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Regularity theorems for a class of degenerate elliptic equations

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Abstract. In this paper we study the regularity of a class of degenerate elliptic equations with special lower order terms. By introducing a proper distance and applying the compactness method, we establish the Hölder type estimates for the weak solutions.

Keywords: degenerate elliptic equations, Hölder estimates, compactness method.

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1 Introduction

We are concerned with the regularity of a class of degenerate elliptic equations:

$$\mathcal{L}u = u_{xx} + |x|^{2\sigma} u_{yy} + u_x + |x|^l u_y - u = f \qquad (x,y) \in \Omega, \tag{1.1}$$

where l and σ are nonnegative numbers and Ω is a bounded domain in R^2 with $(0,0) \in \Omega$.

The investigation of degenerate elliptic equations began in the last century. The paper of Hörmander [5] studied the operators like

$$L = \sum_{i=1}^{n} X_i^2 + X_0 + c, \tag{1.2}$$

where $X_0, X_1, \ldots, X_n$ are smooth vector fields in Ω and satisfy Hörmander's condition that is the vector fields together with their commutators of some finite order span the tangent space at any point. In that paper, Hörmander stated that the operator L satisfies the following subelliptic estimate

$$\|u\|_{H^\varepsilon(K)} \le C\big(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big), \tag{1.3}$$

for compact subsets K of Ω. As a consequence L is hypoelliptic. After that a long series of papers considered many related researches to (1.2), see e.g., [1, 8, 11, 12, 15]. After these,

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some authors have studied the conditions that the vector fields are not smooth. For instance, Wang [10] considered the following equation

$$u_{xx} + |x|^{2\sigma} u_{yy} = f,$$

where σ is an arbitrary positive real number. In this case the vector fields $X = \{\partial x, |x|^{\sigma}\partial y\}$ are Hölder continuous and do not satisfy Hörmander's condition.

Moreover, Hong and Wang [6] studied the regularity of a class of degenerate elliptic Monge–Ampère equation

$$\det(u_{ij}) = K(x,y) f(x,y,u,Du)$$

in $\Omega \subset R^2$ with $u = 0$ on $\partial\Omega$. By Legendre transformation the equation can be rewritten as a degenerate elliptic equation which can be simplified to

$$u_{xx} + x^m u_{yy} + u_x + x^{m-1} u_y + f = 0 \qquad (x,y) \in \Omega \subset R^2_+, \tag{1.4}$$

where $m > 1$ is an integer. Obviously, when $m = 2$, the equation is in the form of (1.2) by taking $X = \{\partial x, x\partial y\}$.

In this paper, we study the local Hölder estimates of (1.1) which is a general form of (1.4). The equation is generated by the vector fields $X = \{\partial x, |x|^{\sigma}\partial y\}$. When σ is a positive integer and $l = \sigma$, the vector fields are smooth and $\mathcal{L}$ belongs to the Hörmander's operator. If σ is a positive integer and $l = 2\sigma - 1$, $\mathcal{L}$ is in the form of (1.4). We assume that σ is an arbitrary nonnegative number, so the vector fields X may not be smooth. We note that $\big||x|^l u_y\big| \le \big||x|^{\sigma} u_y\big|$ in the case $l \ge \sigma$ and $|x| < 1$. That means the lower order terms $\{u_x, |x|^l u_y\}$ can be controlled by the vector fields X. So we can easily have the energy estimate of (1.1). However, in the case $l < \sigma$, the lower order terms $\{u_x, |x|^l u_y\}$ can not be controlled by the vector fields X. Our interest lies in the regularity of the weak solutions of (1.1) in the case that l is an arbitrary nonnegative numbers. The important thing is that if we consider the natural scaling from:

$$u_r(x,y) = u(rx, r^{1+\sigma} y),$$

then we have that the order of the terms u_{xx} and $|x|^{2\sigma} u_{yy}$ is 2, and that of the term $|x|^l u_y$ is $1 + \sigma - l$. So $|x|^l u_y$ is still a lower order term with respect to $|x|^{2\sigma} u_{yy}$ when $\sigma < 1 + l$. In this case, the main result is as follows.

Theorem 1.1. *Let l and σ be nonnegative numbers and $l > \sigma - 1$. Then, there exists a constant $\bar{\alpha} > 0$, such that if $f \in C^{\alpha}_*(\Omega)$ and u is a weak solution of (1.1) in Ω, then $u \in C^{2,\alpha}_*(\Omega')$, for $0 < \alpha < \bar{\alpha}$. Moreover,*

$$\|u\|_{C^{2,\alpha}_*(\Omega')} \le C\big(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{\alpha}_*(\Omega)}\big),$$

where Ω is a bounded domain in R^2 with $(0,0) \in \Omega$ and $\Omega' \subset\subset \Omega$.

Remark 1.2. The spaces $C^{2,\alpha}_*(\Omega'), C^{\alpha}_*(\Omega)$ and the weak solutions are defined in Section 2.

The organization of this paper is as follows. In Section 2, we introduce the definition of the metric related the vector fields $X = \{\partial x, |x|^{\sigma}\partial y\}$ and the spaces such as $C^{k+\alpha}_*(\Omega)$, $W^{1,p}_{0,\sigma}(\Omega)$. In Section 3, we give the regularity of the homogeneous equation near the origin. In Section 4, the regularity of the general equations near the degenerate line is given by using the iteration method. Consequently, the result of Theorem 1.1 is established.

2 Preliminaries

In this section we give some function spaces and results associated to the vector fields. Here we need the intrinsic metric related to the vector fields which is associated with the degenerate elliptic operator. The construction of the intrinsic metric and the modified Hölder spaces appropriate for degenerate parabolic equations, were introduced by Daskalopoulos and Hamilton in [3] for the study of the porous medium equation. A few years later, Feehan and Pop considered the related results for the boundary-degenerate elliptic equations (see [4]).

Now let us review the intrinsic metric and the spaces introduced by Wang in [10]. The metric related to the vector fields $X = \{\partial x, |x|^\sigma \partial y\}$, is given by

$$ds^2 = dx^2 + |x|^{-2\sigma} dy^2.$$

For any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the equivalent metric is defined by

$$d(P_1, P_2) = |x_1 - x_2| + \frac{|y_1 - y_2|}{|x_1|^\sigma + |x_2|^\sigma + |y_1 - y_2|^{\frac{\sigma}{1+\sigma}}}. \tag{2.1}$$

Define the ball with the center point P as

$$B(P, r) = \{X : d(X, P) < r\}.$$

We denote $B(0, r)$ by B_r for simplicity.

The distance and the balls have the following properties:

(1) there exists $\gamma > 1$ such that

$$d(P_1, P_2) \le \gamma\big(d(P_1, P_3) + d(P_3, P_2)\big); \tag{2.2}$$

(2) the measures of the balls are controllable,

$$|B(P, R)| \le \left(\frac{R}{r}\right)^{2+\sigma} |B(P, r)|, \quad R \ge r > 0. \tag{2.3}$$

In the following, we give some useful function spaces related to the vector fields.

For any $0 < \alpha < 1$, we define the Hölder space with respect to the distance defined by (2.1) as

$$C_*^\alpha(\Omega) = \left\{ u \in C(\overline{\Omega}) \colon \sup_{X_1, X_2 \in \Omega} \frac{|u(X_1) - u(X_2)|}{d(X_1, X_2)^\alpha} < \infty \right\},$$

where Ω is a bounded domain in R^2. We define the C_*^α seminorm and norm as

$$[u]_{C_*^\alpha(\Omega)} = \sup_{X_1, X_2 \in \Omega} \frac{|u(X_1) - u(X_2)|}{d(X_1, X_2)^\alpha},$$

$$\|u\|_{C_*^\alpha(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{C_*^\alpha(\Omega)}.$$

When the metric is Euclidean metric, Campanato proved that Campanato space is embedding into the usual Hölder space(see[2]). After that, the similar embedding theorems have been obtained for vector fields of Hörmander's type or the doubling metric measure space (see [7,9,10]). For the distance function defined by (2.1) we also have Campanato type spaces. But here we need a little modification of the usual one.

We denote by $\mathcal{P}^k_{X_0}$ the set of kth order polynomials at $X_0=(x_0,y_0)$ which have the following form

$$P(x,y)=\sum_{0\le i+j\le k} a_{ij}(x-x_0)^i(y-y_0)^j,$$

and

$$\sum_{0\le i+j\le k} |a_{ij}||x_0|^{(i+(1+\sigma)j-(k+\alpha))^+}\le C.$$

We remark that if we consider the point on the degenerate line, i.e., $Y_0=(0,y_0)$, then some terms of the second order polynomials at Y_0 disappear. More specifically, if $\frac{\alpha}{2}\le\sigma<\alpha$ then $a_{02}=0$ and if $\sigma\ge\alpha$ then $a_{02}=a_{11}=0$. Although some terms disappear, we still denote the second order polynomial as $\sum_{0\le i+j\le 2} a_{ij}x^i(y-y_0)^j$. Now we construct Hölder space by the polynomial approximation which is attributed to Safanov (see [13,14]).

Definition 2.1. We say $u\in C^{k,\alpha}_*$ at X_0 if for every $r>0$, there is a polynomial $P(x,y)$ of order k such that

$$|u(x,y)-P(x,y)|\le Cr^{k+\alpha}\quad (x,y)\in B(X_0,r)\cap\Omega,$$

and define

$$[u]_{C^{k,\alpha}_*(X_0,\Omega)}=\sup_{r>0}\inf_{P}\left\{\frac{|u(x,y)-P(x,y)|}{r^{k+\alpha}},(x,y)\in B(X_0,r)\cap\Omega\right\},$$

where P is taking over the set of polynomials at X_0 of order k.

We denote $[u]_{C^{k,\alpha}_*(X_0,\Omega)}$ by $[u]_{C^{k,\alpha}_*(X_0)}$, and define

$$\|P\|_{C^{k,\alpha}_*(X_0)}=\sum_{0\le i+j\le k}|a_{ij}||x_0|^{(i+(1+\sigma)j-(k+\alpha))^+}$$

to be $C^{k,\alpha}_*$ norm of P at $X_0=(x_0,y_0)$. Then, $C^{k,\alpha}_*$ norm of $u(x,y)$ in Ω is

$$\sup_{X\in\Omega}\sup_{r>0}\inf_{P\in\mathcal{P}^k_{X_0}}\left\{\frac{|u(Y)-P(Y)|}{r^{k+\alpha}}+\|P\|_{C^{k,\alpha}_*(X)},Y\in B(X,r)\cap\Omega\right\}.$$

For any $1\le q<\infty$, we define the space $C^{k,\alpha;q}_*(\Omega)$.

Definition 2.2. Let Ω be a bounded domain in R^2 such that there exist positive constants r_0 and c with

$$|B(X,r)\cap\Omega|>c|B(X,r)|\quad\text{for all } X\in\Omega,\ 0<r<r_0.$$

A function $f\in L^q(\Omega)$, $1\le q<\infty$, is $C^{k,\alpha;q}_*$ at $X_0\in\Omega$ if

$$\sup_{r>0}\inf_{P\in\mathcal{P}^k_{X_0}}\left\{\frac{1}{r^{k+\alpha}}\left(\frac{1}{|B(X_0,r)|}\int_{B(X_0,r)\cap\Omega}|f(x,y)-P(x,y)|^q dxdy\right)^{\frac{1}{q}}\right\}<\infty.$$

We denote the left hand side as $[f]_{C^{k,\alpha;q}_*(X_0,\Omega)}$. We say $f\in C^{k,\alpha;q}_*(\Omega)$ if $f\in C^{k,\alpha;q}_*(X_0,\Omega)$, for every point $X_0\in\Omega$, and define

$$[f]_{C^{k,\alpha;q}_*(\Omega)}=\sup_{X_0\in\Omega}\sup_{r>0}\inf_{P\in\mathcal{P}^k_{X_0}}\left\{\frac{1}{r^{k+\alpha}}\left(\frac{1}{|B(X_0,r)|}\int_{B(X_0,r)\cap\Omega}|f(x,y)-P(x,y)|^q dxdy\right)^{\frac{1}{q}}\right\}.$$

We have the following theorem.

Theorem 2.3. *Let $1 \leq q < \infty$. Assume that $u \in L^q(\Omega)$ and that $|B(X,r)| \leq C_1|B(X,r) \cap \Omega|$, for a constant $C_1 > 0$. Then $u \in C_*^{2,\alpha}(\Omega)$ if and only if $u \in C_*^{2,\alpha;q}(\Omega)$.*

For $1 \leq p < \infty$, we define the function spaces

$$W_\sigma^{1,p}(\Omega) = \{u \in L^p(\Omega),\ u_x \in L^p(\Omega),\ |x|^\sigma u_y \in L^p(\Omega)\}.$$

Then, $W_\sigma^{1,p}(\Omega)$ is a Banach space with the norm defined by

$$\|u\|_{W_\sigma^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|u_x\|_{L^p(\Omega)} + \||x|^\sigma u_y\|_{L^p(\Omega)}.$$

Let $W_{0,\sigma}^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W_\sigma^{1,p}(\Omega)$. In particular, we denote $W_\sigma^{1,2}(\Omega), W_{0,\sigma}^{1,2}(\Omega)$ by $H_\sigma^1(\Omega)$, $H_{0,\sigma}^1(\Omega)$.

By Corollary 1 in [10], we have the following lemma.

Lemma 2.4. *For any $\sigma > 0$, there is a small constant $h = h(\sigma) > 0$, such that for any $r < R \leq 2$, there is a constant C depending on σ, r, and R, such that*

$$\|u\|_{H^h(B_r)} \leq C\|u\|_{H_\sigma^1(B_R)}.$$

Now we give the definition of the weak solutions of (1.1). For our convenience, we consider the following equation

$$\tilde{\mathcal{L}}u = u_{xx} + |x|^{2\sigma}u_{yy} + b_1u_x + b_2|x|^l u_y + cu = f. \tag{2.4}$$

Definition 2.5. Let b_1, b_2 and c are constants. We say $u \in H_\sigma^1(\Omega)$ is a weak solution of (2.4) if

$$\int_\Omega (u_x\varphi_x + |x|^{2\sigma}u_y\varphi_y + b_1u\varphi_x + b_2|x|^l u\varphi_y - cu\varphi)dxdy = -\int_\Omega f\varphi dxdy, \tag{2.5}$$

for every $\varphi \in C_0^1(\Omega)$.

3 Regularity of the homogeneous equation

In this section we investigate the estimate of (1.1) when f equals zero.

Lemma 3.1. *Let $f = 0$ and u be a weak solution of (1.1). Then the following inequality holds*

$$\int_{B_1} (|u_x|^2 + |x|^{2\sigma}|u_y|^2)dxdy \leq C\int_{B_2} |u|^2 dxdy.$$

Proof. Let $\eta \in C_0^\infty(B_2)$. Replacing the test function φ by $\eta^2 u$, we have

$$\int_{B_2} \big(u_x(\eta^2u)_x + |x|^{2\sigma}u_y(\eta^2u)_y\big)dxdy + \int_{B_2} \big(u(\eta^2u)_x + |x|^l u(\eta^2u)_y\big)dxdy + \int_{B_2} u(\eta^2u)dxdy = 0.$$

Thus

$$
\begin{aligned}
&\int_{B_2} (|u_x|^2\eta^2 + |x|^{2\sigma}|u_y|^2\eta^2)dxdy \\
&= -2\int_{B_2} (u_x\eta_x\eta u + |x|^{2\sigma}u_y\eta_y\eta u)dxdy - \int_{B_2} \left(\left(\frac{u^2}{2}\right)_x + |x|^l \left(\frac{u^2}{2}\right)_y\right)\eta^2 dxdy \\
&\quad - \int_{B_2} u^2(2\eta\eta_x + 2|x|^l\eta\eta_y + \eta^2)dxdy \\
&= -2\int_{B_2} (u_x\eta_x\eta u + |x|^{2\sigma}u_y\eta_y\eta u)dxdy + \int_{B_2} (u^2\eta\eta_x + |x|^l u^2\eta\eta_y)dxdy \\
&\quad - \int_{B_2} u^2(2\eta\eta_x + 2|x|^l\eta\eta_y + \eta^2)dxdy \\
&\le \epsilon\int_{B_2} (|u_x|^2\eta^2 + |x|^{2\sigma}|u_y|^2\eta^2)dxdy + \frac{1}{\epsilon}\int_{B_2} (\eta_x^2 + |x|^{2\sigma}\eta_y^2)u^2 dxdy \\
&\quad - \int_{B_2} u^2(\eta\eta_x + |x|^l\eta\eta_y + \eta^2)dxdy \\
&\le \epsilon\int_{B_2} (|u_x|^2\eta^2 + |x|^{2\sigma}|u_y|^2\eta^2)dxdy + \frac{1}{\epsilon}\int_{B_2} (\eta_x^2 + |x|^{2\sigma}\eta_y^2)u^2 dxdy \\
&\quad + \int_{B_2} u^2\left(2\eta^2 + \frac{1}{2}\eta_x^2 + \frac{1}{2}|x|^{2l}\eta_y^2\right)dxdy.
\end{aligned}
$$

Taking $\epsilon = \frac{1}{2}$, we find

$$\int_{B_2} (|u_x|^2\eta^2 + |x|^{2\sigma}|u_y|^2\eta^2)dxdy \le 6\int_{B_2} (\eta^2 + \eta_x^2 + |x|^{2\sigma}\eta_y^2 + |x|^{2l}\eta_y^2)u^2 dxdy.$$

Now if we take η such that

$$0 \le \eta \le 1, \quad \eta = 1 \text{ on } \overline{B_1}, \quad \eta = 0 \text{ near } \partial B_2, \quad \text{and} \quad |\nabla\eta| \le C,$$

then we have the lemma. □

Lemma 3.2. *Let $f = 0$ and u be a weak solution of (1.1). Then there is a small constant $h > 0$ such that*

$$\|u\|_{H^h(B_{1/2})} \le C\|u\|_{L^2(B_2)}. \tag{3.1}$$

Proof. By Lemma 3.1, we have

$$\int_{B_1} (|u_x|^2 + |x|^{2\sigma}|u_y|^2)dxdy \le C\int_{B_2} |u|^2 dxdy.$$

Using Lemma 2.4 and taking $r = \frac{1}{2}$ and $R = 1$, we obtain

$$\|u\|_{H^h(B_{1/2})} \le C\big(\|u_x\|_{L^2(B_1)} + \||x|^\sigma u_y\|_{L^2(B_1)} + \|u\|_{L^2(B_1)}\big).$$

The lemma follows by combining the above two inequalities. □

When f equals zero, (1.1) is translation invariant in y direction. So the operator $\mathcal{L}$ is commutative with $|\partial y|^\gamma$, for any $\gamma \in \mathbb{R}^+$. Using Lemma 3.2 and the pseudo-differential calculus and applying the estimate (3.1) to u, $|\partial y|^h u$, $|\partial y|^{2h}u, \ldots$ inductively, we have u is locally smooth in y direction. Since u is a solution of the homogeneous equation we have u is a solution of

$$u_{xx} + u_{yy} + u_x + |x|^l u_y - u = (1 - |x|^{2\sigma})u_{yy}.$$

The right-hand side of the above equation is Hölder continuous and the left hand side is an elliptic operator. By the estimates of the elliptic equations we have the following lemma.

Lemma 3.3. *Let $f = 0$ and u be a weak solution of (1.1) in B_2. Then $u \in C^{2,\bar{\alpha}}(B_{1/4})$ and*

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/4})} \le C\|u\|_{L^2(B_2)}, \tag{3.2}$$

where $\bar{\alpha}$ is a positive constant depending on σ and l.

To obtain the regularity of the nonhomogeneous equation, we need to modify Lemma 3.3 and get a uniform estimate of (2.4).

Lemma 3.4. *Let u be a weak solution of (2.4) in B_1 and $|b_1|, |b_2|, |c| \le 1$. Then there exists a universal constant C, such that*

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/4})} \le C\|u\|_{L^2(B_1)}. \tag{3.3}$$

This lemma can be obtained by applying the same method as in the prove of Lemma 3.3.

4 The estimates near the degenerate line

In this section the estimate of (1.1) near $x = 0$ is given. Sice the equation is translation invariant in y direction, we only need to consider the estimate near the origin.

Theorem 4.1. *Let $\bar{\alpha}$ be the same constant as in Lemma 3.3 and $\alpha < \bar{\alpha}$. Assume that $f \in C^{\alpha}_{*}(B_{10\gamma^3})$ and that u satisfies (1.1) in $B_{10\gamma^3}$. Then $u \in C^{2,\alpha}_{*}(B_1)$, and*

$$\|u\|_{C^{2,\alpha}_{*}(B_1)} \le C\left(\|u\|_{L^\infty(B_{10\gamma^3})} + \|f\|_{C^{\alpha}_{*}(B_{10\gamma^3})}\right).$$

The main techniques are the energy estimates and the iterations. To obtain the estimates of the nonhomogeneous equation, we need the following scaling form

$$\tilde{u}(x,y) = u(rx, r^{1+\sigma}y).$$

Then, $\tilde{u}(x,y)$ satisfies

$$\tilde{u}_{xx} + |x|^{2\sigma}\tilde{u}_{yy} + r\tilde{u}_x + r^{l+1-\sigma}|x|^l\tilde{u}_y - r^2\tilde{u} = r^2 f(rx, r^{1+\sigma}y).$$

So we need the energy estimate of (2.4) when we do the iterations. Since r is small and $\sigma < 1 + l$, it is reasonable to assume that $|b_1|$, $b_2|$ and $|c|$ are less than 1.

We now start proving a series of lemmas that will be used to prove Theorem 4.1.

Lemma 4.2. *If u is a weak solution of (2.4) and $|b_1|, |b_2|, |c| \le 1$, then there is a universal constant C, such that*

$$\int_{B_{\frac{3}{2}}} \left(|u_x|^2 + |x|^{2\sigma}|u_y|^2\right) dxdy \le C\int_{B_2} \left(|u|^2 + |f|^2\right) dxdy.$$

This lemma can be obtained by applying the similar methods as in Lemma 3.1, so we omit the proof.

Lemma 4.3. *Assume that $|b_1|, |b_2|, |c| \le 1$. Then, for every $\varepsilon > 0$, there exists a small constant δ, such that if u is a weak solution of (2.4) in B_2 with*

$$\frac{1}{|B_2|}\int_{B_2} |u|^2 dxdy \le 1, \tag{4.1}$$

$$\frac{1}{|B_2|}\int_{B_2}|f|^2dxdy \le \delta^2, \tag{4.2}$$

then

$$\frac{1}{|B_1|}\int_{B_1}|u-v|^2dxdy \le \varepsilon^2,$$

where v is a weak solution of

$$\tilde{\mathcal{L}}v = 0, \qquad (x,y)\in B_1.$$

Proof. We prove the lemma by contradiction. Suppose there exists an $\varepsilon_0 > 0$, such that for any positive integer k, there exist $u^{(k)}$ and $f^{(k)}$ satisfying

$$\frac{1}{|B_2|}\int_{B_2}|u^{(k)}|^2dxdy \le 1,$$

$$\frac{1}{|B_2|}\int_{B_2}|f^{(k)}|^2dxdy \le \frac{1}{k^2},$$

and

$$\tilde{\mathcal{L}}u^{(k)} = f^{(k)}, \tag{4.3}$$

in the weak sense in B_2, but for any v, which is a weak solution of the equation

$$\tilde{\mathcal{L}}v = 0 \qquad (x,y)\in B_1,$$

we have

$$\frac{1}{|B_1|}\int_{B_1}|u^{(k)}-v|^2dxdy > \varepsilon_0^2. \tag{4.4}$$

Since $u^{(k)}$ is a weak solution of (4.3), by Lemma 4.2, we have

$$\begin{aligned}\int_{B_{\frac{3}{2}}}\left(|u_x^{(k)}|^2+|x|^{2\sigma}|u_y^{(k)}|^2\right)dxdy &\le C\int_{B_2}\left(|u^{(k)}|^2+|f^{(k)}|^2\right)dxdy\\ &\le C.\end{aligned}$$

Thus, $\|u^{(k)}\|_{H^1_\sigma(B_{3/2})} \le C$. By Lemma 2.4 and taking $R=\frac{3}{2}, r=1$, we have $u^{(k)}\in H^h(B_1)$. Since $H^h(B_1)$ is compactly embedded in $L^2(B_1)$, there is a subsequence of $u^{(k)}$, which we still denote as $u^{(k)}$, such that

$$u^{(k)} \longrightarrow v \quad \text{strongly in } L^2(B_1).$$

By the L^2 boundedness of $u_x^{(k)}$ and $|x|^\sigma u_y^{(k)}$, we have

$$u_x^{(k)} \longrightarrow v_x \quad \text{weakly in } L^2(B_1),$$

$$|x|^\sigma u_y^{(k)} \longrightarrow |x|^\sigma v_y \quad \text{weakly in } L^2(B_1).$$

Since $u^{(k)}$ is a weak solution, we have

$$\int_{B_1}\left(u_x^{(k)}\varphi_x+|x|^{2\sigma}u_y^{(k)}\varphi_y+b_1u^{(k)}\varphi_x+b_2|x|^lu^{(k)}\varphi_y-cu^{(k)}\varphi\right)dxdy = -\int_{B_1}f^{(k)}\varphi dxdy.$$

Let $k\to\infty$. Then we have v is a weak solution of equation

$$\tilde{\mathcal{L}}v = 0 \qquad (x,y)\in B_1,$$

which is a contradiction. This finishes the proof. □

Lemma 4.4. *Suppose $|b_1|, |b_2|, |c| \leq 1$. Let $0 < \alpha < \bar{\alpha}$ and r_0 be a small constant. There exists a small constant δ such that if u is a weak solution of (2.4) in B_2 with (4.1) and (4.2) satisfied, then,*

$$\frac{1}{|B_{r_0}|}\int_{B_{r_0}} |u-P|^2 dxdy \leq r_0^{2(2+\alpha)},$$

where $P = \sum_{i+j\leq 2} a_{ij}x^i y^j$ is a second order polynomial at $(0,0)$ such that $\tilde{\mathcal{L}}P = 0$ and $\sum_{i+j\leq 2}|a_{ij}| \leq C$.

Proof. By Lemma 4.3, there exists a $v(x)$ which is a weak solution of

$$\tilde{\mathcal{L}}v = 0 \qquad x \in B_1,$$

such that

$$\frac{1}{|B_1|}\int_{B_1} |u-v|^2 dxdy \leq \varepsilon^2. \tag{4.5}$$

So

$$\begin{aligned}\frac{1}{|B_1|}\int_{B_1}|v|^2 dxdy &\leq \frac{2}{|B_1|}\int_{B_1}\left(|u-v|^2+|u|^2\right)dxdy\\ &\leq 2\left(2^{2+\sigma}+\varepsilon^2\right).\end{aligned}$$

By Lemma 3.4, $v \in C^{2,\bar{\alpha}}(B_{1/4})$, and hence, $v \in C_*^{2,\bar{\alpha}}(B_{1/4})$. So there exists a second order polynomial $P(x,y)$ at $(0,0)$ such that

$$\sup_{0<r<1}\frac{1}{r^{2+\bar{\alpha}}}\left(\frac{1}{|B_r|}\int_{B_r}|v-P|^2 dxdy\right)^{\frac{1}{2}} \leq C\|v\|_{L^2(B_1)}.$$

For $0 < r_0 < \frac{1}{2}$, we have

$$\begin{aligned}\int_{B_{r_0}}|u-P|^2 dxdy &\leq 2\int_{B_{r_0}}\left(|u-v|^2+|v-P|^2\right)dxdy\\ &\leq 2\varepsilon^2|B_1| + 4C\left(2^{2+\sigma}+1\right)r_0^{2(2+\bar{\alpha})}|B_{r_0}|\\ &\leq r_0^{2(2+\alpha)}|B_{r_0}|,\end{aligned}$$

by taking $r_0 = \left(8C(2^{2+\sigma}+1)\right)^{\frac{1}{2(\alpha-\bar{\alpha})}}$ and ε small. □

Lemma 4.5. *Let $0 < \alpha < \bar{\alpha}$ and u be a weak solution of*

$$u_{xx} + |x|^{2\sigma}u_{yy} + u_x + |x|^l u_y - u = f \quad \text{in } B_1, \tag{4.6}$$

with

$$\frac{1}{|B_1|}\int_{B_1}|u|^2 dxdy \leq 1, \tag{4.7}$$

$$[f]_{C_*^{\alpha}(0,0)} \leq \delta, \qquad f(0,0) = 0. \tag{4.8}$$

Then there is a second order polynomial $P(x,y) \in \mathcal{P}^2_{(0,0)}$, such that

$$\sup_{0<r<1}\frac{1}{r^{2+\alpha}}\left(\frac{1}{|B_r|}\int_{B_r}|u-P(x,y)|^2 dxdy\right)^{\frac{1}{2}} \leq C, \tag{4.9}$$

and

$$\sum_{i+j\leq 2}|a_{ij}| \leq C.$$

Proof. Let r_0 be the same constant as in Lemma 4.4. We claim that there exist second order polynomials

$$P_k(x,y) = \sum_{i+j\le 2} a_{ij}^{(k)} x^i y^j \in \mathcal{P}^2_{(0,0)}$$

such that

$$\frac{1}{|B_{r_0^k}|}\int_{B_{r_0^k}} |u-P_k|^2 dxdy \le r_0^{2k(2+\alpha)}, \tag{4.10}$$

and

$$\left|a_{ij}^{(k)} - a_{ij}^{(k-1)}\right| \le C r_0^{(k-1)(2+\alpha-(i+(1+\sigma)j))}.$$

Let $P_0 = 0$ and P_1 be the polynomial P in Lemma 4.4, then the claim holds for $k=1$. Assume that the claim holds for k.

Let

$$u^{(k)}(x,y) = \frac{(u-P_k)(r_0^k x, r_0^{k(1+\sigma)} y)}{r_0^{k(2+\alpha)}}.$$

Then

$$u_{xx}^{(k)} + |x|^{2\sigma} u_{yy}^{(k)} + r_0^k u_x^{(k)} + r_0^{k(l+1-\sigma)} |x|^l u_y^{(k)} - r_0^2 u^{(k)} = f^{(k)} \quad \text{in } B_1,$$

where

$$f^{(k)}(x,y) = \frac{f\left(r_0^k x, r_0^{k(1+\sigma)} y\right)}{r_0^{k\alpha}}.$$

Hence, by (4.10),

$$\begin{aligned}
\frac{1}{|B_1|}\int_{B_1} |u^{(k)}|^2 dxdy &= \frac{1}{|B_1|}\int_{B_1} \left|\frac{(u-P_k)(r_0^k x, r_0^{k(1+\sigma)} y)}{r_0^{k(2+\alpha)}}\right|^2 dxdy \\
&= \frac{1}{r_0^{2k(2+\alpha)}} \frac{1}{|B_{r_0^k}|}\int_{B_{r_0^k}} |(u-P_k)(x,y)|^2 dxdy \\
&\le 1.
\end{aligned}$$

By (4.8),

$$\frac{1}{|B_1|}\int_{B_1} |f^{(k)}|^2 dxdy \le \delta^2.$$

Applying Lemma 4.4 to $u^{(k)}$, we obtain that there is a polynomial

$$P(x,y) \in \mathcal{P}^2_{(0,0)}$$

such that

$$\frac{1}{|B_{r_0}|}\int_{B_{r_0}} |u^{(k)} - P|^2 dxdy \le r_0^{2(1+\alpha)}$$

and

$$\sum_{i+j\le 2} |a_{ij}| \le C.$$

Now substituting $u^{(k)}$ by u, we have

$$\frac{1}{|B_{r_0}|}\int_{B_{r_0}} \left|\frac{(u-P_k)(r_0^k x, r_0^{k(1+\sigma)} y)}{r_0^{k(2+\alpha)}} - P(x,y)\right|^2 dxdy \le r_0^{2(2+\alpha)}.$$

Therefore,

$$\frac{1}{|B_{r_0^{k+1}}|}\int_{B_{r_0^{k+1}}}\left|u-\left(P_k(x,y)+r_0^{k(2+\alpha)}P\left(\frac{x}{r_0^k},\frac{y}{r_0^{k(1+\sigma)}}\right)\right)\right|^2dxdy\le r_0^{2(k+1)(2+\alpha)}.$$

Let

$$P_{k+1}(x,y)=P_k(x,y)+r_0^{k(2+\alpha)}P\left(\frac{x}{r_0^k},\frac{y}{r_0^{k(1+\sigma)}}\right).$$

Then the claim holds. The lemma follows immediately from the claim. □

We note here that by the choice of $P(x,y)$, (4.9) also holds for $r\ge 1$. Since the equation is translation invariant in y direction, we can apply Lemma 4.5 in $B(Y_0,1)$ with $Y_0=(0,y_0)$.

Now we go back to the proof of Theorem 4.1.

Proof. Let $X_0=(x_0,y_0)\in B_1$ and $Y_0=(0,y_0)$. Without lose of generality, we can assume $f(0,y_0)=0$. Multiplying a small number to (1.1), we can assume that (4.7) and (4.8) are satisfied. By Lemma 4.5, there is a second order polynomial

$$\hat{P}(x,y)\in\mathcal{P}^2_{Y_0},$$

such that

$$\frac{1}{|B(Y_0,r)|}\int_{B(Y_0,r)}|u(x,y)-\hat{P}(x,y)|^2dxdy\le Cr^{2(2+\alpha)}.$$

Thus

$$\frac{1}{|B(Y_0,2\gamma|x_0|)|}\int_{B(Y_0,2\gamma|x_0|)}|u(x,y)-\hat{P}(x,y)|^2dxdy\le C|x_0|^{2(2+\alpha)}.$$

Now we give the estimate at the point X_0.

If $r<\frac{1}{4}|x_0|$, then, by (2.2), $B(X_0,r)\subset B(Y_0,2\gamma|x_0|)$. Let

$$v(x,y)=\frac{(u-\hat{P})(|x_0|x,|x_0|^{1+\sigma}y)}{|x_0|^{2+\alpha}}.$$

Then $v(x,y)$ satisfies

$$v_{xx}+|x|^{2\sigma}v_{yy}+|x_0|v_x+|x_0|^{l+1-\sigma}|x|^lv_y-|x_0|^2v=g(x,y),\tag{4.11}$$

where $g(x,y)=\frac{1}{|x_0|^\alpha}f(|x_0|x,|x_0|^{1+\sigma}y)$.

The corresponding good point of $v(x,y)$ is

$$\widetilde{X_0}=\frac{X_0}{|x_0|}=\begin{cases}\left(1,\dfrac{y_0}{|x_0|^{1+\sigma}}\right), & x_0>0,\\ \left(-1,\dfrac{y_0}{|x_0|^{1+\sigma}}\right), & x_0<0.\end{cases}$$

We also have

$$[g]_{C^\alpha_*(B(\widetilde{X_0},\frac{1}{2}))}\le C[f]_{C^\alpha_*(B(X_0,\frac{|x_0|}{2}))}$$

and

$$\frac{1}{|B(\widetilde{Y_0},2)|}\int_{B(\widetilde{Y_0},2)}|v(x,y)|^2dxdy\le C,$$

where $\widetilde{Y_0}=(0,\frac{y_0}{|x_0|^{1+\sigma}})$.

If we consider (4.11) in $B\big(\widetilde{X_0},\frac{1}{2}\big)$, then $\frac{1}{2}<|x|<\frac{3}{2}$. So the equation is uniformly elliptic. Since near $\widetilde{X_0}$ the metric we defined is equivalent to the Euclidean metric, we have

$$[g]_{C^\alpha(B(\widetilde{X_0},\frac{1}{2}))}\le C[g]_{C^\alpha_*(B(\widetilde{X_0},\frac{1}{2}))}.$$

Thus v is $C^{2+\alpha}$ and consequently v is $C^{2+\alpha}_*$ at $\widetilde{X_0}$. So there exists a second order polynomial $P_1(x,y)\in\mathcal{P}^2_{\tilde{X}_0}$ such that

$$\frac{1}{\big|B\big(\widetilde{X_0},\frac{r}{|x_0|}\big)\big|}\int_{B\big(\widetilde{X_0},\frac{r}{|x_0|}\big)}|v(x,y)-P_1(x,y)|^2dxdy\le C\Big(\frac{r}{|x_0|}\Big)^{2(2+\alpha)}.$$

Substituting by u, we have

$$\frac{1}{|B(X_0,r)|}\int_{B(X_0,r)}|u(x,y)-P(x,y)|^2dxdy\le Cr^{2(2+\alpha)},$$

where

$$P(x,y)=\hat{P}(x,y)-|x_0|^{2+\alpha}P_1\Big(\frac{x}{|x_0|},\frac{y}{|x_0|^{1+\sigma}}\Big)$$

and it is easily to verify that $P(x,y)\in\mathcal{P}^2_{X_0}$.

Now we consider $r\ge\frac{1}{4}|x_0|$. By the properties of the distance and the balls, i.e., the inequalities (2.2) and (2.3), we obtain

$$B(X_0,r)\subset B(Y_0,5\gamma r)\subset B(X_0,9\gamma^2r),$$

and

$$\frac{|B(X_0,9\gamma^2r)|}{|B(X_0,r)|}\le(9\gamma^2)^{2+\sigma}.$$

So

$$\begin{aligned}&\frac{1}{|B(X_0,r)|}\int_{B(X_0,r)}|u(x,y)-P_{Y_0}(x,y)|^2dxdy\\&\le\frac{|B(X_0,9\gamma^2r)|}{|B(X_0,r)|}\frac{1}{|B(Y_0,5\gamma r)|}\int_{B(Y_0,5\gamma r)}|u(x,y)-P_{Y_0}(x,y)|^2dxdy\\&\le Cr^{2(2+\alpha)}.\end{aligned}$$

Thus we have the theorem. □

The scaling form of Theorem 4.1 can be stated as the following corollary.

Corollary 4.6. *Let $Y_0=(0,y_0)$, and let u be a solution of (1.1) in $B(Y_0,d)$. Then, for every point $X_0=(x_0,y_0)\in B\big(Y_0,\frac{d}{10\gamma^3}\big)$, there exists a second order polynomial $P_{X_0}(x,y)\in\mathcal{P}^2_{X_0}$ such that*

$$\begin{aligned}&\Big(\frac{1}{|B(X_0,r)|}\int_{B(X_0,r)}|u(x,y)-P_{X_0}(x,y)|^2dxdy\Big)^{\frac{1}{2}}\\&\le C\Big(d^{-(2+\alpha)}\|u\|_{L^\infty(B(Y_0,d))}+d^{-\alpha}\|f\|_{L^\infty(B(Y_0,d))}+[f]_{C^\alpha_*(B(Y_0,d))}\Big),\end{aligned}$$

where

$$P_{X_0}(x,y)=\sum_{i+j\le2}a_{ij}(x-x_0)^i(y-y_0)^j,$$

and

$$\sum_{i+j\le 2} d^{\tau}|a_{ij}||x_0|^{(i+(1+\sigma)j-(2+\alpha))^+} \le C\left(\|u\|_{L^\infty(B(Y_0,d))} + d^2\|f\|_{L^\infty(B(Y_0,d))} + d^{2+\alpha}[f]_{C_*^\alpha(B(Y_0,d))}\right),$$

where $\tau = (2+\alpha)\wedge(i+(1+\sigma)j)$.

Theorem 1.1 is an immediate consequence of this corollary and the estimates of the uniformly elliptic equations.

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