(T)t^{k-1}, \qquad t \in J, \tag{3.2}$$

where $z(t) = f(t,x(t)),\ t \in J$. Note that, since f is continuous, then for every $x \in C(J,E)$, the function z is continuous. Therefore, the operator N is well defined. We shall prove that N satisfies the assumptions of the Schaefer fixed point theorem. We split the proof into several steps.

Step 1. N is continuous.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in $C(J,E)$. For any $t \in J$,

$$\begin{aligned}&\|(N(x_n))(t) - (N(x))(t)\| \\ &\quad\le T^q \max_{t\in J}\|f(t,(x_n(t)) - f(t,(x(t))\|\left[\frac{1}{\Gamma(q+1)} + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{|\gamma_k^{(j+1)|}}{(k-1)!\Gamma(q+1-j)}\right].\end{aligned}$$

Since f is continuous, we get $\|N(x_n) - N(x)\| \to 0$, as, $n \to \infty$.

Step 2. N maps bounded sets into bounded sets.
Let $r > 0$ and $B_r = \{x \in C(J,E) : \|x\|_{C(J,E)} \le r\}$. According to (H_2), for any $x \in B_r$

$$\begin{aligned}\|N(x)\|_{C(J,E)} \le (1+r)\|\varphi\|_{L^1(J,\mathbb{R}^+)}&\left[\frac{3T^{q-1}}{2\Gamma(q)} + \frac{T^{q-1}}{2}\sum_{j=1}^{m-2}\frac{1}{\Gamma(q-j)}\sum_{k=1}^{j+1}\frac{|\gamma_k^{(j+1)}|}{(k-1)!}\right] \\ &+ \frac{(1+r)T^{q-\sigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\sum_{k=1}^{m}\frac{|\gamma_k^{(m)}|}{(k-1)!}\|\varphi\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} = (1+r)\delta.\end{aligned} \tag{3.3}$$

Then, $N(B_r)$ is bounded.

Step 3. N maps bounded sets into relatively compact subsets.
Let $r > 0$, $x \in B_r$, $t_1, t_2 \in J$ $(t_1 < t_2)$. According to (H_2), $\|f(t,x(t))\| \le \varphi(t)(1+r)$ for a.e. $t \in J$. Then,

$$\begin{aligned}&\|N(x)(t_2) - N(x)(t_1)\| \\ &\quad\le \frac{1+r}{\Gamma(q)}\left[\int_0^{t_1}\left[|(t_2-s)^{q-1} - (t_1-s)|^{q-1}\right]\varphi(s)ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}\varphi(s)ds\right] \\ &\qquad+ \frac{r+1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{|\gamma_k^{(j+1)}|T^{j+1-k}}{(k-1)!}I^{q-j}\varphi(T)(t_2^{k-1} - t_1^{k-1}).\end{aligned}$$

Since $q > 1$, then clearly as $t_2 \to t_1$, $\|(N(x_1)(t_2)) - (N(x_1)(t_1))\| \to 0$, independently of x and uniformly for t_1, t_2. Therefore, $N(B_r)$ is equicontinuous.

Now, let $t \in J$ be fixed. We want to show that the subset $G(t) = \{(N(x))(t) : x \in B_r\}$ is relatively compact in E. According to the definition of N, (H_2), the properties of the Hausdorff measure of noncompactness [32], one obtains

$$\begin{aligned}\chi(G(t)) \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \chi(\{f(s,x(s)) : x \in B_r\}) ds \\ & + \frac{1}{2} \sum_{k=1}^{m} \sum_{j=k-1}^{m-1} \frac{|\gamma_k^{(j+1)}| T^j}{(k-1)!\Gamma(q-j)} \int_0^T (T-s)^{q-j-1} \chi(\{f(s,x(s)) : x \in B_r\}) ds.\end{aligned} \tag{3.4}$$

Observe that in view of (H_3), it follows $\chi(\{f(s,x(s)) : x \in B_r\}) = 0$, for almost $s \in J$, Then, (3.4) implies $\chi(G(t)) = 0$.

Step 4. The subset $\{x \in C(J,E) : x = \lambda N(x),\ \lambda \in [0,1)\}$ is bounded.
Let $\lambda \in [0,1)$ and $x \in C(J,E)$ be such that $x = \lambda N(x)$. If $\|x\|_{C(J,E)} = r$, then by (3.3)

$$r = \|x\|_{C(J,E)} \leq \|N(x)\|_{C(J,E)} \leq (1+r)\delta.$$

So

$$\|x\|_{C(J,E)} \leq \frac{\delta}{1-\delta}.$$

As a consequence of steps $1 \to 4$ and the Schaefer fixed point theorem, the function N has a fixed point. In view of (H_2) and Corollary 2.10, this fixed point is a solution for the problem (1.1). The proof is completed. □

In the following theorem, we give another existence result for (1.1).

Theorem 3.2. *Let $m \in \mathbb{N}$, $m \geq 2$ and $q \in (m-1, m)$. Let $f : J \times E \to E$ be a function such that (H_1), (H_3) and the following condition is satisfied.*

(H_4) *There exists $\varphi \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, $\sigma \in (0, m-(q-1))$ such that for any $x \in E$*

$$\|f(t,x)\| \leq \varphi(t) \quad \textit{for a.e. } t \in J.$$

Then, the problem (1.1) has a solution.

Proof. Let us consider the operator $N : C(J,E) \to C(J,E)$ defined as (3.2). Arguing as in the proof of Theorem 3.1, we can show that N is continuous and maps any bounded subset to relatively compact subset. It remains to show that the set $\{x \in C(J,E) : x = \lambda N(x),\ \lambda \in [0,1)\}$ is bounded.

Let $\lambda \in [0,1)$ and $x \in C(J,E)$ be such that $x = \lambda N(x)$. Then, by (H_4) and arguments of (3.3), we have

$$\|x\|_{C(J,E)} \leq \|N(x)\|_{C(J,E)} \leq \delta.$$

This proves that the subset $\{x \in C(J,E) : x = \lambda N(x), \lambda \in [0,1)\}$ is bounded. According to Schaefer's fixed point theorem, the function N has a fixed point. In view of (H_4) and Corollary 2.10, this fixed point is a solution for the problem (1.1). The proof is completed. □

Remark 3.3. The condition (H_3) is satisfied, if the dimension of E is finite.

4 Existence results for (1.2)

4.1 Convex case

At first, we consider the case when values of F are convex. In the following, we give the first existence result for (1.2).

Theorem 4.1. *Let $m \in \mathbb{N}$, $m \geq 2$ and $q \in (m-1, m)$. Let $F : J \times E \to P_{ck}(E)$ be a multifunction. Assume the following conditions.*

(H_5) *For every $x \in E$, $t \to F(t,x)$ is measurable and for a.e. $t \in J$, $x \to F(t,x)$ is upper semicontinuous.*

(H_6) *There exists a function $\varphi \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, $\sigma \in (0, q-(m-1))$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x \in E$*

$$\|F(t,x)\| \leq \varphi(t)\Omega(\|x\|) \quad \text{for a.e. } t \in J.$$

(H_7) *There exists a function $\beta \in L^{\frac{1}{\varsigma}}(J, \mathbb{R}^+)$, $\varsigma \in (0, q-(m-1))$ such that*

$$\begin{aligned} \ell := & \left[\frac{3T^{q-1}}{2\Gamma(q)} + \frac{T^{q-1}}{2} \sum_{j=1}^{m-2} \frac{1}{\Gamma(q-j)} \sum_{k=1}^{j+1} \frac{|\gamma_k^{(j+1)}|}{(k-1)!} \right] \|\beta\|_{L^1(J,\mathbb{R}^+)} \\ & + \frac{T^{q-\varsigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\varsigma}{1-\varsigma})^{1-\varsigma}} \sum_{k=1}^{m} \frac{|\gamma_k^{(m)}|}{(k-1)!} \|\beta\|_{L^{\frac{1}{\varsigma}}(J,\mathbb{R}^+)} < 1 \end{aligned} \tag{4.1}$$

and for every bounded subset $D \subseteq E$, $\chi(F(t,D)) \leq \beta(t)\chi(D)$ for a.e. $t \in J$, where χ is the Hausdorff measure of noncompactness in E.

(H_8) *There is a positive number r such that*

$$\delta\Omega(r) \leq r \tag{4.2}$$

for δ in (3.1).

Then the problem (1.2) has a solution.

Proof. In view of (H_5) and [27, Theorems 1.3.1 and 1.3.5] for every $x \in C(J,E)$, the multifunction $t \to F(t,x(t))$ has a measurable selection and by (H_6) this selection belongs to $S^1_{F(\cdot,x(\cdot))}$. So, we can define a multifunction $R : C(J,E) \to 2^{C(J,E)}$ as follows: For any $x \in C(J,E)$, a function $y \in R(x)$ if and only if

$$y = \mathcal{N}(f),$$

where $f \in S^1_{F(\cdot,x(\cdot))}$ and $\mathcal{N} : C(J,E) \to C(J,E)$ is defined as (3.2) with $z = f$. According to (H_6) and Corollary 2.10, any fixed point for R is a solutions for (1.2). So, our aim is to prove that the multivalued function R satisfies the assumptions of Lemma 2.14. Denote $D = B_r$. It is clear since the values of F are convex, the values of R are convex also.

Step 1. $R(D) \subseteq D$.
Let $x \in D$ and $y \in R(x)$. Then, by using the same arguments in Step 2 of the proof of Theorem 3.1, we can show that, for any $t \in J$

$$\|y(t)\|_{C(J,E)} \leq \delta\Omega(r). \tag{4.3}$$

This inequality with (4.2) imply $||y||_\infty \le r$.

Step 2. $R(D)$ is a equicontinuous set in $C(J,E)$.

Let $y \in R(D)$ and $t_1, t_2 \in J$, $(t_1 < t_2)$. Then there is $x \in D$ with $y \in R(x)$. By using the same arguments in Step 3 of the proof of Theorem 3.1 with (H_6), and recalling the definition of R, there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that

$$\begin{aligned}\|y(t_2)-y(t_1)\| &= \|\mathcal{N}(f)(t_2)-\mathcal{N}(f)(t_1)\|\\ &\le \frac{\Omega(r)}{\Gamma(q)}\left[\int_0^{t_1}\left[|(t_2-s)^{q-1}-(t_1-s)|^{q-1}\right]\varphi(s)ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}\varphi(s)ds\right]\\ &\quad+\frac{r+1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{|\gamma_k^{(j+1)}|T^{j+1-k}}{(k-1)!}I^{q-j}\varphi(T)(t_2^{k-1}-t_1^{k-1}),\quad t\in J.\end{aligned}$$

Then, the right hand side does not depend on x and uniformly tends to zero when $t_2 \to t_1$.

Step 3. The implication (2.22) holds with $x_0 = 0$.

Let $Z \subseteq D$, $Z = \text{conv}(\{x_0\} \cup R(Z))$, $\overline{Z} = \overline{C}$ with $C \subseteq Z$ countable. We claim that Z is relatively compact. Since C is countable and since $C \subseteq Z = \text{conv}(\{x_0\} \cup R(Z))$, we can find a countable set $H = \{y_n : n \ge 1\} \subseteq R(Z)$ with $C \subseteq \text{conv}(\{x_0\} \cup H)$. Then, for any $n \ge 1$, there exists $x_n \in Z$ with $y_n \in R(x_n)$. This means that there is $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that $y_n = \mathcal{N}(f_n)$. From $Z \subseteq \overline{C} \subseteq \overline{\text{conv}}\ (\{x_0\} \cup H)$ we find that $\chi(Z(t)) \le \chi(\overline{C}(t)) \le \chi(H(t))$, $t \in J$. Observe that, by (H_6), for every natural number n, $\|f_n(s)\| \le \varphi(s)\ \Omega(r)$ a.e. Then, by using (H_6) and the properties of the measure of noncompactness, one has for $t \in J$ (see [9,21,32])

$$\begin{aligned}\chi(Z(t)) &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\chi(\{f_n(s):n\ge 1\})ds\\ &\quad+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{|\gamma_k^{(j+1)}|T^j}{(k-1)!\Gamma(q-j)}\int_0^T(T-s)^{q-j-1}\chi(\{f_n(s):n\ge 1\})ds.\end{aligned} \tag{4.4}$$

Observe that, since $Z \subseteq \overline{C} \subseteq \overline{\text{conv}}\ (\{x_0\} \cup H)$, then from Step 2, Z is equicontinuous. Moreover, according to condition (H_7) for a.e. $t \in J$,

$$\chi(\{f_n(t):n\ge 1\}) \le \beta(t)\chi(\{x_n(t):n\ge 1\}) \le \beta(t)\chi(Z(t)) \le \beta(t)\chi_{C(J,E)}(Z)$$

So, we find from (4.4) that

$$\chi_{C(J,E)}(Z) = \max_{t\in J}\chi(Z(t)) \le \ell\chi_{C(J,E)}(Z).$$

This inequality with (4.1) imply that Z is relatively compact.

Step 4. R maps compact sets into relatively compact sets.

Let G be a compact subset of D. From Step 2, $R(G)$ is equicontinuous. Let $\{y_n\}_{n\in\mathbb{N}}$, be a sequence in $R(G)$. Then, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in G such that $y_n \in R(x_n)$. This means that

$$y_n = \mathcal{N}(f_n) \tag{4.5}$$

for $f_n \in S^1_{F(\cdot,x_n(\cdot))}$. Arguing as above, one obtains from (4.5), for $t \in J$

$$\begin{aligned}\chi(\{y_n(t):n\ge 1\}) &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\chi(\{f_n(s):n\ge 1\}ds)\\ &\quad+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1}\frac{|\gamma_k^{(j+1)}|T^j}{(k-1)!\Gamma(q-j)}\int_0^T(T-s)^{q-j-1}\chi(\{f_n(s):n\ge 1\})ds.\end{aligned} \tag{4.6}$$

Now, since G is compact in $C(J,E)$, then for $t \in J$, the subset $G(t)$ is relatively compact in E. Therefore, $\chi(\{x_n(t) : n \geq 1\}) = 0$, $t \in J$. Hence, by (H_7) we get for a.e. $t \in J$,

$$\chi(\{f_n(t) : n \geq 1\}) \leq \beta(t)\chi(\{x_n(t) : n \geq 1\}) = 0.$$

This with (4.6) imply

$$\chi(\{y_n(t) : n \geq 1\}) = 0, \quad \text{for all } t \in J.$$

This together with the equicontinuity of $\{y_n : n \geq 1\}$, we conclude, from the Ascoli–Arzelà theorem, the set $\{y_n : n \geq 1\}$ is relatively compact. Then, the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. This shows that $R(G)$ is relatively compact.

Step 5. The graph R is closed.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in D with $x_n \to x$ in D and let $y_n \in R(x_n)$ with $y_n \to y$ in $C(J,E)$. We have to show that $y \in R(x)$. Then, for any $n \geq 1$, there is $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$y_n = \mathcal{N}(f_n). \tag{4.7}$$

Observe that for every $n \geq 1$ and for a.e. $t \in J$

$$\|f_n(t)\| \leq \varphi(t)\Omega\left(\|x_n\|_{C(J,E)}\right) \leq \varphi(t)\Omega(r).$$

This show that the set $\{f_n : n \geq 1\}$ is integral bounded. In addition, the set $\{f_n(t) : n \geq 1\}$ is relatively compact for a.e. $t \in J$ because of assumption (H_7) both with the convergence of $\{x_n\}_{n\in\mathbb{N}}$ imply that

$$\chi(\{f_n\ (t) : n \geq 1\}) \leq \chi(F(t, \{x_n\ (t) : n \geq 1\})) \leq \beta(t)\chi(\{x_n\ (t) : n \geq 1\}) = 0.$$

Hence, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi compact, hence, it is weakly compact in $L^1(J,E)$ (see, [27, Proposition 4.2.1]). So, without loss of generality we can assume that f_n converges weakly to a function $f \in L^1(J,\mathbb{R}^+)$. From Mazur's lemma, for every natural number j there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = j,\ldots,k_0(j)$ such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k}\ f_k$, $j \geq 1$ converges strongly to f in $L^1(J,\mathbb{R}^+)$ as $j \to \infty$. Then we get for a.e. $t \in J$

$$f(t) \in \bigcap_{j\geq 1} \overline{\{z_k(t) : k \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{\text{conv}} \left\{ \bigcup_{k\ \geq j} F(t, x_k(t)) \right\}.$$

Since F is upper semicontinuous with convex and compact values, using [44, Lemma 2.6], we conclude that $f(t) \in F(t, x(t))$, for a.e. $t \in J$. Note that, by (H_6) for every $t \in J$, $s \in (0,t]$ and every $n \geq 1$

$$\left\|(t-s)^{\alpha-1} z_n(s)\right\| \leq |t-s|^{\alpha-1}\ \varphi(s)\Omega(r) \in L^1((0,t],\mathbb{R}^+), \qquad \alpha = q-(m-1),\ldots,q. \tag{4.8}$$

Next taking $\overline{y}_n = \sum_{k=n}^{k_0(n)} \lambda_{n,k}\ y_k$. Then by (4.7)

$$\overline{y}_n(t) = I^q z_n(t) - \frac{1}{2}\sum_{k=1}^{m}\sum_{j=k-1}^{m-1} \frac{\gamma_k^{(j+1)} T^{j+1-k}}{(k-1)!} I^{q-j} z_n(T) t^{k-1}, \qquad t \in J. \tag{4.9}$$

But $\overline{y}_n(t) \to y(t)$ and $z_n(t) \to f(t)$. Therefore, by passing to the limit as $n \to \infty$ in (4.9) we obtain, from the Lebesgue dominated convergence theorem (see (4.8)), that $y = \mathcal{N}(f)$, i.e., $y \in R(x)$.

As a result of the Steps $1 \to 5$ the multivalued R satisfies all assumptions of Lemma 2.14. Then, R has a fixed point which is a mild solution of (1.2). The proof is completed. □

Now we present our second result for the problem (1.2).

Theorem 4.2. *Let $m \in \mathbb{N}$, $m \geq 2$ and $q \in (m-1, m)$. Let $F : J \times E \to P_{ck}(E)$ be a multifunction. We suppose the following assumptions.*

(H_9) *For every $x \in E$, $t \longrightarrow F(t,x)$ is measurable.*

(H_{10}) *There is a function $\varsigma \in L^{\frac{1}{\sigma}}(J,E)$, $\sigma \in (0, q-(m-1))$ such that for every $x, y \in E$*

$$h\left(F(t,x), F(t,y)\right) \leq \varsigma(t)\|x-y\| \quad \text{for a.e. } t \in J,$$

where h is the Hausdorff metric, and

$$\sup\{\|x\| : x \in F(t,0)\} \leq \varsigma(t) \quad \text{for a.e. } t \in J.$$

Then the problem (1.2) has a solution provided that

$$\begin{aligned} c = &\left[\frac{3T^{q-1}}{2\Gamma(q)} + \frac{T^{q-1}}{2}\sum_{j=1}^{m-2}\frac{1}{\Gamma(q-j)}\sum_{k=1}^{j+1}\frac{|\gamma_k^{(j+1)}|}{(k-1)!}\right]\|\varsigma\|_{L^1(J,\mathbb{R}^+)} \\ &+ \frac{T^{q-\sigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\sum_{k=1}^{m}\frac{|\gamma_k^{(m)}|}{(k-1)!}\|\varsigma\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} < 1. \end{aligned} \tag{4.10}$$

Proof. According to (H_9), (H_{10}) and [27, Theorems 1.1.9, 1.3.1 and 1.3.5] for every $x \in C(J,E)$, the multifunction $t \to F(t, x(t))$ has an integrable selection. So, we can define a multifunction $R : C(J,E) \to 2^{C(J,E)}$ as in Theorem 4.1. Now, we show that R satisfies the assumptions of the theorem of Covitz and Nadler [13]. The proof will be given in two steps.

Step 1. The values of R are nonempty and closed.
Since $S^1_{F(\cdot,x(\cdot))}$ is nonempty, the values of R are nonempty. In order to prove the values of R are closed, let $x \in C(J,E)$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $R(x)$ such that $u_n \to u$ in $C(J,E)$. Then, according to the definition of R there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $S^1_{F(\cdot,x(\cdot))}$ such that $u_n = \mathcal{N}(f_n)$. Now, let $t \in J$ be a fixed. In view of (H_{10}), for every $n \geq 1$, and for a.e. $t \in J$

$$\begin{aligned} \|F(t,x)\| &= h\left(F(t,x(t)), \{0\}\right) \leq h\left(F(t,x(t)), F(t,0)\right) + h\left(F(t,0), \{0\}\right) \\ &\leq \varsigma(t)\|x(t)\| + \varsigma(t) \leq \varsigma(t)(1 + \|x\|_{C(J,E)}). \end{aligned} \tag{4.11}$$

Then, for every $n \geq 1$, and for a.e. $t \in J$, $\|f_n(t)\| \leq \varsigma(t)(1+\|x\|_{C(J,E)})$. This show that the set $\{f_n : n \geq 1\}$ is integrably bounded. Arguing as in Step 5 in the proof of Theorem 4.1, we can show that $u \in R(x)$.

Step 2. R is a contraction.
Let $z_1, z_2 \in C(J,E)$ and $y_1 \in R(z_1)$. Then, there is $f \in S^1_{F(\cdot,z_1(\cdot))}$ such that

$$y_1 = \mathcal{N}(f). \tag{4.12}$$

Consider a multifunction $Z : J \to 2^E$ defined by $Z(t) = \{u \in E : \|f(t) - u\| \leq \varsigma(t)\|z_1(t) - z_2(t)\|\}$. For each $t \in J$, $Z(t) \cap F(t, z_2(t))$ is nonempty. Indeed, let $t \in J$. From (H_{10}), we have $h\left(F(t,z_2(t)), F(t,z_1(t))\right) \leq \varsigma(t)\|z_1(t) - z_2(t)\|$. Hence, there exists $u_t \in F(t, z_2(t))$ such that

$$\|u_t - f(t)\| \leq \varsigma(t)\|z_1(t) - z_2(t)\|.$$

Moreover, since the functions $\varsigma, z_1, z_2 f$, are measurable ([10, Proposition III.4], [27, Corollary 1.3.1]) the multifunction $V : t \to Z(t) \cap F(t, z_1(t))$ is measurable. Then there is $h \in S^1_{F(\cdot, z_2(\cdot))}$ such that for a.e. $t \in J$

$$\|h(t) - f(t)\| \leq \varsigma(t)\|z_1(t) - z_2(t)\|. \tag{4.13}$$

Let us define

$$y_2 = \mathcal{N}(h). \tag{4.14}$$

Obviously $y_2 \in R(z_2)$. Furthermore, following the proof of Step 2 of Theorem 3.1, we get from (4.12)–(4.14) and Hölder's inequality

$$\|y_1(t) - y_2(t)\| \leq c\|z_1 - z_2\|_{C(J,E)}, \qquad t \in J,$$

by interchanging the role of y_2 and y_1 we obtain

$$h(R(z_1), R(z_2)) \leq c\|z_1 - z_2\|_{C(J,E)}.$$

Therefore, the multivalued function R is a contraction due to the condition (4.10).

Thus, by the theorem of Covitz and Nadler [13], R has a fixed point which is a solution for (1.2). The proof is finished. □

4.2 Nonconvex case

Now we give an existence result for (1.2) when the values of F are not necessarily convex. This result extend [2, Theorem 4.2] to infinite dimensional spaces.

Theorem 4.3. *Let $m \in \mathbb{N}$, $m \geq 2$ and $q \in (m-1, m)$. Assume that the condition (H_7) and the following condition are satisfied:*

(H_{11}) *$F : J \times E \to P_{cl}(E)$ is a multifunction such that*

- *(i) $F(t, x)$ has a measurable graph and for almost $t \in J$, $x \to F(t, x)$ is lower semicontinuous.*
- *(ii) There exists a function $p \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, $\sigma \in (0, q-(m-1))$ such that for any $x \in E$*

$$\|F(t, x)\| \leq p(t) \quad \text{for a.e. } t \in J.$$

Then, the problem (1.2) has a solution.

Proof. Consider the multivalued Nemitsky operator $\mathrm{N} : C(J, E) \to 2^{L^1(J,E)}$ defined by

$$\mathrm{N}(x) = S^1_{F(\cdot, x(\cdot))} = \{f \in L^1(J, E) : f(t) \in F(t, x(t))) \ \text{ for a.e. } t \in J\}.$$

We prove that N has a continuous selection. To achieve this aim, we show that N satisfies the assumptions of [8, Theorem 3]. That is, N is *l.s.c.* and possessing a nonempty closed decomposable value. Since F has closed values, S^1_F is closed. Because F is integrably bounded, S^1_F is nonempty (see [22, Theorem 3.2]). It is easy to see that, S^1_F is decomposable (see [22, Theorem 3.1]). To check the lower semicontinuity of N, we need to show that, for every $u \in L^1(J, E)$, $x \to d(u, \mathrm{N}(x))$ is upper semicontinuous. This is equivalent to show that for any $\lambda \geq 0$, the set

$$u_\lambda = \{x \in C(J, E) : d(u, \mathrm{N}(x)) \geq \lambda\}$$

is closed. For this purpose, let $\{x_n\}_{n\in\mathbb{N}} \subseteq u_\lambda$ and assume that $x_n \to x$ in $C(J,E)$. Then, for all $t \in J$, $x_n(t) \to x(t)$ in E. Note that from [22, Theorem 2.2] we have

$$\begin{aligned} d(u, \mathrm{N}(x_n)) &= \inf_{v\in \mathrm{N}(x_n)} \|u-v\|_{L^1(J,E)} = \inf_{v\in \mathrm{N}(x_n)} \int_0^T \|u(t)-v(t)\| dt \\ &= \int_0^T \inf_{v\in \mathrm{N}(x_n)} \|u(t)-v(t)\| dt = \int_0^T d\left(u(t), F(t,x_n(t))\right) dt. \end{aligned} \tag{4.15}$$

Therefore,

$$\lambda \le \lim_{n\to\infty} \sup d(u,\mathrm{N}(x_n)) = \lim_{n\to\infty} \sup \int_0^b d(u(t),F(t,x_n(t))dt. \tag{4.16}$$

Observe that, as a result of the Fatou lemma, we have

$$\lim_{n\to\infty} \sup \int_0^b d\left(u(t),F(t,x_n(t))\right) dt \le \int_0^b \lim_{n\to\infty} \sup d\left(u(t),F(t,x_n(t))\right) dt.$$

Thus (4.16) gives us

$$\lambda \le \int_0^b \lim_{n\to\infty} \sup d\left(u(t),F(t,x_n(t))\right) dt.$$

On the other hand, by virtue of (H_{11})(i), the function $z \to d(u(t),F(t,z))$ is *u.s.c.* Then the last inequality with (4.15) imply

$$\lambda \le \int_0^b d(u(t),F(t,x(t))dt = d(u,\mathrm{N}(x)).$$

This proves that u_λ is closed. By applying Theorem of Bressan–Colombo [8, Theorem 3], there is a continuous map $Z : C(J,E) \to L^1(J,E)$ such that $Z(x) \in \mathrm{N}(x)$, for every $x \in C(J,E)$. Then, $Z(x)(s) \in F(s,x(s))$ a.e. for $s \in J$. Now consider a map $\pi : C(J,E) \to C(J,E)$ defined by

$$\pi(x) = \mathcal{N}(Z(x)). \tag{4.17}$$

Our aim now is to prove that the function π satisfies the assumptions of Mönch's fixed point theorem. Arguing as in (3.3) we can show that, by using (H_{11})(ii), for any $x \in C(J,E)$, we have $\|\pi(x)\|_{C(J,E)} \le \Upsilon$ for

$$\begin{aligned} \Upsilon = \|p\|_{L^1(J,\mathbb{R}^+)} &\left[\frac{3T^{q-1}}{2\Gamma(q)} + \frac{T^{q-1}}{2}\sum_{j=1}^{m-2}\frac{1}{\Gamma(q-j)}\sum_{k=1}^{j+1}\frac{|\gamma_k^{(j+1)}|}{(k-1)!}\right] \\ &+ \frac{(1+r)T^{q-\sigma}}{2\Gamma(q-m+1)(\frac{q-m+1-\sigma}{1-\sigma})^{1-\sigma}}\sum_{k=1}^{m}\frac{|\gamma_k^{(m)}|}{(k-1)!}\|p\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}. \end{aligned}$$

Let $D = B_\Upsilon$. Then $\pi(D) \subseteq D$. Also, arguing as in the proof of Theorem 4.1, we can show that π is continuous and satisfies (2.22). So, as result of the Mönch fixed point theorem, there is $x \in C(J,E)$ such that $x = \pi(x)$. Since $p \in L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)$ and $0 < \sigma < q-(m-1)$, by Corollary 2.10, the function x is a solution for (1.2). The proof is completed. □

5 Applications to fractional lattice inclusions

In this section, we apply the above abstract results to a model of fractional differential inclusions on lattices with local neighborhood interactions represented by

$${}^cD^q_{0,t}x_n(t) \in f_n(t,x_{n-1}(t),x_n(t),x_{n+1}(t)), \qquad n \in \mathbb{Z}, \text{ a.e. on } J, \tag{5.1}$$

where $q \in (m-1, m)$, $m \in \mathbb{N}$, $m \geq 2$, $f_n : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ be specified latter and $x_n \in \mathbb{R}$ with $\lim_{n\to\pm\infty} x_n = 0$. We are looking for forced localized solutions of (5.1). We are motivated by [14, 15, 31, 37], where countable systems of ODEs are studied. Here we study countable systems of fractional differential inclusions.

Set $E = c_0 = \{x = \{x_n\}_{n\in\mathbb{Z}} : x_n \in \mathbb{R}, \lim_{n\to\pm\infty} x_n = 0\}$ with the norm $\|x\| = \sup_{n\in\mathbb{Z}} |x_n|$. Then E is a separable Banach space. The following two examples will illustrate the feasibility of our assumptions in the above theorems.

Example 5.1. Let $f_n : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ be multifunctions defined by

$$f_n(t, x_{n-1}, x_n, x_{n+1}) = a_1 \frac{\cos \pi t}{|n| + 1} + [-a_2 x_n, a_2 x_n] + a_3 x_{n-1} + a_4 x_{n+1}$$

for any $n \in \mathbb{Z}$ where a_i, $i = 1, 2, 3, 4$ are constants. Then we define $F : J \times E \to 2^E$ as $F(t, x) = \{f_n(t, x_{n-1}, x_n, x_{n+1})\}_{n\in\mathbb{Z}}$, and the condition (H_5) is easily verified. Note $T = 1$. Next, for $(t, x) \in J \times E$, we have

$$\|F(t, x)\| = \sup\{\|y\| : y \in F(t, x)\} \leq 3A(\|x\| + 1),$$

for $A = \max\{|a_i| : i = 1, 2, 3, 4\}$, so the condition (H_6) is satisfied with $\varphi(t) = 3A$ and $\Omega(z) = z + 1$ for $t \in J$ and $z \in \mathbb{R}^+$. Now let $t \in J$, $x, y \in E$ and $u \in F(t, x)$. Then $u_n = a_1 \frac{\cos 2\pi t}{|n|+1} + \lambda_n x_n + a_3 x_{n-1} + a_4 x_{n+1}$, $|\lambda_n| \leq |a_2|$, $n \in \mathbb{Z}$. Taking $v = \{a_1 \frac{\cos 2\pi t}{|n|+1} + \lambda_n y_n + a_3 y_{n-1} + a_4 y_{n+1}\}_{n\in\mathbb{Z}} \in F(t, y)$, we get

$$d(u, F(t, y)) \leq \|u - v\| \leq 3A\|x - y\|.$$

This yields that

$$\sup_{u\in F(t,x)} d(u, F(t, y)) \leq 3A\|x - y\|.$$

Similarly, we can show that

$$\sup_{v\in F(t,y)} d(v, F(t, x)) \leq 3A\|x - y\|.$$

Therefore,

$$h(F(t, x), F(t, y)) \leq 3A\|x - y\|,$$

where h denotes the standard Hausdorff distance. So by [27, Corollary 2.2.1, p. 47], for every bounded subset $D \subseteq E$, $\chi(F(t, D)) \leq \beta(t)\chi(D)$, for a.e. $t \in J$, with $\beta(t) = 3A$ for $t \in J$. By assuming

$$\left[\frac{3}{\Gamma(q)} + \sum_{j=1}^{m-2} \frac{1}{\Gamma(q-j)} \sum_{k=1}^{j+1} \frac{|\gamma_k^{(j+1)}|}{(k-1)!}\right] A + \frac{1}{\Gamma(q-m+1)(\frac{q-m+1-\varsigma}{1-\varsigma})^{1-\varsigma}} \sum_{k=1}^{m} \frac{|\gamma_k^{(m)}|}{(k-1)!} A < \frac{2}{3} \tag{5.2}$$

for some $\varsigma \in (0, q - (m-1))$, then (H_7) and (H_8) are satisfied, since also $\delta < 1$. Finally, applying Theorem 4.1, the problem (5.1) has a 1-antiperiodic solution.

Example 5.2. Let $f_n : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ be multifunctions defined as:

$$f_n(t, x_{n-1}, x_n, x_{n+1}) = [-a_2 x_n, a_2 x_n] + a_3 x_{n-1} + a_4 x_{n+1}$$

when $t \in [0, \frac{1}{4})$, and $f_n(t, x_{n-1}, x_n, x_{n+1}) = \big[-a_1 \frac{\cos \pi t}{|n|+1}, a_1 \frac{\cos \pi t}{|n|+1}\big]$ when $t \in [\frac{1}{4}, 1]$ for any $n \in \mathbb{Z}$ and a_i, $i = 1, 2, 3, 4$ are constants. Then we define $F : J \times E \to 2^E$ as $F(t, x) = \{f_n(t, x_n)\}_{n\in\mathbb{Z}}$, and the condition (H_9) is easily verified. Note $T = 1$. Clearly $\|F(t,0)\| = 0$ for $t \in [0, \frac{1}{4})$ and $\|F(t,0)\| \leq |a_1|$ for $t \in [\frac{1}{4}, 1]$. By arguing as in the previous example we can show that $h(F(t,x), F(t,y)) \leq 3\max\{|a_2|, |a_3|, |a_4|\}\|x-y\|$. Hence the condition (H_{10}) is satisfied with $\varsigma(t) = 3A$. By assuming (5.2), then (4.10) is satisfied. Finally, applying Theorem 4.2 the problem (5.1) has a 1-antiperiodic solution.

Acknowledgements

The authors are grateful to the referee for careful reading of the manuscript and valuable comments.

The first author's work was partially supported by Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Unite Foundation of Guizhou Province ([2015]7640), Outstanding Scientific and Technological Innovation Talent Award of Education Department of Guizhou Province ([2014]240) and Graduate Course of Guizhou University (ZDKC[2015]003). The third author's work was partially supported by Grants VEGA-MS 1/0071/14, VEGA-SAV 2/0153/16 and by the Slovak Research and Development Agency under the contract No. APVV-14-0378.

References

[1] R. P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62**(2011), 1200–1214. MR2824708; url

[2] R. P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109**(2010), 973–1033. MR2596185; url

[3] B. Ahmad, J. J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray–Schauder degree theory, *Topol. Methods Nonlinear Anal.* **35**(2010), 295–304. MR2676818

[4] B. Ahmad, Existence of solutions for fractional differential equations of order $q \in (2,3]$ with anti-periodic boundary conditions, *J. Appl. Math. Comput.* **34**(2010), 385–391. MR2718793; url

[5] A. Alsaedi, B. Ahmad, A. Assolami, On anti-periodic boundary value problems for higher-order fractional differential equations, *Abstr. Appl. Anal.* **2012**, Art. ID 325984, 15 pp. MR2959741

[6] J. P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 2009. MR2458436; url

[7] M. Benchohra, J. Henderson, D. Seba, Boundary value problems for fractional differential inclusions in Banach spaces, *Fract. Differ. Calc.* **2**(2012), 99–108. MR3003005; url

[8] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90**(1988), 69–86. MR947921

[9] D. Bothe, Multivalued perturbation of *m*-accerative differential inclusions, *Israel J. Math.* **108**(1998), 109–138. MR1669396; url

[10] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Springer-Verlag, Berlin, 1977. MR0467310

[11] A. Cernea, On the existence of solutions for fractional differential inclusions with anti-periodic boundary conditions, *J. Appl. Math. Comput.* **38**(2012), 133–143. MR2886671; url

[12] Y. Chen, J. J. Nieto, D. O'Regan, Antiperiodic solutions for fully nonlinear first-order differential equations, *Math. Comput. Modelling* **46**(2007), 1183–1190. MR2376702; url

[13] H. Covitz, S. B. Nadler, Multi-valued contraction mapping in generalized metric spaces, *Israel J. Math.* **8**(1970), 5–11. MR0263062

[14] J. L. Daleckij, M. G. Krein, *Stability of solutions of differential equations in Banach space*, American Mathematical Society, Providence, R.I., 1974. MR0352639

[15] K. Deimling, *Ordinary differential equations in Banach spaces*, Springer-Verlag, Berlin–New York, 1977. MR0463601

[16] A. Granas, J. Dugundji, *Fixed point theory*, Springer, New York, 2003. MR1987179; url

[17] A. M. A. El-Sayed, A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comput.* **68**(1995), 15–25. MR1320806; url

[18] W. G. Glöckle, T. F. Nonnemacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68**(1995), 46–53. url

[19] A. M. Gomaa, On the solutions sets of three-point boundary value problems for nonconvex differential inclusions, *J. Egyptian Math. Soc.* **12**(2004), 97–107. MR2101529

[20] A. M. Gomaa, On the solutions sets of four-point boundary value problems for nonconvex differential inclusions, *Int. J. Geom. Methods Mod. Phys.* **8**(2011), 23–37. MR2782872; url

[21] H. P. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Anal.* **7**(1983), 1351–1371. MR726478; url

[22] F. Hiai, H. Umegaki, Integrals, conditional expectation, and martingales of multivalued functions, *J. Multivariate Anal.* **7**(1977), 149–182. MR0507504

[23] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. MR1890104; url

[24] S. Hu, N. S. Papageorgiou, *Handbook of multivalued analysis, Vol. I: Theory*, Kluwer Academic Publishers, Dordrecht, 1997. MR1485775

[25] A. G. Ibrahim, A. M. Gomaa, Existence theorems for functional multivalued three-point boundary value problem of second order, *J. Egyptian Math. Soc.* **8**(2000), 155–168. MR1798938

[26] A. G. Ibrahim, Fractional differential inclusions with anti-periodic boundary conditions in Banach spaces, *Electron. J. Qual. Theory Differ. Equ.* **65**(2014), 1–32. MR3304191

[27] M. I. Kamenskii, V. V. Obukhowskii, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, Walter de Gruyter & Co., Berlin, 2001. MR1831201; url

[28] E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, *Nonlinear Anal. Real World Appl.* **13**(2012), 1489–1497. MR2863974; url

[29] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, 2006. MR2218073

[30] Q. Lan, W. Lin, Positive solutions of systems of Caputo fractional differential equations, *Commun. Appl. Anal.* **17**(2013), 61–86. MR3075769

[31] R. S. MacKay, S. Aubry, Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators, *Nonlinearity* **7**(1994), 1623–1643. MR1304442

[32] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4**(1980), 985–999. MR586861; url

[33] S. K. Ntouyas, B. Ahmad, Existence results for boundary value problem for fractional differential inclusions with three point integral boundary conditions, *ENOC-2011, 24–29 July 2011, Rome, Italy.*

[34] A. Ouahab, Some results for fractional boundary value problem of differential inclusions, *Nonlinear Anal.* **69**(2008), 3877–3896. MR2463341; url

[35] D. O'Regan, R. Precup, Fixed point theorems for set-valued maps and existence principles for integral inclusions, *J. Math. Anal. Appl.* **245**(2000), 594–612. MR1758558; url

[36] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993. MR1347689

[37] A. M. Samoilenko, Yu. V. Teplinskii, *Countable systems of differential equations*, VSP, Utrecht, 2003. MR2225426; url

[38] J. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Anal. Real World Appl.* **12**(2011), 262–272. MR2728679; url

[39] J. Wang, Y. Zhou, Analysis of nonlinear fractional control systems in Banach spaces, *Nonlinear Anal.* **74**(2011), 5929–5942. MR2833364; url

[40] J. Wang, M. Fečkan, Y. Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dyn. Partial Differ. Equ.* **8**(2011), 345–361. MR2901608; url

[41] J. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. Real World Appl.* **12**(2011), 3642–3653. MR2832998; url

[42] J. Wang, A. G. Ibrahim, Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces, *J. Funct. Space Appl.* **2013**, Art. ID 518306, 1–16. MR3125085

[43] J. Wang, Y. Zhang, On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives, *Appl. Math. Lett.* **39**(2015), 85–90. MR3269223; url

[44] J. Wang, A. G. Ibrahim, M. Fečkan, Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces, *Appl. Math. Comput.* **257**(2015), 103–118. MR3320652; url

[45] J. Wang, A. G. Ibrahim, M. Fečkan, Nonlocal cauchy problems for semilinear differential inclusions with fractional order in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.* **27**(2015), 281–293. MR3341562; url