On a class of invariant algebraic curves for Kukles systems

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Abstract. In this paper we give a new upper bound for the degree of a class of transversal to infinity invariant algebraic curves for polynomial Kukles systems of arbitrary degree. Moreover, we prove that a quadratic Kukles system having at least one transversal to infinity invariant algebraic curve is integrable.

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1   Introduction

Darboux in 1878 published his seminal works [4] and [5], where he showed that a planar polynomial differential system with a sufficient number of invariant algebraic curves has a first integral. Since that time, the research and computation of invariant curves in planar polynomial vector fields has been intensive. See [2,3,6–8] and references there in. However, to determine whether a concrete planar polynomial system has invariant algebraic curves or not, as well as the properties of such curves: degree, connected components, etc. can be extremely difficult problems.

In this work, we consider real Kukles systems of the form

$$\dot{x} = -y, \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where $Q(x,y)$ is a real polynomial of degree at least two and without y as a divisor.

Our main result is the following.

Theorem 1.1. *Let $\{F = 0\}$ be a transversal to infinity invariant algebraic curve of degree n of the Kukles system* (1.1) *of degree $d \geq 3$. Supposes that x is not a divisor of the higher degree homogeneous part of F. Then*

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1. if $d = 3$, then $n \leq 2$;

2. if $d \geq 4$, then $n \leq d - 2$.

Theorem 1.1 is an improvement of the following result: if a Kukles system of degree $d \geq 2$ admits a transverse to infinity invariant algebraic curve, then the degree of the curve is at most d. This assertion was showed in [1] and we will give an elementary proof in Section 3.

We also study the roll that transverse to infinity algebraic curves plays in the integrability of Kukles systems. We prove the following result.

Theorem 1.2. *Any quadratic Kukles system supporting one transversal to infinity invariant algebraic curve is integrable.*

The paper is organized as follows. In Section 2 we recall some basic definitions. Theorem 1.1 will be proved in Section 3. Finally, Section 4 is devoted to quadratic Kukles systems and we will prove Theorem 1.2.

2   Preliminaries

As usual we denote by $\mathbb{R}[x,y]$ the ring of the polynomials in the real variables x and y with real coefficients. We recall that an *algebraic curve of degree n* is the zero-locus

$$\{F = 0\} := \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = 0\}$$

of a polynomial $F \in \mathbb{R}[x,y]$ of degree n, and that a *polynomial vector field $\mathcal{X}$ of degree d* in $\mathbb{R}^2$ is an expression of the form

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{2.1}$$

where $P, Q \in \mathbb{R}[x,y]$, and $d = \max\{\deg(P), \deg(Q)\}$. Thus, each polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{2.2}$$

has associated a polynomial vector field $\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$.

An *invariant algebraic curve* of system (2.2) or equivalently of the vector field (2.1) is an algebraic curve $\{F = 0\}$ such that the polynomial F satisfies the linear partial differential equation

$$PF_x + QF_y = KF, \tag{2.3}$$

for some $K \in \mathbb{R}[x,y]$. Here F_x and F_y denote the partial derivatives of F respect to x and y, respectively.

The left-hand side of (2.3) is the scalar product between $\mathcal{X}$ and the gradient of F. As the gradient of F is orthogonal to $\{F = 0\}$ and the right-hand side of (2.3) vanishes on $\{F = 0\}$, then $\mathcal{X}$ is tangent to $\{F = 0\}$, this fact implies that $\{F = 0\}$ is invariant by the flow of $\mathcal{X}$. This property justifies that $\{F = 0\}$ be called an invariant curve.

The polynomial K is called the *cofactor* of $\{F = 0\}$. From (2.3) we have that if $\mathcal{X}$ has degree d, then each invariant algebraic curve has cofactor of degree at most $d - 1$.

An important tool that connects integrability and limit cycles of a vector field is the inverse integrating factor. Recall that a function $V : U \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be an *inverse integrating*

factor of system (2.2) or equivalently of $\mathcal{X}$ if it is of class $C^1(U)$, it is not locally null, and it satisfies the following partial differential equation:

$$\mathcal{X}\,V = PV_x + QV_y = V \operatorname{div} \mathcal{X}, \tag{2.4}$$

where $\operatorname{div} \mathcal{X} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is the divergence of the vector field $\mathcal{X}$.

The name "inverse integrating factor" for the function V comes from the fact that its reciprocal $1/V$ is an integrating factor for system (1.1) on $U \setminus V^{-1}(0)$.

2.1 Homogeneous decomposition of invariant algebraic curves

Recall that each polynomial $F \in \mathbb{R}[x, y]$ of degree n can be written as

$$F = \sum_{i=0}^{n} F_i$$

where F_i means the homogeneous part of degree i of F. By writing

$$P = \sum_{i=0}^{d} P_i, \qquad Q = \sum_{i=0}^{d} Q_i, \quad \text{and} \quad K = \sum_{i=0}^{d-1} K_i,$$

the equation (2.3) becomes

$$\sum_{l=0}^{d+n-1} \left(\sum_{\substack{i+j=l \\ i \le d,\, j \le n-1}} P_i(F_{j+1})_x + Q_i(F_{j+1})_y \right) = \sum_{l=0}^{d+n-1} \left(\sum_{\substack{i+j=l \\ i \le d-1,\, j \le n}} K_i F_j \right). \tag{2.5}$$

This equation (2.5) for the case $l = d + n - 1$ is

$$P_d(F_n)_x + Q_d(F_n)_y = K_{d-1}F_n, \tag{2.6}$$

which implies the following result.

Lemma 2.1. *If* (2.6) *does not have a solution* (K_{d-1}, F_n), *then $\mathcal{X}$ does not have invariant algebraic curves.*

An algebraic curve $\{F = 0\}$ is called transversal to infinity if F_n factors as a product of n pairwise different linear forms. An algebraic curve $\{F = 0\}$ is said to be non-singular if there is not any point (x_0, y_0) such that $F(x_0, y_0) = F_x(x_0, y_0) = F_y(x_0, y_0) = 0$.

Lemma 2.2. *Let H be a transversal to infinity homogeneous polynomial of degree n. Then we have either $\gcd(H, H_y) = 1$ or $\gcd(H, H_y) = x$.*

Proof. Let $ax + by$ be a linear factor of H such that $ax + by \mid H_y$. From the Euler identity $xH_x + yH_y = nH$ it follows that $ax + by \mid xH_x$. Since $\gcd(H_x, H_y) = 1$ because H is transversal to infinity, $ax + by \mid x$. Hence, $b = 0$ and as H is transversal to infinity x have multiplicity one. Therefore, $\gcd(H, H_y) = x$ otherwise $\gcd(H, H_y) = 1$. $\qquad\square$

3   Some general results of Kukles systems

A simple computation shows that equation (2.5) for a Kukles system (1.1) of degree $d \geq 2$ can be written as

$$C_0 + \sum_{l=1}^{n} (C_l - y(F_l)_x) + \sum_{l=n+1}^{d+n-1} C_l = \sum_{l=0}^{d+n-1} D_l, \tag{3.1}$$

where

$$C_l = \sum_{\substack{i+j=l \\ i \leq d, j \leq n-1}} Q_i(F_{j+1})_y \quad \text{and} \quad D_l = \sum_{\substack{i+j=l \\ i \leq d-1, j \leq n}} K_i F_j.$$

In particular, we have $C_l = D_l$ for $l = d + n - 1, d + n - 2, \ldots, n + 1$, that is,

$$Q_d(F_n)_y = K_{d-1}F_n$$
$$Q_{d-1}(F_n)_y + Q_d(F_{n-1})_y = K_{d-1}F_{n-1} + K_{d-2}F_n$$
$$\vdots \tag{3.2}$$
$$Q_2(F_n)_y + \cdots + Q_d(F_{n-d+2})_y = K_{d-1}F_{n-d+2} + \cdots + K_1 F_n.$$

Our first result on the kind of invariant algebraic curves that a Kukles system can support is the following.

Lemma 3.1. *Let F be a transversal to infinity invariant algebraic curve of degree two of the Kukles system (1.1) of degree $d = 2$, then $x \nmid F_2$, i.e., $\gcd(F_2, (F_2)_y) = 1$.*

Proof. Suppose that $F = xR_1 + F_1 + F_0$ is a transversal to infinity invariant curve of the quadratic Kukles system with cofactor K of degree at most one. Direct computations on the corresponding equation (2.3) yields $R_1 \equiv 0$, contradiction.   □

By following similar ideas we can prove the next result which is not new in this paper since it follows from [1, Proposition 6], but we will give an elementary proof by using (3.2).

Proposition 3.2. *If $\{F = 0\}$ is a transversal to infinity invariant algebraic curve of degree n of the Kukles system (1.1) of degree $d \geq 2$, then $n \leq d$.*

Proof. By using the Euler identity $x(F_n)_x + y(F_n)_y = nF_n$ we can write the first identity in (3.2) as

$$nQ_d(F_n)_y = K_{d-1}(x(F_n)_x + y(F_n)_y)$$

or equivalently

$$(nQ_d - yK_{d-1})(F_n)_y = (xK_{d-1})(F_n)_x.$$

Since F is transversal to infinity, $(F_n)_x$ and $(F_n)_y$ does not have factors in common. That implies that there exists a polynomial S such that $xK_{d-1} = S(F_n)_y$ and $nQ_d - yK_{d-1} = S(F_n)_x$. Moreover, we note that S is a non zero polynomial. Thus, $n - 1 = \max\{\deg(F_n)_x, \deg(F_n)_y\} \leq \deg Q_d = d$. Hence $n \leq d + 1$.

From Lemma 2.2 we have two cases: $\gcd(F_n, (F_n)_y) = 1$ or $\gcd(F_n, (F_n)_y) = x$. In the first case the first equation in (3.2) holds if and only if there exists a polynomial S such that $K_{d-1} = S(F_n)_y$ and $Q_d = SF_n$. Hence, $n = \deg F = \deg F_n \leq \deg Q_d = d$.

To complete the proof we will prove that if $\gcd(F_n, (F_n)_y) = x$ and $n = d + 1$, then $\{F = 0\}$ is not an invariant algebraic curve of (1.1).

Suppose $F = F_0 + F_1 + \cdots + F_{d+1}$, $F_{d+1} = xR_d$, with R_d a homogeneous polynomial of degree d which is transversal to infinity. Moreover, as F_{d+1} is transversal to infinity it follows from Lemma 2.2 that $\gcd(R_d, (R_d)_y) = 1$. From the first identity in (3.2) we get

$$Q_d(R_d)_y = K_{d-1}R_d.$$

Hence, there exists a nonzero constant c_1 such that $Q_d = c_1 R_d$ and $K_{d-1} = c_1(R_d)_y$. Thus, $K_{d-1} = (Q_d)_y$. From the second identity in (3.2) and using the previous expressions we obtain

$$Q_{d-1}(x(R_d)_y) + c_1 R_d(F_d)_y = c_1(R_d)_y F_d + K_{d-2}xR_d,$$

which can be written as

$$(xQ_{d-1} - c_1 F_d)(R_d)_y = (xK_{d-2} - c_1(F_d)_y)R_d.$$

Since $\gcd(R_d, (R_d)_y) = 1$, there exists a constant c_2 such that

$$xQ_{d-1} - c_1 F_d = c_2 R_d \quad \text{and} \quad xK_{d-2} - c_1(F_d)_y = c_2(R_d)_y.$$

Thus, as $R_d = Q_d/c_1$ we get

$$F_d = -\frac{c_2}{c_1^2}Q_d + \frac{x}{c_1}Q_{d-1}, \quad (F_d)_y = -\frac{c_2}{c_1^2}(Q_d)_y + \frac{x}{c_1}K_{d-2}, \quad \text{and} \quad K_{d-2} = (Q_{d-1})_y.$$

By using these last expressions of F_d, $(F_d)_y$, and K_{d-2}, as well as, the expressions of F_{d+1}, $(F_{d+1})_y$, K_{d-1}, and Q_d in the third identity of (3.2), and reordering terms we obtain

$$\left(xQ_{d-2} - \frac{c_2}{c_1}Q_{d-1} - c_1 F_{d-1}\right)(R_d)_y = \left(xK_{d-3} - \frac{c_2}{c_1}(Q_{d-1})_y - c_1(F_{d-1})_y\right)R_d.$$

Again, since $\gcd(R_d, (R_d)_y) = 1$, the previous equation holds if and only if

$$F_{d-1} = -\frac{c_2}{c_1^2}Q_{d-1} + \frac{x}{c_1}Q_{d-2}, \quad (F_{d-1})_y = -\frac{c_2}{c_1^2}(Q_{d-1})_y + \frac{x}{c_1}K_{d-3}, \quad \text{and} \quad K_{d-3} = (Q_{d-2})_y.$$

By applying the same idea in the subsequent identities of (3.2) is easy to see that

$$F_i = -\frac{c_2}{c_1^2}Q_i + \frac{x}{c_1}Q_{i-1}, \quad (F_i)_y = -\frac{c_2}{c_1^2}(Q_i)_y + \frac{x}{c_1}K_{i-2}, \quad \text{and} \quad K_{i-2} = (Q_{i-1})_y$$

for $i = 3, 4, \ldots, d$.

By replacing all these expressions in the case $l = d + 1$ of (3.1) we get, after a straightforward computation, the following

$$y(F_{d+1})_x + \left(xQ_1 - \frac{c_2}{c_1}Q_2 - c_1 F_2\right)(R_d)_y = \left(xK_0 - \frac{c_2}{c_1}(Q_2)_y - c_1(F_2)_y\right)R_d.$$

Since $F_{d+1} = xR_d$ and $x(R_d)_x + y(R_d)_y = dR_d$ is homogeneous, $(F_{d+1})_x = (d+1)R_d - y(R_d)_y$. By replacing this in previous equation we obtain

$$\left(xQ_1 - \frac{c_2}{c_1}Q_2 - c_1 F_2 - y^2\right)(R_d)_y = \left(xK_0 - \frac{c_2}{c_1}(Q_2)_y - c_1(F_2)_y - (d+1)y\right)R_d.$$

Hence, since $\gcd(R_d, (R_d)_y) = 1$, there exists a constant c_3 such that

$$F_2 = -\frac{c_3}{c_1}R_d - \frac{c_2}{c_1^2}Q_2 + \frac{x}{c_1}Q_1 - \frac{1}{c_1}y^2$$

and

$$(F_2)_y = -\frac{c_3}{c_1}(R_d)_y - \frac{c_2}{c_1^2}(Q_2)_y + \frac{x}{c_1}K_0 - \frac{d+1}{c_1}y^2,$$

which is a contradiction because these two expressions are incompatible for $d \geq 2$. We notice that for the case $d = 2$ the constant c_2 does not exists because in such a case (3.2) has only one identity, and hence we can take $c_2 = 0$ in the two previous equations. □

Remark 3.3. There are differential systems of degree d having transversal to infinity invariant algebraic curves of degree $d + 1$. For instance, the system

$$\dot{x} = 2x^2 - 3xy - x - 2, \qquad \dot{y} = -2xy + y^2 - 2x + y,$$

has the invariant algebraic curve $\{V(x,y) = -x^2y + xy^2 - x^2 + xy + y + 1 = 0\}$, which is transversal to infinity, with cofactor $R = 2x - y$.

3.1   Characterization of the Kukles systems

Next result gives some conditions on the Kukles systems that have invariant algebraic curves which are transversal to infinity, as well as, on the cofactors of the invariant curves.

Proposition 3.4. *If F is a transversal to infinity invariant algebraic curve of degree $2 \leq n \leq d$ for the Kukles system (1.1) of degree d such that $x \nmid F_n$, then there are $d - 1$ homogeneous polynomials S_0 and $S_1, \ldots, S_{d-2}$ of degrees $d - n$ and at most $d - n - 1, \ldots, -n + 2$, respectively, such that for each $s \in \{0, \ldots, d - 2\}$ we have*

$$Q_{d-s} = \sum_{i=0}^{s} S_i F_{n-s+i} \quad and \quad K_{d-1-s} = \sum_{i=0}^{s} S_i \left(F_{n-s+i}\right)_y.$$

Proof. By assumption F is transversal to infinity and $x \nmid F_n$, then Lemma 2.2 implies that F_n and $(F_n)_y$ do not have common factors. Hence, from the first identity in (3.2) it follows that there is a homogeneous polynomial S_0 of degree $d - n$ such that

$$Q_d = S_0 F_n \quad and \quad K_{d-1} = S_0(F_n)_y. \tag{3.3}$$

Replacing in second identity in (3.2) we obtain

$$Q_{d-1}(F_n)_y + S_0 F_n (F_{n-1})_y = S_0(F_n)_y F_{n-1} + K_{d-2} F_n$$

which is equivalent to

$$(Q_{d-1} - S_0 F_{n-1}) (F_n)_y = (K_{d-2} - S_0(F_{n-1})_y) F_n.$$

Again as F_n and $(F_n)_y$ do not have common factors there exists a homogeneous polynomial S_1 of degree at most $d - n - 1$ such that

$$Q_{d-1} - S_0 F_{n-1} = S_1 F_n \quad and \quad K_{d-2} - S_0(F_{n-1})_y = S_1(F_n)_y.$$

Thus,

$$Q_{d-1} = S_0 F_{n-1} + S_1 F_n \quad and \quad K_{d-2} = S_0(F_{n-1})_y + S_1(F_n)_y.$$

The rest of the proof follows by applying the previous idea in the subsequent equations in (3.2). □

Corollary 3.5. *If Kukles system* (1.1) *is of degree d and admits a transversal to infinity invariant algebraic curve of degree d, then Q is transversal to infinity.*

Proof. Follows from (3.3) because S_0 must be a non-zero constant. ☐

Remark 3.6. In Proposition 3.4 only we can guarantee that the polynomial S_0 is non zero. The rest of the polynomials S_i maybe some or all of them could be zero.

3.2 Proof of Theorem 1.1

Proof of Theorem 1.1. Suppose that $\{F = 0\}$ is a transversal to infinity invariant algebraic curves of degree n of the Kukles system (1.1) of degree d such that $x \nmid F_n$. The proof will be split into two cases: $d \geq 3$ and $d \geq 4$. For these two cases, we will prove that the Kukles system does not support transversal to infinity invariant algebraic curves of degree $n = d$ and $n = d - 1$, respectively. The proof follows from these two assertions and Proposition 3.2.

Case $d \geq 3$. Suppose $n = d$. From Proposition 3.4 it follows that $S_2 = \cdots = S_{n-1} = 0$ and S_1 is a constant. Thus,

$$Q_d = S_1 F_n, \ Q_{d-1} = S_1 F_{n-1}, \ \dots, \ Q_2 = S_1 F_2$$

and

$$K_{d-1} = S_1(F_n)_y, \ K_{d-2} = S_1(F_{n-1})_y, \ \dots, \ K_1 = S_1(F_2)_y.$$

By replacing these expressions in the case $l = n = d$ of (3.1) and reordering terms we obtain

$$(Q_1 - S_1 F_1)(F_d)_y - y(F_d)_x = (K_0 - S_1(F_1)_y)F_d.$$

On the other hand, since F_d is homogeneous we have

$$y(F_d)_y + x(F_d)_x = d\, F_d$$

Hence, from these two last equations we get

$$\left(d(Q_1 - S_1 F_1) - (K_0 - S_1(F_1)_y)y\right)(F_d)_y = \left(d\, y + (K_0 - S_1(F_1)_y)x\right)(F_d)_x$$

Since $(F_d)_x$ and $(F_d)_y$ do not have common factors (F is transversal to infinity), there is a polynomial T such that

$$d(Q_1 - S_1 F_1) - (K_0 - S_1(F_1)_y)y = T(F_d)_x$$

and

$$d\, y + (K_0 - S_1(F_1)_y)x = T(F_d)_y,$$

which is a contradiction because $d\, y + (K_0 - S_1(F_1)_y)x$ is a linear polynomial and $(F_d)_y$ is of degree at least two. Therefore, we have proved that $n \leq d - 1$.

Case $d \geq 4$. Suppose $n = d - 1$. From Proposition 3.4 it follows that $S_3 = \cdots = S_{n-1} = 0$, S_1 is a polynomial of degree 1 and S_2 is a constant. Thus,

$$Q_d = S_1 F_n, \ Q_{d-1} = S_1 F_{n-1} + S_2 F_n, \ \dots, \ Q_3 = S_1 F_2 + S_2 F_3 \tag{3.4}$$

and

$$K_{d-1} = S_1(F_n)_y, \ K_{d-2} = S_1(F_{n-1})_y + S_2(F_n)_y, \ \dots, \ K_2 = S_1(F_2)_y + S_2(F_3)_y. \tag{3.5}$$

By replacing these expressions in the case $l = d = n + 1$ of (3.1), and following same ideas as in previous case we get

$$[Q_2 - S_1 F_1 - S_2 F_2](F_{d-1})_y = [K_1 - S_1 (F_1)_y - S_2 (F_2)_y] F_{d-1}.$$

$(F_{d-1})_y$ and (F_{d-1}) do not have common factors because F is transversal to infinity. Moreover, from the hypothesis, $d - 1 \geq 3$. Thus, the previous equation holds if and only if

$$Q_2 = S_1 F_1 + S_2 F_2 \quad \text{and} \quad K_1 = S_1 (F_1)_y + S_2 (F_2)_y. \tag{3.6}$$

By replacing (3.4), (3.5), and (3.6) in the case $l = n = d - 1$ of (3.1), and reducing and reordering terms we have

$$[Q_1 - S_1 F_0 - S_2 F_1](F_{d-1})_y - y(F_{d-1})_x = [K_0 - S_2 (F_1)_y] F_{d-1}.$$

Since F_{d-1} is homogeneous,

$$y(F_{d-1})_y + x(F_{d-1})_x = (d - 1) F_{d-1}.$$

Hence, from these two last equations we get

$$\begin{aligned}
[(d - 1)(Q_1 - S_1 F_0 - S_2 F_1) - y (K_0 - S_2 (F_1)_y)] (F_{d-1})_y \\
= [(d - 1)y + x (K_0 - S_2 (F_1)_y)] (F_{d-1})_x.
\end{aligned}$$

$(F_{d-1})_x$ and $(F_{d-1})_y$ are polynomials of degree at least two and do not have common factors. Hence, by comparing degrees, previous equation is impossible. This proves that $n \leq d - 2$. $\quad \square$

4   Kukles systems of degree two

In this section we will consider the Kukles systems of degree two

$$\dot{x} = -y, \qquad \dot{y} = Q(x, y), \tag{4.1}$$

where $Q = Q(x, y) = q_{00} + q_{10}x + q_{01}y + q_{20}x^2 + q_{11}yx + q_{02}y^2$ and $y \nmid Q$.

Lemma 4.1. *Each Kukles system* (4.1) *with Q transversal to infinity can be transformed in one, and only one, of the following three systems*

$$\dot{x} = -y, \qquad \dot{y} = q_{00} + q_{10}x + q_{01}y + q_{20}x^2 + q_{11}yx + y^2, \tag{4.2}$$

with $q_{20} \neq 0$ and $q_{11}^2 - 4q_{20} \neq 0$;

$$\dot{x} = -y, \qquad \dot{y} = q_{00} + q_{10}x + q_{11}yx + y^2, \tag{4.3}$$

with $q_{11} \neq 0$ and $q_{00}^2 + q_{10}^2 > 0$; or

$$\dot{x} = -y, \qquad \dot{y} = q_{00} + q_{10}x + q_{20}x^2 + yx, \tag{4.4}$$

with $q_{00}^2 + q_{10}^2 + q_{20}^2 > 0$.

Proof. If $q_{02} \neq 0$ and $q_{20} \neq 0$ in Q, then by using the linear transformation $(x, y) \to (q_{02}x, q_{02}y)$ system (4.1) becomes (4.2). The condition $q_{11}^2 > 4q_{20}$ follows from the transversality hypothesis on Q.

If $q_{02} \neq 0$ and $q_{20} = 0$ in Q, then we must assume $q_{11} \neq 0$, otherwise Q is no transversal to infinity. Hence, by using the linear transformation $(x, y) \to ((q_{02}(q_{11} + q_{01})/q_{11})x, q_{02}y)$ system (4.1) becomes (4.3). Finally, the condition $q_{00}^2 + q_{10}^2 > 0$ follows from $y \nmid Q$.

If $q_{02} = 0$ in Q, then $q_{11} \neq 0$; otherwise Q is no transversal to infinity. By using the linear transformation $(x, y) \to (q_{11}x + q_{01}, q_{11}y)$ system (4.1) becomes (4.4). Again, the condition $q_{00}^2 + q_{10}^2 + q_{20}^2 > 0$ follows from $y \nmid Q$. $\qquad\square$

Proposition 4.2. *A Kukles system* (4.2) *with* $q_{11} \neq 0$ *has a transversal to infinity invariant algebraic curves* $\{F = 0\}$ *of degree* 2 *if and only if the following conditions hold:*

(i) $E := 2q_{01}q_{20} - q_{10}q_{11} - q_{11}q_{20} = 0$;

(ii) $D := 4q_{00}q_{20} - q_{10}^2 + q_{20}^2 = 0$.

Proof. We suppose that $\{F = 0\}$ is a transversal to infinity invariant algebraic curve of degree 2 of (4.2) and we will prove that (i) and (ii) hold.

From Lemma 3.1 and Proposition 3.4 we know that there is a nonzero constant S such that

$$Q_2 = SF_2 \quad \text{and} \quad K_1 = S(F_2)_y.$$

By replacing these expressions in case $l = 2$ of (3.1) and reordering terms we get

$$(Q_1 - SF_1)(F_2)_y - y(F_2)_x = (K_0 - S(F_1)_y)F_2;$$

moreover, since F_2 is homogeneous we have

$$y(F_2)_y + x(F_2)_x = 2F_2.$$

Hence, by combining these two last equations we get

$$\big(2(Q_1 - SF_1) - y(K_0 - S(F_1)_y)\big)(F_2)_y = \big(2y + x(K_0 - S(F_1)_y)\big)(F_2)_x.$$

Since F is transversal to infinity, $(F_2)_x$ and $(F_2)_y$ do not have common factors. Thus, there exists a nonzero constant T such that

$$2(Q_1 - SF_1) - y(K_0 - S(F_1)_y) = T(F_2)_x \tag{4.5}$$

and

$$2y + x(K_0 - S(F_1)_y) = T(F_2)_y. \tag{4.6}$$

We multiply (4.5) and (4.6) by x and y, respectively. The addition of the resulting expressions yields

$$2x(Q_1 - SF_1) + 2y^2 = 2TF_2. \tag{4.7}$$

The derivative of (4.6) with respect to y gives $2 = T(F_2)_{yy}$. Since $2 = (Q_2)_{yy} = S(F_2)_{yy}$, $T = S$. Hence (4.7) becomes

$$x(Q_1 - SF_1) + y^2 = Q_2,$$

which, by using the expression of Q_2, reduces to $Q_1 - SF_1 = q_{20}x + q_{11}y$, whence

$$F_1 = \frac{q_{10} - q_{20}}{S}x + \frac{q_{01} - q_{11}}{S}y \tag{4.8}$$

We now compute the derivative of (4.5) and (4.6) with respect to y and x, respectively. As $[T(F_2)_x]_y = [T(F_2)_y]_x$ we obtain

$$\left[2(Q_1 - SF_1) - y(K_0 - S(F_1)_y)\right]_y = \left[2y + x(K_0 - S(F_1)_y)\right]_x,$$

that is

$$2[(Q_1)_y - S(F_1)_y] - (K_0 - S(F_1)_y) = K_0 - S(F_1)_y,$$

whence $(Q_1)_y - S(F_1)_y = K_0 - S(F_1)_y$, which is equivalent to

$$K_0 = (Q_1)_y.$$

The cases $l = 1$ and $l = 0$ of (3.1) are

$$Q_0(F_2)_y + Q_1(F_1)_y - y(F_1)_x = K_0 F_1 + K_1 F_0$$

and

$$Q_0(F_1)_y = K_0 F_0.$$

By using the expressions of F_2, F_1, K_1, K_0 and Q the previous two equations are equivalent to

$$q_{00}q_{11} + q_{01}q_{20} - q_{10}q_{11} - q_{11}f_{00}S = 0, \tag{4.9}$$

$$2q_{00} - q_{10} + q_{20} - 2f_{00}S = 0, \tag{4.10}$$

and

$$q_{01}(q_{00} - Sf_{00}) - q_{00}q_{11} = 0 \tag{4.11}$$

We multiply (4.9) and (4.10) by 2 and $-q_{11}$, respectively, then the addition of the resulting equations is

$$2q_{01}q_{20} - q_{10}q_{11} - q_{11}q_{20} = 0, \tag{4.12}$$

which is condition (i) in the theorem. Now, we multiply (4.11) by 2 and by using (4.10) we have

$$q_{01}(q_{10} - q_{20}) - 2q_{00}q_{11} = 0.$$

Then we multiply last equation by $2q_{20}$, and by using (4.12) we obtain

$$-q_{11}\left(4q_{00}q_{20} - q_{10}^2 + q_{20}^2\right) = 0. \tag{4.13}$$

Finally, since $q_{11} \neq 0$, we obtain the condition (ii) given in the theorem.

For the converse, we consider the polynomials

$$F = \frac{(q_{20} - q_{10})^2}{4q_{20}} - (q_{20} - q_{10})x - \frac{q_{11}(q_{20} - q_{10})y}{2q_{20}} + q_{20}x^2 + q_{11}xy + y^2$$

and

$$K = \frac{q_{11}(q_{20} + q_{10})}{2q_{20}} + q_{11}x + 2y.$$

A simple computation shows that $\mathcal{X}F - KF$ has the form

$$\frac{q_{11}(q_{10} - q_{20})D}{8q_{20}^2} + \frac{q_{11}Dx}{4q_{20}} + \frac{(2Dq_{20} + E(q_{10} - q_{20})q_{11})y}{4q_{20}^2} + \frac{q_{11}Exy}{2q_{20}} + \frac{Ey^2}{q_{20}}.$$

Therefore, if (i) and (ii) hold, then $\{F = 0\}$ is an invariant algebraic curve of $\mathcal{X}$ with cofactor K. Moreover, F is transversal to infinity because $q_{11}^2 - 4q_{20} \neq 0$. This completes the proof. $\qquad \square$

Corollary 4.3. *A Kukles system* (4.2) *with* $q_{11} = 0$ *has a transversal to infinity invariant algebraic curves* $\{F = 0\}$ *of degree* 2 *if and only if* $q_{01} = 0$.

Proof. The necessity part follows from (4.9).

Sufficiency. From (4.10) we get that

$$F = q_{00} - \frac{q_{10}}{2} + \frac{q_{20}}{2} - (q_{20} - q_{10})\, x + q_{20}x^2 + y^2$$

is an invariant algebraic curve of the system with cofactor $K = 2\, y$. $\qquad\square$

Proposition 4.4. *A Kukles system* (4.3) *does not support transversal to infinity invariant algebraic curves of degree two.*

Proof. Suppose that $\{F = 0\}$ is a transversal to infinity invariant algebraic curves of degree two of (4.3). All equations obtained in the proof of previous proposition, except (4.13), can be applied in this case. Hence, as in this case $q_{20} = 0$ and $q_{11} \neq 0$, then from (4.12) we get $q_{10} = 0$. Thus, (4.10) reduces to $q_{00} - f_{00}S = 0$. Therefore, from (4.11) we obtain $q_{00} = 0$. This is a contradiction to the condition $q_{00}^2 + q_{10}^2 > 0$. $\qquad\square$

Proposition 4.5. *A Kukles system* (4.4) *does not support transversal to infinity invariant algebraic curves of degree* 2.

Proof. Any transversal to infinity invariant algebraic curve of a Kukles system must satisfy (4.6). However, that equation does not hold under the conditions on system (4.4). $\qquad\square$

4.1 Proof of Theorem 1.2

Proof of Theorem 1.2. From Corollary 3.5, Lemma 4.1, Proposition 4.2, Proposition 4.3, Proposition 4.4, and Proposition 4.5 of previous section we obtain that a quadratic Kukles system with a transversal to infinity invariant algebraic curve of degree two is of the form

$$\dot{x} = -y, \qquad \dot{y} = \frac{q_{20}^2 - q_{10}^2}{4q_{20}} + q_{10}x + \frac{q_{11}(q_{20} + q_{10})}{2q_{20}}y + q_{20}x^2 + q_{11}yx + y^2, \tag{4.14}$$

with $q_{11}q_{20} \neq 0$ and $q_{11}^2 - 4q_{20} \neq 0$, or

$$\dot{x} = -y, \qquad \dot{y} = q_{00} + q_{10}x + q_{20}x^2 + y^2, \tag{4.15}$$

with $q_{20} \neq 0$.

It follows from the proof of Proposition 4.2 that the zero locus of the polynomial

$$F_1 = \frac{(q_{20} - q_{10})^2}{4q_{20}} - (q_{20} - q_{10})\, x - \frac{q_{11}\,(q_{20} - q_{10})\, y}{2q_{20}} + q_{20}x^2 + q_{11}xy + y^2$$

is the unique invariant algebraic curve of (4.15), whose cofactor is

$$K = \frac{q_{11}\,(q_{20} + q_{10})}{2q_{20}} + q_{11}x + 2\, y.$$

Analogously, from the proof of Proposition 4.3 it follows that the zero locus of the polynomial

$$F_2 = q_{00} - \frac{q_{10}}{2} + \frac{q_{20}}{2} - (q_{20} - q_{10})\, x + q_{20}x^2 + y^2$$

is the unique invariant algebraic curve of (4.15), whose cofactor is $K = 2y$.

If $\mathcal{X}$ denotes the vector field associated with system (4.14), then we see that $K = \operatorname{div} \mathcal{X}$, which implies that F_1 is also an inverse integrating factor. Therefore, (4.14) is integrable. Also, if $\mathcal{X}$ denotes the vector field associated with system (4.15), then we see that $K = \operatorname{div} \mathcal{X}$, which implies that F_2 is also an inverse integrating factor. Therefore, (4.15) is integrable. □

Corollary 4.6. *Any Kukles system (4.1) does not have quadratic limit cycles.*

Proof. Each quadratic limit cycle of (4.1) must be a connected component, homeomorphic to the unit circle, of a nonsingular invariant algebraic curve $\{V = 0\}$ defined by a polynomial V of degree two. We have two cases, either V is transversal to infinity or V takes the form $V = a_0 + a_1 x + a_2 y + a_3 y^2$ after a linear transformation if necessary. In the former, we can suppose that V is one of the polynomials F_1 or F_2 given in previous proof. Thus, the assertion follows because $\{F_1 = 0\}$ is singular and system (4.15) has the symmetry $(x, y, t) \mapsto (x, -y, -t)$. In the latter, $\{V = 0\}$ does not have connected components homeomorphic to the unit circle. □

Acknowledgments

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