Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **32**, 1–23; doi: 10.14232/ejqtde.2016.1.32 http://www.math.u-szeged.hu/ejqtde/

Multiplicity of positive solutions to a singular (p_1, p_2)-Laplacian system with coupled integral boundary conditions

Jeongmi Jeong[1], **Chan-Gyun Kim**✉[2] and **Eun Kyoung Lee**[2]

[1]Department of mathematics, Pusan National University, Busan, 609-735, Korea
[2]Department of Mathematics Education, Pusan National University, Busan, 609-735, Korea

Received 14 July 2015, appeared 1 June 2016
Communicated by Gennaro Infante

Abstract. In this work, we investigate the existence and multiplicity results for positive solutions to a singular (p_1, p_2)-Laplacian system with coupled integral boundary conditions and a parameter $(\mu, \lambda) \in \mathbb{R}^3_+$. Using sub-super solutions method and fixed point index theorems, it is shown that there exists a continuous surface $\mathcal{C}$ which separates $\mathbb{R}^2_+ \times (0, \infty)$ into two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the problem under consideration has two positive solutions for $(\mu, \lambda) \in \mathcal{O}_1$, at least one positive solution for $(\mu, \lambda) \in \mathcal{C}$, and no positive solutions for $(\mu, \lambda) \in \mathcal{O}_2$.

Keywords: nonlocal boundary condition, multiplicity, positive solution, singular weight function.

2010 Mathematics Subject Classification: 34B10, 35J57, 35J92.

1 Introduction

Nonlocal boundary conditions appear when the information on the boundary are connected to values inside the domain. Various types of boundary value problems involving nonlocal conditions have been extensively studied by various methods such as fixed point theorems on cones and the Leray–Schauder alternative, etc. We refer the reader to [11, 14, 15, 24, 30–33, 37] and the references therein.

For example, Ma [24] considered

$$\begin{cases} u''(t) = g_1(t, u(t), u'(t)) + e(t), \ a.e. \ t \in (0,1), \\ u'(0) = 0, \ u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \end{cases}$$

where $g_1 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $e \in C[0,1]$, $m \in \mathbb{N}$, $\xi_i \in (0, \infty)$ with $0 < \xi_1 < \cdots < \xi_m < 1$, and $\alpha_i \in (0, \infty)$. Using the Leray–Schauder alternative, the existence of at least one

✉Corresponding author. Email: cgkim75@gmail.com

solution is obtained for two cases:

$$\sum_{i=1}^{m} \alpha_i \neq 1 \quad (Nonresonance)$$

and

$$\sum_{i=1}^{m} \alpha_i = 1 \quad (Resonance).$$

Webb and Infante [31, 32] studied the existence of multiple positive solutions of nonlinear differential equations of the form

$$-u''(t) = w_1(t) g_2(t, u(t)), \quad \text{a.e. } t \in (0,1)$$

with various nonlocal boundary conditions involving linear functionals on $C[0,1]$ including the following conditions: either

$$u(0) = \alpha_1[u], \quad u(1) = \alpha_2[u] \quad \text{or} \quad u(0) = \alpha_1[u], \quad u'(1) = \alpha_2[u].$$

Here, $w_1 \in L^1((0,1), \mathbb{R}_+)$, $\mathbb{R}_+ := [0,\infty)$, $g_2 : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ is continuous, and for $i \in \{1,2\}$, α_i is bounded linear functionals on $C[0,1]$ involving Stieltjes integrals with signed measures. Recently, Zhang and Feng [37] studied the following one-dimensional singular p-Laplacian problems of the form

$$\begin{cases} \lambda(\varphi_p(u'(t)))' + w_2(t) g_3(t, u(t)) = 0, \ \ t \in (0,1), \\ au(0) - bu'(0) = \int_0^1 w_3(t) u(t) dt, \ u'(1) = 0, \end{cases}$$

where λ is a positive parameter, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $a, b > 0$, and $w_2, w_3 \in L^1((0,1), \mathbb{R}_+)$. Using fixed point index theory on cones of Banach spaces, they obtained several results about the existence, multiplicity, and nonexistence of positive solutions under various assumptions on the nonlinearity $g_3(t,s)$ which satisfies L^1-Carathéodory condition.

The systems of differential equations equipped with a variety of boundary conditions have been extensively studied by many authors, see, e.g., [2–7, 10, 19, 21, 23, 25, 26, 34]. For example, in [25], do Ó et al. considered a class of system of second-order differential equations of the form

$$\begin{cases} -u'' = g_4(t, u, v, a, b), \ \ \text{in } (0,1), \\ -v'' = g_5(t, u, v, a, b), \ \ \text{in } (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.1}$$

where the nonlinearities g_4 and g_5 are superlinear at the origin as well as at infinity, and $a, b \in \mathbb{R}_+$. Using fixed point theorems of cone expansion/compression type, the upper-lower solutions method and degree argument, it was shown that there exists a continuous curve Γ which splits the positive quadrant of the $(a$-$b)$-plane into disjoint sets S_1 and S_2 such that (1.1) has at least two positive solutions in S_1, has at least one positive solution on the boundary of S_1, and has no positive solutions in S_2. The result was applied to establish the existence and multiplicity of positive radial solutions for a certain class of semilinear elliptic systems in annular domains.

We are concerned with the existence of positive solutions to the following singular (p_1, p_2)-Laplacian system with coupled integral boundary conditions

$$\begin{cases} (\Phi(u'(t)))' + \lambda h(t) \bullet f(t, u(t)) = \theta, \quad t \in (0,1), \\ u(0) = \alpha[u], \ u(1) = \mu, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\Phi(s_1, s_2) = (\varphi_{p_1}(s_1), \varphi_{p_2}(s_2))$, $\varphi_{p_i}(s) := |s|^{p_i-2}s$ with $p_i > 1$ for $i \in \{1,2\}$, θ is the origin of $\mathbb{R}^2$, $\alpha : C[0,1] \times C[0,1] \to \mathbb{R}^2$ is a linear transformation which is defined by, for $u \in C[0,1] \times C[0,1]$,

$$\alpha[u] := \int_0^1 u(s)k(s)ds$$

and $k = (k_{ij})_{2\times 2}$ with $k_{ij} \in L^1((0,1), \mathbb{R}_+)$ for $i, j \in \{1,2\}$, $(\mu, \lambda) = (\mu_1, \mu_2, \lambda) \in \mathbb{R}^3_+$ is a parameter, and $\bullet$ denotes the entrywise product, i.e., $(a_1, a_2) \bullet (b_1, b_2) := (a_1 b_1, a_2 b_2)$.

Throughout this paper, we assume the following hypotheses are satisfied unless otherwise stated:

(H_1) $f = (f_1, f_2) : [0,1] \times \mathbb{R}^2 \to (0,\infty)^2$ with $f_i \in C([0,1] \times \mathbb{R}^2, (0,\infty))$ and $h = (h_1, h_2) : (0,1) \to (0,\infty)^2$ with $h_i \in C((0,1), (0,\infty))$ satisfying $h_i \in \mathcal{A}_i$ for $i \in \{1,2\}$, where

$$\mathcal{A}_i = \left\{ \hat{h} : \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1} \left(\int_s^{\frac{1}{2}} \hat{h}(\tau)d\tau \right) ds + \int_{\frac{1}{2}}^1 \varphi_{p_i}^{-1} \left(\int_{\frac{1}{2}}^s \hat{h}(\tau)d\tau \right) ds < \infty \right\};$$

(H_2) for $i \in \{1,2\}$, $\int_0^1 (1-s)k_{ii}(s)ds \in [0,1)$;

(H_3) $\det K > 0$, where

$$K := I - \left(\int_0^1 (1-s)k(s)ds \right) \quad \text{and } I \text{ is the identity matrix of size 2.}$$

For convenience, we identify $(a, b) \in \mathbb{R}^2$ with the 1-by-2 matrix $\begin{pmatrix} a & b \end{pmatrix}$ if necessary. Consequently, $\alpha[u]$ is well defined for $u \in C[0,1] \times C[0,1]$.

The main purpose of this paper is to study the existence and multiplicity results for positive solutions to problem ($P_{\lambda,\mu}$) using sub-super solutions method and fixed point index theorems. For sub-super solutions method concerning semilinear problems with nonlocal boundary conditions, we refer to [27–29]. It seems not obvious that sub-super solutions method can be applicable to our problem with (p_1, p_2)-Laplacian due to the coupled integral boundary condition in ($P_{\lambda,\mu}$). Thus we prove a theorem for sub-super solutions (see Theorem 2.12), and it is shown that there exists a continuous surface $\mathcal{C}$ which separates $\mathbb{R}^2_+ \times (0,\infty)$ into two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ such that ($P_{\lambda,\mu}$) has two positive solutions for $(\mu, \lambda) \in \mathcal{O}_1$, at least one positive solution for $(\mu, \lambda) \in \mathcal{C}$, and no positive solutions for $(\mu, \lambda) \in \mathcal{O}_2$ (see Theorem 3.10).

Deng and Li [9] considered a semilinear elliptic problem of the form

$$\begin{cases} \Delta u + A(x)u^q = 0 \text{ in } \Omega_1, \\ u > 0 \text{ in } \Omega_1, \ u \in H^1_{\text{loc}}(\Omega_1) \cap C(\bar{\Omega}_1), \\ u|_{\partial\Omega_1} = 0, \ u \to \mu_1 > 0 \text{ as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $\Omega_1 = \mathbb{R}^N \backslash \omega$ is an exterior domain in $\mathbb{R}^N$, $\omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $N > 2$, and $q > 1$. Among other results, when $q = (N+2)/(N-2)$ and $0 \le A \in L^1(\Omega_1)$ satisfies certain additional conditions, it was shown that there exists $\mu^* > 0$ such that (1.2) has at least two positive solutions for $\mu_1 \in (0, \mu^*)$, exactly one positive solution for $\mu_1 = \mu^*$, and no positive solutions for $\mu_1 \in (\mu^*, \infty)$. The existence, multiplicity and nonexistence of positive radial solutions to p-Laplacian problems similar to (1.2) were studied in [16–18].

As applications, we study existence results for positive radial solutions to p-Laplacian systems defined in an exterior domain as follows:

$$\begin{cases} \text{div}\left(|\nabla z_1|^{p-2}\nabla z_1\right) + \lambda K_1(|x|)\hat{f}_1(|x|, z_1, z_2) = 0 & \text{in } \Omega, \\ \text{div}\left(|\nabla z_2|^{p-2}\nabla z_2\right) + \lambda K_2(|x|)\hat{f}_2(|x|, z_1, z_2) = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

subject to coupled integral boundary conditions

$$\begin{cases} z_1(x) = \int_\Omega [l_{11}(|y|)z_1(y) + l_{21}(|y|)z_2(y)]dy & \text{on } |x| = r_0, \\ z_2(x) = \int_\Omega [l_{12}(|y|)z_1(y) + l_{22}(|y|)z_2(y)]dy & \text{on } |x| = r_0, \end{cases} \tag{1.4}$$

or

$$\begin{cases} z_1(x) = \int_\Omega [l_{11}(|y|)z_1(y) + l_{21}(|y|)z_2(y)]dy & \text{as } |x| \to \infty, \\ z_2(x) = \int_\Omega [l_{12}(|y|)z_1(y) + l_{22}(|y|)z_2(y)]dy & \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $r_0 > 0$, $N > p > 1$, $K_i \in C((r_0, \infty), (0, \infty))$, $l_{ij} \in L^1((r_0, \infty), \mathbb{R}_+)$, and $\hat{f}_i \in C([r_0, \infty) \times \mathbb{R}_+^2, (0, \infty))$ for $i, j \in \{1, 2\}$.

Using the main result (Theorem 3.10), we investigate the existence, multiplicity and nonexistence of positive solutions $z = (z_1, z_2)$ to (1.3)+(1.4) (resp. (1.3)+(1.5)) satisfying $z(x) \to \mu$ as $|x| \to \infty$ (resp. $z(x) = \mu$ on $|x| = r_0$) for given $\mu \in \mathbb{R}_+^2$ and $\lambda \in (0, \infty)$ (see Corollary 4.1).

For $u, v \in \mathbb{R}^2$, $u \le v$ (resp. $u < v$) means $u_i \le v_i$ (resp. $u_i < v_i$) for all $i \in \{1, 2\}$, where u_i and v_i are i-th coordinates of u and v, respectively. For functions $w^1, w^2 : [0, 1] \to \mathbb{R}^n$ with $n \in \{1, 2\}$, $w^1 \le w^2$ (resp. $w^1 < w^2$) also means $w^1(t) \le w^2(t)$ (resp. $w^1(t) < w^2(t)$) for $t \in [0, 1]$. We also denote θ the zero function from $[0, 1]$ to $\mathbb{R}^2$ as well as the origin of $\mathbb{R}^2$.

This paper is organized as follows. In Section 2, well-known theorems such as generalized Picone identity and a fixed point index theorem are recalled, and a solution operator and a theorem for sub-super solutions related to problem ($P_{\lambda,\mu}$) are also introduced. In Section 3, the main result in this paper is given (see Theorem 3.10). Finally, in Section 4, applications for problem (1.3)+(1.4) or (1.3)+(1.5) are given (see Corollary 4.1).

2 Preliminaries

For semilinear problems, we usually use integration by parts twice in order to obtain the useful information for solutions such as a block for parameters λ and *a priori* estimates for solutions. However, it is not effective for the p-Laplacian problem. The following generalized Picone identity can be used to overcome the difficulty (see Lemma 3.1 and Lemma 3.3). The identity can be verified by straightforward differentiation, but for completeness, we give the proof of it.

Theorem 2.1 (generalized Picone identity, see, e.g., [13, 20]). *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$

$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z),$$

where $\varphi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R}$, $p > 1$ and b_1, b_2 are continuous functions on an interval I. Let y and z be functions such that y, z, $\varphi_p(y')$, $\varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$. Then the *generalized Picone identity can be written as*

$$\begin{aligned}\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} &= (b_1 - b_2)|y|^p \\ &\quad - \left[|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - py'\varphi_p\left(\frac{yz'}{z}\right)\right] \\ &\quad - yl_p[y] + \frac{|y|^p}{\varphi_p(z)}L_p[z]. \end{aligned} \tag{2.1}$$

Proof. By straightforward differentiation,

$$\begin{aligned} &\frac{d}{dt}\left(\frac{|y|^p\varphi_p(z')}{\varphi_p(z)}\right) \\ &\quad = \frac{(|y|^p\varphi_p(z'))'\varphi_p(z) - |y|^p\varphi_p(z')(\varphi_p(z))'}{(\varphi_p(z))^2} \\ &\quad = \frac{py'\varphi_p(y)\varphi_p(z') + |y|^p(L^p[z] - b_2(t)\varphi_p(z))}{\varphi_p(z)} - (p-1)\left|\frac{yz'}{z}\right|^p \\ &\quad = py'\varphi_p\left(\frac{yz'}{z}\right) + \frac{|y|^p}{\varphi_p(z)}L_p[z] - b_2(t)|y|^p - (p-1)\left|\frac{yz'}{z}\right|^p \end{aligned} \tag{2.2}$$

and

$$\frac{d}{dt}(y\varphi_p(y')) = |y'|^p + y(l_p[y] - b_1(t)\varphi_p(y)). \tag{2.3}$$

Then subtracting (2.3) from (2.2) yields the identity (2.1). □

Remark 2.2. By Young's inequality,

$$y'\varphi_p\left(\frac{yz'}{z}\right) \le \frac{|y'|^p}{p} + \left(1 - \frac{1}{p}\right)\left|\frac{yz'}{z}\right|^p,$$

and the equality holds if and only if $\operatorname{sgn} y' = \operatorname{sgn}(yz'/z)$ and $|y'|^p = |yz'/z|^p$. Thus,

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - py'\varphi_p\left(\frac{yz'}{z}\right) \ge 0,$$

which implies, by (2.1),

$$\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} \le (b_1 - b_2)|y|^p - yl_p[y] + \frac{|y|^p}{\varphi_p(z)}L_p[z]. \tag{2.4}$$

Now we recall a well-known theorem for the existence of a global continuum of solutions by Leray and Schauder [22] and a fixed point index theorem:

Theorem 2.3 (see, e.g., [35, Corollary 14.12])**.** *Let X be a Banach space with $X \neq \{0\}$ and let $\mathcal{P}$ be an order cone in X. Consider*

$$x = H(\lambda, x), \tag{2.5}$$

where $\lambda \in \mathbb{R}_+$ and $x \in \mathcal{P}$. If $H : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ is completely continuous and $H(0, x) = 0$ for all $x \in \mathcal{P}$, then $\mathcal{C}_+(\mathcal{P})$, the component of the solution set of (2.5) containing $(0, 0)$, is unbounded.

Theorem 2.4 (see, e.g., [12])**.** *Let X be a Banach space, $\mathcal{P}$ be a cone in X and $\mathcal{O}$ be a bounded open set containing θ in X, where θ is the origin of X. Let $A : \mathcal{P} \cap \overline{\mathcal{O}} \to \mathcal{P}$ be completely continuous. Suppose that $Ax \neq \nu x$ for all $x \in \mathcal{P} \cap \partial\mathcal{O}$ and all $\nu \ge 1$. Then $i(A, \mathcal{O} \cap \mathcal{P}, \mathcal{P}) = 1$.*

2.1 Solution operator

In this subsection, we define an operator related to problem ($P_{\lambda,\mu}$) and prove the complete continuity of it.

Denote $X := C[0,1] \times C[0,1]$ with norm $\|(u_1, u_2)\|_X := \|u_1\|_\infty + \|u_2\|_\infty$, and $\mathcal{P} := \{u = (u_1, u_2) \in X : u_1, u_2 \in \mathcal{K}\}$. Then $(X, \|\cdot\|_X)$ is a Banach space and $\mathcal{P}$ is a cone in X. Here, $C[0,1]$ denotes the Banach space of continuous functions u defined on $[0,1]$ with usual maximum norm $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$ and $\mathcal{K} := \{u \in C[0,1] : u$ is a nonnegative concave function$\}$.

By (H_3), $\det K > 0$ and

$$K^{-1} = \frac{1}{\det K}\begin{pmatrix} 1 - \int_0^1 (1-s)k_{22}(s)ds & \int_0^1 (1-s)k_{12}(s)ds \\ \int_0^1 (1-s)k_{21}(s)ds & 1 - \int_0^1 (1-s)k_{11}(s)ds \end{pmatrix}.$$

Then all entries of K^{-1} are nonnegative by (H_2) and nonnegativity of k_{ij} for $i, j \in \{1,2\}$.

Define $\beta : \mathbb{R}^2_+ \times X \to \mathbb{R}^2$ by, for $(\mu, v) \in \mathbb{R}^2_+ \times X$,

$$\beta[\mu, v] := (\beta_1[\mu, v], \beta_2[\mu, v]) := \int_0^1 (v(s) + \mu s)k(s)K^{-1}ds.$$

Then $\beta[\mu_n, v_n] \to \beta[\mu_0, v_0]$ in $\mathbb{R}^2$ as $(\mu_n, v_n) \to (\mu_0, v_0)$ in $\mathbb{R}^2_+ \times X$, and $\beta[\mu, v] \in \mathbb{R}^2_+$ for all $(\mu, v) \in \mathbb{R}^2_+ \times \mathcal{P}$.

Consider the following problem

$$\begin{cases} (\Phi(v'(t) - \beta[\mu, v] + \mu))' + \lambda h(t) \bullet F(\mu, v)(t) = \theta, \ t \in (0,1), \\ v(0) = v(1) = \theta, \end{cases} \tag{$\hat{P}_{\lambda,\mu}$}$$

where $F := (F_1, F_2) : \mathbb{R}^2_+ \times X \to X$ is defined by, for $(\mu, v) \in \mathbb{R}^2_+ \times X$,

$$F(\mu, v)(t) := f(t, v(t) + (1-t)\beta[\mu, v] + t\mu), \quad t \in [0,1]. \tag{2.6}$$

Then $F(\mu, v) > \theta$, since $f(t, s) > \theta$ for all $(t, s) \in [0,1] \times \mathbb{R}^2$.

For $i = 1, 2$, define continuous transformations $L^i : \mathbb{R}^2_+ \times X \to X$ by, for $t \in [0,1]$ and $(\mu, u), (\mu, v) \in \mathbb{R}^2_+ \times X$,

$$L^1(\mu, u)(t) := (L^1_1(\mu, u), L^1_2(\mu, u))(t) := u(t) - ((1-t)\alpha[u] + t\mu) \tag{2.7}$$

and

$$L^2(\mu, v)(t) := (L^2_1(\mu, v), L^2_2(\mu, v))(t) := v(t) + ((1-t)\beta[\mu, v] + t\mu). \tag{2.8}$$

With the above transformations (2.7) and (2.8), we have the following lemma.

Lemma 2.5. *Assume that (H_3) holds. Then $\alpha[u] = \beta[\mu, L^1(\mu, u)]$ for $(\mu, u) \in \mathbb{R}^2_+ \times X$ and $\beta[\mu, v] = \alpha[L^2(\mu, v)]$ for $(\mu, v) \in \mathbb{R}^2_+ \times X$.*

Proof. We only show that $\alpha[u] = \beta[\mu, L^1(\mu, u)]$, since the other case can be proved in a similar manner. For $(\mu, u) \in \mathbb{R}^2_+ \times X$,

$$\begin{aligned}\beta[\mu, L^1(\mu,u)] &= \int_0^1 (L^1(\mu,u)(s) + \mu s)k(s)K^{-1}ds \\ &= \int_0^1 (u(s) - (1-s)\alpha[u])\, k(s)K^{-1}ds \\ &= \int_0^1 u(s)k(s)K^{-1}ds - \alpha[u]\int_0^1 (1-s)k(s)K^{-1}ds \\ &= \alpha[u]\left(I - \int_0^1 (1-s)k(s)ds\right)K^{-1} = \alpha[u],\end{aligned}$$

and the proof is complete. □

By a non-negative solution (resp. a positive solution) u to problem $(P_{\lambda,\mu})$ or $(\hat{P}_{\lambda,\mu})$, we mean $u = (u_1, u_2)$ is a solution to problem $(P_{\lambda,\mu})$ or $(\hat{P}_{\lambda,\mu})$ which satisfies $u_i(t) \geq 0$ (resp. $u_i(t) > 0$) for all $t \in (0,1)$ and all $i \in \{1,2\}$.

Remark 2.6. (1) Let $v = (v_1, v_2)$ be a solution to problem $(\hat{P}_{\lambda,\mu})$. Then for $i \in \{1,2\}$, v_i' is decreasing in $(0,1)$, and v_i is concave on $(0,1)$. Since $v(0) = v(1) = \theta$, $v \in \mathcal{P}$. Moreover, v is a positive solution to problem $(\hat{P}_{\lambda,\mu})$ if $\lambda > 0$, and $v = \theta$ only if $\lambda = 0$. Similarly, let u be a non-negative solution to problem $(P_{\lambda,\mu})$. Then $u \in \mathcal{P}$, and u is a positive solution to problem $(P_{\lambda,\mu})$ if $\lambda > 0$.

(2) By Lemma 2.5, for fixed $(\lambda, \mu) \in \mathbb{R}^3_+$, if v is a (non-negative) solution to problem $(\hat{P}_{\lambda,\mu})$, then $u = L^2(\mu, v)$ is a non-negative solution to problem $(P_{\lambda,\mu})$, and conversely if u is a solution to problem $(P_{\lambda,\mu})$, then $v = L^1(\mu, u)$ is a (non-negative) solution to problem $(\hat{P}_{\lambda,\mu})$.

Lemma 2.7. *Assume that (H_1)–(H_3) hold. For fixed $(\lambda, \mu, v) \in (0,\infty) \times \mathbb{R}^2_+ \times X$ with $\mu = (\mu_1, \mu_2)$ and $i \in \{1,2\}$, there exists a unique constant $M_i = M_i(\lambda, \mu, v) \in (0,1)$ satisfying*

$$\begin{aligned}(\beta_i[\mu,v] - \mu_i)M_i &+ \int_0^{M_i} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu,v]) + \lambda\int_s^{M_i} h_i(\tau)F_i(\mu,v)(\tau)d\tau\right)ds \\ &= -(\beta_i[\mu,v] - \mu_i)(1 - M_i) \\ &\quad + \int_{M_i}^1 \varphi_{p_i}^{-1}\left(\varphi_{p_i}(-\mu_i + \beta_i[\mu,v]) + \lambda\int_{M_i}^s h_i(\tau)F_i(\mu,v)(\tau)d\tau\right)ds.\end{aligned} \tag{2.9}$$

Proof. Let $(\lambda, \mu, v) \in (0,\infty) \times \mathbb{R}^2_+ \times X$ with $\mu = (\mu_1, \mu_2)$ and $i \in \{1,2\}$ be fixed. Define a continuous function $x_i = x_i(\lambda, \mu, v) : (0,1) \to \mathbb{R}$ by

$$x_i(t) = \beta_i[\mu,v] - \mu_i + x_i^1(t) + x_i^2(t), \qquad t \in (0,1),$$

where

$$x_i^1(t) = \int_0^t \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu,v]) + \lambda\int_s^t h_i(\tau)F_i(\mu,v)(\tau)d\tau\right)ds$$

and

$$x_i^2(t) = \int_t^1 \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu,v]) - \lambda\int_t^s h_i(\tau)F_i(\mu,v)(\tau)d\tau\right)ds.$$

We claim that x_i is strictly increasing in $(0,1)$. Indeed, by (H_1), $\lim_{t\to 0^+} x_i^1(t) = \lim_{t\to 1^-} x_i^2(t) = 0$, $\lim_{t\to 1^-} x_i^1(t) > \mu_i - \beta_i[\mu,v]$, and $\lim_{t\to 0^+} x_i^2(t) < \mu_i - \beta_i[\mu,v]$. Consequently,

$$\lim_{t\to 0^+} x_i(t) < 0 \quad \text{and} \quad \lim_{t\to 1^-} x_i(t) > 0.$$

For $0 < t_1 < t_2 < 1$, one has

$$x_i^1(t_2) - x_i^1(t_1) > \int_{t_1}^{t_2} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu, v]) + \lambda \int_s^{t_2} h_i(\tau)F_i(\mu, v)(\tau)d\tau\right) ds$$

and

$$\begin{aligned} x_i^2(t_2) - x_i^2(t_1) &> -\int_{t_1}^{t_2} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu, v]) + \lambda \int_s^{t_1} h_i(\tau)F_i(\mu, v)(\tau)d\tau\right) ds \\ &> -\int_{t_1}^{t_2} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i[\mu, v]) + \lambda \int_s^{t_2} h_i(\tau)F_i(\mu, v)(\tau)d\tau\right) ds. \end{aligned}$$

Thus, $x_i(t_2) - x_i(t_1) > 0$ for $0 < t_1 < t_2 < 1$, and there exists a unique $M_i = M_i(\lambda, \mu, v) \in (0, 1)$ such that $x_i(M_i) = 0$. Consequently, M_i satisfies (2.9). $\square$

For $i = 1, 2$, define $T_i : \mathbb{R}_+^3 \times X \to C[0, 1]$ by, for $(\lambda, \mu, v) \in \mathbb{R}_+^3 \times X$,

$$T_i(\lambda, \mu, v)(t) := \begin{cases} (\beta_i[\mu, v] - \mu_i)t + \int_0^t \varphi_{p_i}^{-1}\Big(\varphi_{p_i}(\mu_i - \beta_i[\mu, v]) \\ \qquad +\lambda \int_s^{M_i} h_i(\tau)F_i(\mu, v)(\tau)d\tau\Big)ds, \quad 0 \le t \le M_i, \\ -(\beta_i[\mu, v] - \mu_i)(1 - t) + \int_t^1 \varphi_{p_i}^{-1}\Big(\varphi_{p_i}(-\mu_i + \beta_i[\mu, v]) \\ \qquad +\lambda \int_{M_i}^s h_i(\tau)F_i(\mu, v)(\tau)d\tau\Big)ds, \quad M_i \le t \le 1, \end{cases}$$

where $M_i = M_i(\lambda, \mu, v) \in (0, 1)$ is a constant satisfying (2.9) for $\lambda > 0$ and M_i may be taken arbitrary number in $(0, 1)$ for $\lambda = 0$, since $T_i(0, \mu, v) = 0$ for all $(\mu, v) \in \mathbb{R}_+^2 \times X$.

For $i \in \{1, 2\}$, by Lemma 2.7, we can see that T_i is well-defined, $\|T_i(\lambda, \mu, v)\|_\infty = T_i(\lambda, \mu, v)(M_i)$, and $(T_i(\lambda, \mu, v))'(M_i) = 0$. Moreover, since $h_i(t)f_i(t, s) \ge 0$ for all $(t, s) \in [0, 1] \times \mathbb{R}^2$, $(T_i(\lambda, \mu, v))'$ is decreasing in $(0, 1)$ for all $(\lambda, \mu, v) \in \mathbb{R}_+^3 \times X$. Since $T_i(\lambda, \mu, v)(0) = T_i(\lambda, \mu, v)(1) = 0$, $T_i(\mathbb{R}_+^3 \times X) \subseteq \mathcal{K}$.

Define $T : \mathbb{R}_+^3 \times X \to \mathcal{P}$ by $T(\lambda, \mu, v) := (T_1(\lambda, \mu, v), T_2(\lambda, \mu, v))$ for $(\lambda, \mu, v) \in \mathbb{R}_+^3 \times X$. Then T is well defined, and by standard argument we have the following lemma.

Lemma 2.8. *Assume that (H_1)–(H_3) hold. Let $(\lambda, \mu) \in \mathbb{R}_+^3$ be fixed. Then problem $(\hat{P}_{\lambda,\mu})$ has a (non-negative) solution v if and only if $T(\lambda, \mu, \cdot)$ has a fixed point v in $\mathcal{P}$. Moreover, if θ is a solution to problem $(\hat{P}_{\lambda,\mu})$, then $\lambda = 0$, and if $\lambda = 0$, then θ is a unique solution to problem $(\hat{P}_{\lambda,\mu})$.*

To show the complete continuity of T, we first prove the following lemma.

Lemma 2.9. *Assume that (H_1)–(H_3) hold, let $\{(\lambda^n, \mu^n, v^n)\}$ be a bounded sequence in $\mathbb{R}_+^3 \times X$ and let $i \in \{1, 2\}$ be fixed. If $M_i^n = M_i(\lambda^n, \mu^n, v^n) \to 0$ (or 1) as $n \to \infty$, then $\|T_i(\lambda^n, \mu^n, v^n)\|_\infty \to 0$ as $n \to \infty$.*

Proof. We only prove the case $M_i^n \to 0$, since the other case can be proved in a similar manner. Since $\{(\lambda^n, \mu^n, v^n)\}$ is a bounded sequence in $\mathbb{R}_+^3 \times X$, there exists $C > 0$ such that $\mu_i^n + \lambda^n\|F_i(\mu^n, v^n)\|_\infty + |\beta_i[\mu^n, v^n]| \le C$ for all $n \in \mathbb{N}$. Here, μ_i^n is the i-th component of μ^n. Then

$$\begin{aligned} \|T_i(\lambda^n, \mu^n, v^n)\|_\infty &= T_i(\lambda^n, \mu^n, v^n)(M_i^n) \\ &= (\beta_i[\mu^n, v^n] - \mu_i^n)M_i^n \\ &\quad + \int_0^{M_i^n} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i^n - \beta_i[\mu^n, v^n]) + \lambda^n \int_s^{M_i^n} h_i(\tau)F_i(\mu^n, v^n)(\tau)d\tau\right)ds \\ &\le CM_i^n + \gamma_{\frac{1}{p_i-1}} \int_0^{M_i^n} \left(C + \varphi_{p_i}^{-1}(C)\varphi_{p_i}^{-1}\left(\int_s^{M_i^n} h_i(\tau)d\tau\right)\right)ds, \end{aligned}$$

where $\gamma_q = \max\{1, 2^{q-1}\}$ for $q > 0$. It follows from $h_i \in \mathcal{A}$ that, as $n \to \infty$, $\|T_i(\lambda_n, \mu_n, v_n)\|_\infty \to 0$, and thus the proof is complete. □

Combining Lemma 2.9 with the standard arguments (see, e.g., [1, Lemma 3] and [14, Lemma 2.4]), we have the following lemma. We omit the proof of it.

Lemma 2.10. *Assume that (H_1)–(H_3) hold. Then $T : \mathbb{R}^3_+ \times X \to \mathcal{P}$ is completely continuous.*

2.2 Sub-super solutions theorem

In this subsection, we give a theorem for sub-super solutions to problem $(P_{\lambda,\mu})$.

Definition 2.11. Let $\lambda > 0$ and $\mu \in \mathbb{R}^2_+$ be given. We say that $\zeta = (\zeta_1, \zeta_2)$ is a *supersolution* to problem $(P_{\lambda,\mu})$ if $\zeta_i \in C^1(0,1)$ with $\varphi_{p_i}(\zeta_i')$ absolutely continuous for $i = 1, 2$, and

$$\begin{cases} (\Phi(\zeta'(t)))' + \lambda h(t) \bullet f(t, \zeta(t)) \le \theta, & t \in (0,1), \\ \zeta(0) \ge \alpha[\zeta], \ \zeta(1) \ge \mu. \end{cases}$$

We also say that $\psi = (\psi_1, \psi_2)$ is a *subsolution* to problem $(P_{\lambda,\mu})$ if $\psi_i \in C^1(0,1)$ with $\varphi_{p_i}(\psi_i')$ absolutely continuous for $i = 1, 2$, and it satisfies the reverse of the above inequalities.

To get a theorem for sub-super solutions to problem $(P_{\lambda,\mu})$, we make the following hypotheses:

(H_2') $\displaystyle \max_{i \in \{1,2\}} \left\{ \int_0^1 (1-s)[k_{1i}(s) + k_{2i}(s)]ds \right\} =: C_k \in [0,1);$

(F_1) For fixed $(t, u) \in [0,1] \times \mathbb{R}_+$, $f_1 = f_1(t, u, v)$ is quasi-monotone nondecreasing with respect to v, i.e., $f_1(t, u, v_1) \le f_1(t, u, v_2)$ whenever $0 \le v_1 \le v_2$. For fixed $(t, v) \in [0,1] \times \mathbb{R}_+$, $f_2 = f_2(t, u, v)$ is quasi-monotone nondecreasing with respect to u.

Note that (H_2') implies (H_2).

Now, a theorem for sub-super solutions for the problem $(P_{\lambda,\mu})$ is given as follows.

Theorem 2.12. *Let $\lambda > 0$ and $\mu \in \mathbb{R}^2_+$ be fixed. Assume that $(H_1), (H_2'), (H_3)$ and (F_1) hold, and that there exist ψ and ζ, respectively, a subsolution and a supersolution to problem $(P_{\lambda,\mu})$ such that $\psi \le \zeta$. Then problem $(P_{\lambda,\mu})$ has at least one solution u such that $\psi \le u \le \zeta$.*

Proof. Define $\gamma : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2_+$ by, for $t \in [0,1]$ and $u = (u_1, u_2) \in \mathbb{R}^2$,

$$\gamma(t, u) := (\gamma_1(t, u_1), \gamma_2(t, u_2)),$$

where, for $i \in \{1, 2\}$, $\gamma_i : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ by

$$\gamma_i(t, s) := \begin{cases} \zeta_i(t), & s \ge \zeta_i(t), \\ s, & \psi_i(t) \le s \le \zeta_i(t), \\ \psi_i(t), & s \le \psi_i(t). \end{cases}$$

Let $\lambda > 0$ and $\mu \in \mathbb{R}^2_+$ be fixed, and consider the following modified problem

$$\begin{cases} (\Phi(u'(t)))' + \lambda h(t) \bullet f(t, \gamma(t, u(t))) = \theta, & t \in (0,1), \\ u(0) = \alpha^\gamma[u], \ u(1) = \mu, \end{cases} \tag{2.10}$$

where $\alpha^\gamma[u] := \int_0^1 \gamma(s, u(s))k(s)ds$.

For given $v \in X$, define $g_v : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g_v(x) := \int_0^1 \gamma(s, v(s) + (1-s)x + s\mu)k(s)ds.$$

Then g_v is a contraction mapping on $(\mathbb{R}^2, |\cdot|_{\mathbb{R}_2})$. Here, for $(x_1, x_2) \in \mathbb{R}^2$, $|(x_1, x_2)|_{\mathbb{R}_2} := \max(|x_1|, |x_2|)$. Indeed, since $|\gamma_i(t,s) - \gamma_i(t,\tau)| \le |s-\tau|$ for any $s, \tau \in \mathbb{R}$ and $t \in [0,1]$, for $\bar{v} = (\bar{v}_1, \bar{v}_2), \tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \mathbb{R}^2$,

$$\begin{aligned} |\gamma(t,\bar{v}) - \gamma(t,\tilde{v})|_{\mathbb{R}_2} &= \max\{|\gamma_1(t,\bar{v}_1) - \gamma_1(t,\tilde{v}_1)|, |\gamma_2(t,\bar{v}_2) - \gamma_2(t,\tilde{v}_2)|\} \\ &\le \max\{|\bar{v}_1 - \tilde{v}_1|, |\bar{v}_2 - \tilde{v}_2|\} = |\bar{v} - \tilde{v}|_{\mathbb{R}_2}. \end{aligned}$$

Then, for $s \in [0,1]$ and $x, y \in \mathbb{R}^2$,

$$|\gamma(s, v(s) + (1-s)x + s\mu) - \gamma(s, v(s) + (1-s)y + s\mu)|_{\mathbb{R}_2} \le (1-s)|x-y|_{\mathbb{R}_2},$$

and $|g_v(x) - g_v(y)|_{\mathbb{R}_2} \le C_k|x-y|_{\mathbb{R}_2}$. Then g_v is a contraction mapping on $(\mathbb{R}^2, |\cdot|_{\mathbb{R}_2})$ by (H_2'). Thus, for given $v \in X$, there is a unique solution $\beta^\gamma[v] \in \mathbb{R}^2$ of the equation $x = g_v(x)$, in other words, it is the unique element of $\mathbb{R}^2$ which satisfies that

$$\beta^\gamma[v] = \int_0^1 \gamma(s, v(s) + (1-s)\beta^\gamma[v] + s\mu)k(s)ds.$$

From this fact, it follows that

$$\alpha^\gamma[u] = \beta^\gamma[v]$$

under the transformations

$$u(t) := v(t) + (1-t)\beta^\gamma[v] + t\mu, \ \ v(t) := u(t) - ((1-t)\alpha^\gamma[u] + t\mu). \tag{2.11}$$

Thus (2.10) can be equivalently rewritten as follows:

$$\begin{cases} (\Phi(v'(t) - \beta^\gamma[v] + \mu))' + \lambda h(t) \bullet F^\gamma(v)(t) = \theta, \ t \in (0,1), \\ v(0) = v(1) = \theta, \end{cases} \tag{2.12}$$

where $F^\gamma := (F_1^\gamma, F_2^\gamma) : X \to X$ is defined by, for $v \in X$,

$$F^\gamma(v)(t) := f(t, \gamma(t, v(t) + (1-t)\beta^\gamma[v] + t\mu)), \quad t \in [0,1].$$

Consequently, v is a solution to problem (2.12) if and only if u is a solution to problem (2.10) under the transformations (2.11), respectively.

Now, define $T^\gamma = (T_1^\gamma, T_2^\gamma) : X \to \mathcal{P}$ by, for each $i = 1,2$ and $v \in X$,

$$T_i^\gamma(v)(t) := \begin{cases} (\beta_i^\gamma[v] - \mu_i)t + \int_0^t \varphi_{p_i}^{-1}\Big(\varphi_{p_i}(\mu_i - \beta_i^\gamma[v]) \\ \qquad + \lambda \int_s^{M_i^\gamma} h_i(\tau)F_i^\gamma(v)(\tau)d\tau\Big)ds, \quad 0 \le t \le M_i^\gamma, \\ -(\beta_i^\gamma[v] - \mu_i)(1-t) + \int_t^1 \varphi_{p_i}^{-1}\Big(\varphi_{p_i}(-\mu_i + \beta_i^\gamma[v]) \\ \qquad + \lambda \int_{M_i^\gamma}^s h_i(\tau)F_i^\gamma(v)(\tau)d\tau\Big)ds, \quad M_i^\gamma \le t \le 1, \end{cases}$$

where $M_i^\gamma = M_i^\gamma(v)$ is the constant satisfying

$$(\beta_i^\gamma[v] - \mu_i)M_i^\gamma + \int_0^{M_i^\gamma} \varphi_{p_i}^{-1}\left(\varphi_{p_i}(\mu_i - \beta_i^\gamma[v]) + \lambda \int_s^{M_i^\gamma} h_i(\tau)F_i^\gamma(v)(\tau)d\tau\right) ds$$
$$= -(\beta_i^\gamma[v] - \mu_i)(1 - M_i^\gamma) + \int_{M_i^\gamma}^1 \varphi_{p_i}^{-1}\left(\varphi_{p_i}(-\mu_i + \beta_i^\gamma[v]) + \lambda \int_{M_i^\gamma}^s h_i(\tau)F_i^\gamma(v)(\tau)d\tau\right) ds.$$

Then v is a fixed point of T^γ in X if and only if v is a solution to problem (2.12). It follows that T^γ is completely continuous on X and $T^\gamma(X)$ is bounded in X. Then, by Theorem 2.4, T^γ has a fixed point v, and consequently (2.10) has a solution u under the first transformation (2.11). Now if we prove that $\psi \le u \le \zeta$, then, by the definition of γ, ($P_{\lambda,\mu}$) has a solution u such that $\psi \le u \le \zeta$ and the proof is complete. In order to show $u \le \zeta$, assume on the contrary that $u_1 \not\le \zeta_1$ or $u_2 \not\le \zeta_2$. We only consider the case $u_1 \not\le \zeta_1$, since the case $u_2 \not\le \zeta_2$ can be dealt in a similar manner. Set $X_1(t) := u_1(t) - \zeta_1(t)$ for $t \in [0,1]$. Then, since $X_1(0) = u_1(0) - \zeta_1(0) \le \int_0^1 \sum_{i=1}^2 [\gamma_i(s, u_i(s)) - \zeta_i(s)]k_{i1}(s)ds \le 0$ and $X_1(1) = u_1(1) - \zeta_1(1) \le 0$, there exists $\sigma \in (0,1)$ such that $X_1(\sigma) = \max_{t\in[0,1]} X_1(t) > 0$. Then $X_1'(\sigma) = 0$ and we may assume that there is $a \in (\sigma, 1)$ such that $X_1'(t) < 0$ and $X_1(t) > 0$ for $t \in (\sigma, a]$, which imply that

$$u_1'(\sigma) = \zeta_1'(\sigma), \quad u_1'(t) < \zeta_1'(t), \quad \text{and} \quad u_1(t) > \zeta_1(t) \quad \text{for } t \in (\sigma, a]. \tag{2.13}$$

By the quasi-monotonicity of f_1 and (2.13), for $t \in (\sigma, a]$,

$$\begin{aligned} -(\varphi_{p_1}(u_1'(t)))' &= \lambda h_1(t) f_1(t, \gamma_1(t, u_1(t)), \gamma_2(t, u_2(t))) \\ &= \lambda h_1(t) f_1(t, \zeta_1(t), \gamma_2(t, u_2(t))) \\ &\le \lambda h_1(t) f_1(t, \zeta_1(t), \zeta_2(t)) \le -(\varphi_{p_1}(\zeta_1'(t)))'. \end{aligned}$$

For $t \in (\sigma, a]$, integrating this inequality from σ to t, we get

$$\varphi_{p_1}(u_1'(t)) \ge \varphi_{p_1}(\zeta_1'(t)), \quad \text{for } t \in (\sigma, a].$$

Since φ_{p_1} is monotone increasing, $u_1'(t) \ge \zeta_1'(t)$ for $t \in (\sigma, a]$, and it contradicts (2.13). Thus $u_1 \le \zeta_1$, and we can show that $u \le \zeta$. In a similar manner, it is shown that $\psi \le u$, and thus the proof is complete. □

3 Main results

First, we give a hypothesis which will be used in this section:

(F_∞) For each $i \in \{1, 2\}$, there exists an interval $I_i := [\theta_1^i, \theta_2^i] \subset (0,1)$ with $\theta_1^i < \theta_2^i$ such that

$$f_i^\infty := \lim_{|s_1|+|s_2|\to\infty} \frac{f_i(t, s_1, s_2)}{s_i^{p_i - 1}} = \infty \quad \text{uniformly in } t \in I_i.$$

Set

$$m_h := \min_{t \in I_1 \cup I_2} \{h_1(t), h_2(t)\} > 0. \tag{3.1}$$

Now we give *a priori* estimates for solutions as follows.

Lemma 3.1. *Assume that* (H_1)–(H_3) *and* (F_∞) *hold. Let* $\alpha \in (0, \infty)$ *be given. Then there exists* $M = M(\alpha) > 0$ *such that* $\|u\|_X < M$ *for any non-negative solutions* u *to problem* ($P_{\lambda,\mu}$) *for all* $\lambda \in [\alpha, \infty)$ *and* $\mu \in \mathbb{R}_+^2$.

Proof. Assume, on the contrary, that there exists a sequence $\{(\lambda^n, \mu^n, u^n)\} \subseteq [\alpha, \infty) \times \mathbb{R}^2_+ \times X$ such that $u^n = (u^n_1, u^n_2)$ is a positive solution of ($P_{\lambda,\mu}$) with $\lambda = \lambda^n$ and $\mu = \mu^n$ satisfying $\|u^n\|_X \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that $\|u^n_1\|_\infty \to \infty$ as $n \to \infty$. Set $A_\alpha := \frac{1}{\alpha m_h}\big(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}\big)^{p_1} + 1$, where θ^1_1, θ^1_2 are the constants defined in (F_∞), m_h is the constant defined in (3.1) and

$$\pi_{p_1} = \frac{2\pi(p_1 - 1)^{\frac{1}{p_1}}}{p_1 \sin\left(\frac{\pi}{p_1}\right)}.$$

By (F_∞), there exists $M = M(\alpha) > 0$ such that

$$f_1(t, s_1, s_2) > A_\alpha s_1^{p_1 - 1} \quad \text{for } (t, s_1, s_2) \in I_1 \times [\theta M, \infty) \times \mathbb{R}_+, \tag{3.2}$$

where $\theta = \min\{\theta^1_1, 1 - \theta^1_2\} > 0$. For large n, $\|u^n_1\|_\infty > M$ and it follows from the concavity of u^n_1 that

$$u^n_1(t) \geq \theta \|u^n_1\|_\infty \geq \theta M, \qquad t \in I_1.$$

Combining this with (3.2), $(\varphi_{p_1}((u^n_1)'(t)))' + \alpha A_\alpha m_h \varphi_{p_1}(u^n_1(t)) \leq 0,\ t \in I_1$. It is easy to check that $w_1(t) = S_{q_1}\big(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}(t - \theta^1_1)\big)$ is a solution of

$$\begin{cases} (\varphi_{p_1}(w_1'(t)))' + \left(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}\right)^{p_1} \varphi_{p_1}(w_1(t)) = 0, & t \in I_1, \\ w_1(\theta^1_1) = w_1(\theta^1_2) = 0, \end{cases}$$

where S_{q_1} is the q_1-sine function with $\frac{1}{p_1} + \frac{1}{q_1} = 1$ (e.g., see [8, 36]). Applying $y = w_1$, $z = u^n_1$, $b_1 = \big(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}\big)^{p_1}$ and $b_2 = \alpha A_\alpha m_h$ in (2.4) and integrating it from θ^1_1 to θ^1_2, we have $\int_{\theta^1_1}^{\theta^1_2} \big(\big(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}\big)^{p_1} - \alpha A_\alpha m_h\big)|w_1|^{p_1} dt \geq 0$. Thus $A_\alpha \leq \frac{1}{\alpha m_h}\big(\frac{\pi_{p_1}}{\theta^1_2 - \theta^1_1}\big)^{p_1}$. This contradicts the choice of A_α. □

Remark 3.2. (1) By Lemma 3.1, for any fixed $\alpha \in (0, \infty)$, there exist a positive constant M such that ($P_{\lambda,\mu}$) has no positive solutions if $\lambda \in [\alpha, \infty)$ and $\mu = (\mu_1, \mu_2)$ with $|\mu_1| + |\mu_2| \geq M$.

(2) Combining Lemma 3.1 with Remark 2.6, for a fixed $\alpha \in (0, \infty)$, there exists $M_1 = M_1(\alpha) \in (0, \infty)$ such that $\|v\|_X < M_1$ for any non-negative solutions v to problem ($\hat{P}_{\lambda,\mu}$) with $\lambda \in [\alpha, \infty)$ and $|\mu|_{\mathbb{R}^2} \leq \alpha$.

By (F_∞), there exists $b_0 > 0$ such that, for $i = 1, 2$,

$$f_i(t, s) > b_0 s_i^{p_i - 1} \quad \text{for } (t, s) \in I_i \times \mathbb{R}^2_+ \text{ with } s = (s_1, s_2).$$

By similar arguments as in the proof of Lemma 3.1, we have the following lemma. We omit the proof of it.

Lemma 3.3. *Assume that* (H_1)–(H_3) *and* (F_∞) *hold. Then there exists a positive constant* $\bar{\lambda} > 0$ *such that problem* ($P_{\lambda,\mu}$) *(or* ($\hat{P}_{\lambda,\mu}$)*) has no non-negative solutions for all* $\lambda > \bar{\lambda}$ *and all* $\mu \in \mathbb{R}^2_+$.

Theorem 3.4. *Assume that* (H_1)–(H_3) *and* (F_∞) *hold. For fixed* $\mu \in \mathbb{R}^2_+$*, there exists* $\underline{\lambda}(\mu) \in (0, \bar{\lambda})$ *such that* $(P_{\lambda,\mu})$ *has two positive solutions* $\bar{u}^\mu_\lambda$ *and* $\underline{u}^\mu_\lambda$ *for* $\lambda \in (0, \underline{\lambda}(\mu))$*. Moreover,* $\|\bar{u}^\mu_\lambda\|_X \to \infty$ *and* $\underline{u}^\mu_\lambda \to L^2[\mu, \theta]$ *in* X *as* $\lambda \to 0^+$*. Here,* L^2 *is the transformation defined in* (2.8).

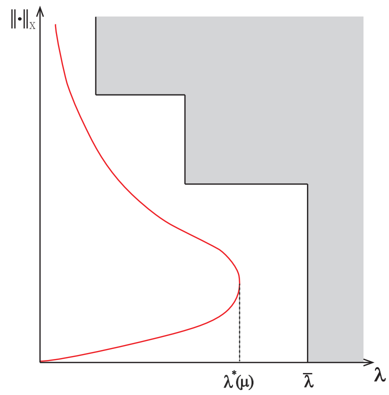

Figure 3.1: A possible bifurcation diagram of $\mathcal{C}_\mu(\mathcal{P})$ when μ is fixed in $\mathbb{R}^2_+$.

Proof. Let $\mu \in \mathbb{R}^2_+$ be fixed. Define $H_\mu(\lambda, v) = T(\lambda, \mu, v)$, then by Lemma 2.10, $H_\mu : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ is completely continuous and $H_\mu(0, v) = \theta$ for all $v \in \mathcal{P}$. By Theorem 2.3, there exists an unbounded continuum $\mathcal{C}_\mu(\mathcal{P})$, the component of the solution set of $H_\mu(\lambda, v) = v$ containing $(0, \theta) \in \mathbb{R}_+ \times X$. Since $H_\mu(\lambda, v) \neq \theta$ for $\lambda > 0$, for any solutions (λ, v) in $\mathcal{C}_\mu(\mathcal{P})$, v is a positive solution to problem $(\hat{P}_{\lambda,\mu})$ if $\lambda > 0$. By Remark 3.2 (2) and Lemma 3.3, $(\hat{P}_{\lambda,\mu})$ has at least two positive solutions $\bar{v}^\mu_\lambda$ and $\underline{v}^\mu_\lambda$ for small $\lambda > 0$ satisfying that $\|\bar{v}^\mu_\lambda\|_X \to \infty$ and $\underline{v}^\mu_\lambda \to \theta$ in X as $\lambda \to 0^+$ (see Figure 3.1). By Remark 2.6, the proof is complete. □

Define $\lambda^* : \mathbb{R}^2_+ \to (0, \infty)$ by, for $\mu \in \mathbb{R}^2_+$,

$$\lambda^*(\mu) = \sup\{\lambda > 0 : (P_{\lambda,\mu}) \text{ has a positive solution }\}. \tag{3.3}$$

Then, by Lemma 3.3 and Theorem 3.4, λ^* is well defined and $\lambda^*(\mu) \in (0, \infty)$ for all $\mu \in \mathbb{R}^2_+$.

Theorem 3.5. *Assume that (H_1), (H_2'), (H_3), (F_1) and (F_∞) hold. Let $\mu \in \mathbb{R}^2_+$ be fixed. Then $(P_{\lambda,\mu})$ has at least two positive solutions for $\lambda \in (0, \underline{\lambda}(\mu))$, has one positive solution for $\lambda \in [\underline{\lambda}(\mu), \lambda^*(\mu)]$, and has no positive solutions for $\lambda > \lambda^*(\mu)$.*

Proof. Let $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2_+$ be fixed. Clearly, there is no positive solutions for $\lambda > \lambda^*(\mu)$. Now we prove that $(\hat{P}_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu)$ has a positive solution $v^*(\mu)$. By definition of $\lambda^*(\mu)$, there is a sequence $\{\lambda_n\} \subset \mathbb{R}_+$ such that $(\hat{P}_{\lambda,\mu})$ with $\lambda = \lambda_n$ has a positive solution v_n, i.e., $T(\lambda_n, \mu, v_n) = v_n$ in $\mathcal{P}$ and $\lambda_n \to \lambda^*(\mu)$. By Lemma 3.1, $\|v_n\|_X < C$ for some $C > 0$. By compactness of T, $\{v_n\}$ has a convergent subsequence converging to, say $v^*(\mu)$ and by continuity of T, we see that $v^*(\mu)$ is a positive solution to problem $(\hat{P}_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu)$, since $\lambda^*(\mu) > 0$. Then $u^*(\mu) = L^2(\mu, v^*(\mu))$ is a positive solution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu)$. Since $u^*(\mu)$ is a supersolution and θ is a trivial subsolution to problem $(P_{\lambda,\mu})$ for $\lambda \in (0, \lambda^*(\mu)]$, by Theorem 2.12, $(P_{\lambda,\mu})$ has at least one positive solution for $\lambda \in (0, \lambda^*(\mu)]$, and thus the proof is complete in view of Theorem 3.4. □

Lemma 3.6. *Assume that (H_1), (H_2'), (H_3), (F_1) and (F_∞) hold. Then:*

(1) *if $\mu^1 \le \mu^2$ in $\mathbb{R}^2$, then $0 < \lambda^*(\mu^2) \le \lambda^*(\mu^1)$;*

(2) *$\lambda^*(\mu) \to 0$ as $|\mu|_{\mathbb{R}^2} \to \infty$;*

(3) *if $\mu^n \to \mu^0$ in $\mathbb{R}^2_+$, then $\limsup_{n\to\infty} \lambda^*(\mu^n) \le \lambda^*(\mu^0)$.*

Proof. (1) Let $\mu^1 \leq \mu^2$ in $\mathbb{R}^2$. Then it suffices to show that $(P_{\lambda,\mu})$ has a positive solution for $\lambda = \lambda^*(\mu^2)$ and $\mu = \mu^1$. Let u^2 be a positive solution of $(P_{\lambda,\mu})$ with $\mu = \mu^2$ and $\lambda = \lambda^*(\mu^2)$, then u^2 is a supersolution and θ is a trivial subsolution of $(P_{\lambda,\mu})$ with $\mu = \mu^1$ and $\lambda = \lambda^*(\mu^2)$. Then, by Theorem 2.12, $(P_{\lambda,\mu})$ has a positive solution u^1 satisfying $\theta \leq u^1 \leq u^2$ for $\lambda = \lambda^*(\mu^2) > 0$ and $\mu = \mu^1$. Consequently, $\lambda^*(\mu^2) \leq \lambda^*(\mu^1)$ by (3.3).

(2) From Remark 3.2 (1), it follows that $\lambda^*(\mu) \to 0$ as $|\mu|_{\mathbb{R}^2} \to \infty$.

(3) Let $\mu^n \to \mu^0$ in $\mathbb{R}^2_+$ as $n \to \infty$. By (1), $\{\lambda^*(\mu^n)\}$ is a bounded sequence in $\mathbb{R}_+$. Then there exists a subsequence of $\{\lambda^*(\mu^n)\}$, denote it again $\{\lambda^*(\mu^n)\}$, such that, as $n \to \infty$, $\lambda^*(\mu^n) \to \limsup_{n\to\infty} \lambda^*(\mu^n) > 0$, and there exists a sequence $\{v^n\}$ such that v^n is a positive solution to problem $(\hat{P}_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu^n)$ and $\mu = \mu^n$, that is, $T(\lambda^*(\mu^n), \mu^n, v^n) = v^n$. By Lemma 3.1, $\{(\lambda^*(\mu^n), \mu^n, v^n)\}$ is a bounded sequence in $\mathbb{R}^3_+ \times \mathcal{P}$. Then, by the complete continuity T, there exists a subsequence of $\{(\lambda^*(\mu^n), \mu^n, v^n)\}$, denote it again $\{(\lambda^*(\mu^n), \mu^n, v^n)\}$, such that, as $n \to \infty$, $v^n = T(\lambda^*(\mu^n), \mu^n, v^n) \to V$ in X and $T\left(\limsup_{n\to\infty} \lambda^*(\mu^n), \mu^0, V\right) = V$. Thus, V is a positive solution to problem $(\hat{P}_{\lambda,\mu})$ with $\lambda = \limsup_{n\to\infty} \lambda^*(\mu^n) > 0$ and $\mu = \mu^0$, and by definition of λ^*, the proof is complete. □

Lemma 3.7. *Assume that (H_1), (H_3), (F_1) and (F_∞) hold. Assume in addition that*

$$(H_2'')\ \ 0 \leq \max_{i\in\{1,2\}} \left\{ \int_0^1 [k_{1i}(s) + k_{2i}(s)]ds \right\} \leq 1.$$

Let $\mu^0 \in \mathbb{R}^2_+$ be fixed. Then for any $\lambda \in (0, \lambda^*(\mu^0))$, there exists a positive constant $\delta = \delta(\lambda, \mu^0)$ such that $\zeta = u^* + (\delta, \delta)$ is a supersolution to problem $(P_{\lambda,\mu})$ with $\mu = \mu_\delta$. Here $\mu_\delta = \mu^0 + (\delta, \delta)$ and u^* is a positive solution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu^0)$ and $\mu = \mu^0$.

Proof. Note that (H_2'') implies (H_2'). Let $\mu^0 \in \mathbb{R}^2_+$ and $\lambda_0 \in (0, \lambda^*(\mu^0))$ be fixed. Let u^* be a positive solution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu^0)$ and $\mu = \mu^0$. Put

$$\epsilon = \left(\frac{\lambda^*(\mu^0)}{\lambda_0} - 1 \right) \min_{t\in[0,1]} \{ f_1(t, u^*(t)), f_2(t, u^*(t)) \} > 0. \tag{3.4}$$

Then there exists $\delta = \delta(\lambda_0, \mu^0) > 0$ such that, for $i = 1, 2$,

$$|f_i(t, u + (\delta, \delta)) - f_i(t, u)| < \epsilon \quad \text{for } t \in [0,1] \quad \text{and} \quad u \in [0, \|u^*\|_X] \times [0, \|u^*\|_X]. \tag{3.5}$$

Set $\zeta := u^* + (\delta, \delta)$. Then, by (3.4) and (3.5), for $i \in \{1, 2\}$,

$$\begin{aligned}
&(\varphi_{p_i}(\zeta_i'(t)))' + \lambda_0 h_i(t) f_i(t, \zeta(t)) \\
&\quad = -\lambda^*(\mu^0) h_i(t) f_i(t, u^*(t)) + \lambda_0 h_i(t) f_i(t, \zeta(t)) \\
&\quad < -\lambda^*(\mu^0) h_i(t) f_i(t, u^*(t)) + \lambda_0 h_i(t) [f_i(t, u^*(t)) + \epsilon] \\
&\quad = -\lambda_0 h_i(t) \left[\left(\frac{\lambda^*(\mu^0)}{\lambda_0} - 1 \right) f_i(t, u^*(t)) - \epsilon \right] \leq 0,
\end{aligned}$$

which implies

$$(\Phi(\zeta'(t)))' + \lambda_0 h(t) f(t, \zeta(t)) < \theta, \qquad t \in (0, 1). \tag{3.6}$$

By (H_2''), $\zeta(0) = \int_0^1 u^*(s)k(s)ds + (\delta, \delta) \geq \int_0^1 \zeta(s)k(s)ds$, and since $\zeta(1) = \mu + (\delta, \delta) > \mu$, ζ is a supersolution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda_0$ and $\mu = \mu_\delta$. Thus the proof is complete. □

Remark 3.8. (1) If we assume that

$$(H_2''')\ 0 \leq \max_{i\in\{1,2\}} \left\{ \int_0^1 [k_{1i}(s) + k_{2i}(s)]ds \right\} < 1$$

holds in Lemma 3.7 instead of (H_2''), then the supersolution ζ satisfies $\zeta(0) > \int_0^1 \zeta(s)k(s)ds$.

(2) Assume (H_2''') holds. Then (H_2'') and (H_3) are satisfied. In fact, clearly (H_2'') is satisfied. Since

$$1 - \int_0^1 (1-s)k_{11}(s)ds \geq 1 - \int_0^1 k_{11}(s)ds > \int_0^1 k_{21}(s)ds$$

and

$$1 - \int_0^1 (1-s)k_{22}(s)ds \geq 1 - \int_0^1 k_{22}(s)ds > \int_0^1 k_{12}(s)ds,$$

(H_3) is also satisfied.

On the other hand, if $k_{11} \equiv k_{22} \equiv 0$ and

$$\int_0^1 k_{12}(s)ds = 1 \quad \text{and} \quad 0 \leq \int_0^1 k_{21}(s)ds < 1,$$

then (H_2'') and (H_3) are satisfied, but (H_2''') is not satisfied.

Proposition 3.9. *Assume that $(H_1), (H_2''), (H_3)$, (F_1) and (F_∞) hold. Then $\lambda^* : \mathbb{R}^2_+ \to (0, \infty)$ is a continuous function satisfying $0 < \lambda^*(\mu^2) \leq \lambda^*(\mu^1)$ for any $\mu^1 \leq \mu^2$ in $\mathbb{R}^2_+$ and $\lambda^*(\mu) \to 0$ as $|\mu|_{\mathbb{R}^2_+} \to \infty$ (see, e.g., Figure 3.2).*

Proof. Let $\mu^0 \in \mathbb{R}^2_+$ be fixed and let $m \in \mathbb{N}$ be fixed such that $\lambda_m = \lambda^*(\mu^0) - 1/m > 0$. Then by Lemma 3.7, there exists $\delta_m = \delta(\lambda_m, \mu^0) > 0$ such that problem $(P_{\lambda,\mu})$ has an upper solution $u(\delta_m)$ for $\lambda = \lambda_m$ and $\mu = \mu_{\delta_m}$. Here, $\mu_{\delta_m} = \mu^0 + (\delta_m, \delta_m)$. Let $\{\mu^n\}$ be a sequence in $\mathbb{R}^2_+$ such that $\mu^n \to \mu^0$ in $\mathbb{R}^2_+$ as $n \to \infty$. For sufficiently large n, $\mu^n \leq \mu_{\delta_m}$, and $u(\delta_m)$ is also a supersolution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda_m$ and $\mu = \mu^n$. Since θ is a trivial subsolution to problem $(P_{\lambda,\mu})$ for all $\lambda \in (0, \infty)$ and $\mu \in \mathbb{R}^2_+$, by Theorem 2.12, $(P_{\lambda,\mu})$ has at least one positive solution for $\lambda = \lambda_m$ and $\mu = \mu^n$. Thus $\lambda_m \leq \lambda^*(\mu^n)$ for all large n and

$$\lambda^*(\mu^0) - \frac{1}{m} = \lambda_m \leq \liminf_{n\to\infty} \lambda^*(\mu^n).$$

Letting $m \to \infty$, $\lambda^*(\mu^0) \leq \liminf_{n\to\infty} \lambda^*(\mu^n)$. Combining this with Lemma 3.6 (3), it follows that $\lambda^* : \mathbb{R}^2_+ \to (0, \infty)$ is a continuous function. Thus the proof is complete in view of Lemma 3.6 (1) and (2). □

Let $\mathcal{C} := \lambda^*(\mathbb{R}^2_+)$. Then, by Proposition 3.9, $\mathcal{C}$ is a continuous surface in $\mathbb{R}^3_+$, and it separates $\mathbb{R}^3_+$ into two regions

$$\mathcal{O}_1 := \{(\mu, \lambda) \in \mathbb{R}^3_+ : \lambda \in (0, \lambda^*(\mu)),\ \mu \in \mathbb{R}^2_+\}$$

and

$$\mathcal{O}_2 := \{(\mu, \lambda) \in \mathbb{R}^3_+ : \lambda > \lambda^*(\mu),\ \mu \in \mathbb{R}^2_+\}.$$

Moreover, by Theorem 3.5, $(P_{\lambda,\mu})$ has at least one positive solutions for $(\mu, \lambda) \in \mathcal{O}_1 \cup \mathcal{C}$ and no positive solution for $(\mu, \lambda) \in \mathcal{O}_2$.

Finally we give the main result in this paper.

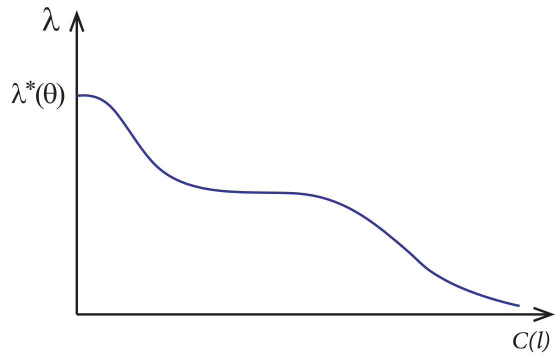

Figure 3.2: A possible graph of λ^* on $C(l)$, where $C(l) = \{(x, lx) : x \geq 0\}$ for $l \in \mathbb{R}_+$ and $C(\infty) = \{(0, y) : y \geq 0\}$.

Theorem 3.10. *Assume that* $(H_1), (H_2''), (H_3)$, (F_1) *and* (F_∞) *hold. Then* $(P_{\lambda,\mu})$ *has two positive solutions for* $(\mu, \lambda) \in \mathcal{O}_1$*, at least one positive solution for* $(\mu, \lambda) \in \mathcal{C}$*, and no positive solutions for* $(\mu, \lambda) \in \mathcal{O}_2$.

Proof. For fixed $\mu \in \mathbb{R}^2_+$, we will show that $(P_{\lambda,\mu})$ has at least two positive solutions for $\lambda \in (0, \lambda^*(\mu))$, and thus the proof is complete by Theorem 3.5.

Let $\mu \in \mathbb{R}^2_+$ be fixed. Define $T_\mu : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ by, for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$ and $t \in [0, 1]$,

$$T_\mu(\lambda, u)(t) := T(\lambda, \mu, L^1(\mu, u))(t) + (1 - t)\alpha[u] + t\mu.$$

Here, $L^1 : \mathbb{R}^2_+ \times X \to X$ is the continuous mapping defined in (2.7), and it maps bounded sets in $\mathbb{R}_+ \times X$ into bounded sets in X. Then T_μ is completely continuous, $T_\mu(\lambda, u)(0) = \alpha[u]$ and $T_\mu(\lambda, u)(1) = \mu$. Moreover, by Lemma 2.5 and Remark 2.6, $T_\mu(\lambda, u) = u$ in $\mathcal{P}$ if and only if u is a non-negative solution to problem $(P_{\lambda,\mu})$.

Let $\lambda_0 \in (0, \lambda^*(\mu))$ be fixed and u^* be a positive solution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda^*(\mu)$. Then, by Lemma 3.7, there exists $\delta_1 \in (0, 1)$ such that $\zeta = u^* + (\delta_1, \delta_1)$ is a supersolution to $(P_{\lambda,\mu})$ with $\lambda = \lambda_0$. Moreover, by (3.6), ζ satisfies

$$\begin{cases} (\Phi(\zeta'(t)))' + \lambda_0 h(t) \bullet f(t, \zeta(t)) < \theta, \quad t \in (0, 1), \\ \zeta(0) = u^*(0) + (\delta_1, \delta_1) \geq \alpha[\zeta],\ \zeta(1) = \mu + (\delta_1, \delta_1) > \mu. \end{cases} \tag{3.7}$$

Consider the following problem

$$\begin{cases} (\Phi(u'(t)))' + \lambda_0 h(t) \bullet f(t, \gamma(t, u(t))) = \theta, \quad t \in (0, 1), \\ u(0) = \alpha^\gamma[u],\ u(1) = \mu, \end{cases} \tag{3.8}$$

where $\alpha^\gamma[u]$ and γ are defined in the same way as in the proof of Theorem 2.12 with $\psi = \theta$ and $\zeta = u^* + (\delta_1, \delta_1)$. Then if u is a solution to problem (3.8), then $\theta \leq u \leq \zeta$ by the same argument as in the proof of Theorem 2.12, and $\gamma(t, u(t)) = u(t)$ for $t \in (0, 1)$. Thus u must be a positive solution to problem $(P_{\lambda,\mu})$ with $\lambda = \lambda_0$.

Set $\Gamma = \{u \in X : -(\delta_1, \delta_1) < u < \zeta\}$. Then Γ is an open set containing θ in X. Now we will show that if u is a solution to problem (3.8), then $u \in \Gamma \cap \mathcal{P}$. Let u be a solution to problem (3.8). First, we show that $u(t) < \zeta(t)$ for $t \in (0, 1]$. If not, since $u(1) = \mu < \mu + (\delta_1, \delta_1) = \zeta(1)$, there exist $t_0 \in (0, 1)$ and $j \in \{1, 2\}$ such that $u_j(t_0) = \zeta_j(t_0)$, $u_j'(t_0) = \zeta_j'(t_0)$ and $u_j(t) < \zeta_j(t)$ for $t \in (t_0, 1]$. We take $j = 1$ for convenience. By (3.7), for fixed $t_0' \in (t_0, 1)$,

$$\max_{t \in [t_0, t_0']} \{(\varphi_{p_1}(\zeta_1'(t)))' + \lambda_0 h_1(t) f_1(t, \zeta(t))\} =: -\epsilon_1 < 0. \tag{3.9}$$

Since f_1 is uniformly continuous on $[0,1] \times [0, \|u_1^*\|_\infty + 1] \times [0, \|u_2^*\|_\infty + 1]$, there exists $\delta_2 > 0$ such that if $|v - w|_{\mathbb{R}^2} < \delta_2$ and $v, w \in [0, ||\zeta_1||_\infty] \times [0, ||\zeta_2||_\infty]$, then

$$|f_1(t,v) - f_1(t,w)| < \epsilon_2, \tag{3.10}$$

where $\epsilon_2 = \epsilon_1 \big(\lambda_0 \max_{t\in[t_0,t_0']} h_1(t)\big)^{-1} > 0$. Then there exists $t_1 \in (t_0, t_0')$ such that $-\delta_2 < u_1(t) - \zeta_1(t) < 0$ for $t \in [t_0, t_1]$ and $u_1'(t_1) < \zeta_1'(t_1)$, which imply

$$\varphi_{p_1}(u_1'(t_0)) - \varphi_{p_1}(\zeta_1'(t_0)) = 0, \varphi_{p_1}(u_1'(t_1)) - \varphi_{p_1}(\zeta_1'(t_1)) < 0. \tag{3.11}$$

By (F_1) and (3.10), for $t \in [t_0, t_1]$,

$$f_1(t, u(t)) - f_1(t, \zeta(t)) \le f_1(t, u_1(t), \zeta_2(t)) - f_1(t, \zeta_1(t), \zeta_2(t)) < \epsilon_2, \tag{3.12}$$

since $u_2 \le \zeta_2$. By (3.9), (3.11), (3.12) and the choice of ϵ_2,

$$\begin{aligned}
0 &> \varphi_{p_1}(u_1'(t_1)) - \varphi_{p_1}(\zeta_1'(t_1)) - \varphi_{p_1}(u_1'(t_0)) + \varphi_{p_1}(\zeta_1'(t_0)),\\
&= \int_{t_0}^{t_1} \left\{(\varphi_{p_1}(u_1'(t)))' - (\varphi_{p_1}(\zeta_1'(t)))'\right\} dt\\
&= \int_{t_0}^{t_1} \left[-\lambda_0 h_1(t) f_1(t, u(t)) - (\varphi_{p_1}(\zeta_1'(t)))'\right] dt\\
&> \int_{t_0}^{t_1} \left[-\lambda_0 h_1(t)(f_1(t,\zeta(t)) + \epsilon_2) - (\varphi_{p_1}(\zeta_1'(t)))'\right] dt\\
&= \int_{t_0}^{t_1} \left(-\lambda_0 h_1(t)\epsilon_2 - \left[(\varphi_{p_1}(\zeta_1'(t)))' + \lambda_0 h_1(t) f_1(t,\zeta(t))\right]\right) dt\\
&\ge \int_{t_0}^{t_1} (-\lambda_0 h_1(t)\epsilon_2 + \epsilon_1) dt \ge 0.
\end{aligned}$$

This is a contradiction, and thus $u(t) < \zeta(t)$ for $t \in (0, 1]$.

Now we show that $u(0) < \zeta(0)$. We only show that $u_1(0) < \zeta_1(0)$, since the case $u_2(0) < \zeta_2(0)$ can be proved in a similar manner. If $k_{11} \equiv k_{21} \equiv 0$, $u(0) = 0 < \delta = \zeta_1(0)$. If $k_{11} \not\equiv 0$ or $k_{21} \not\equiv 0$, then

$$u_1(0) - \zeta_1(0) = \int_0^1 [(u_1(s) - \zeta_1(s))k_{11}(s) + (u_2(s) - \zeta_2(s))k_{21}(s)] ds < 0.$$

Thus $u \in \Gamma \cap \mathcal{P}$ for any solution u to problem (3.8).

Define $\overline{T}^\gamma : \mathcal{P} \to \mathcal{P}$ by, for $u \in \mathcal{P}$ and $t \in [0, 1]$,

$$\overline{T}^\gamma(u)(t) = T^\gamma(L_\gamma^1(u))(t) + (1-t)\alpha^\gamma[u] + t\mu,$$

where T^γ is the operator with $\lambda = \lambda_0$ defined in the proof of Theorem 2.12 and L_γ^1 is an operator defined by $L_\gamma^1(u)(t) := u(t) - ((1-t)\alpha^\gamma[u] + t\mu)$. Then $\overline{T}^\gamma$ is completely continuous on $\mathcal{P}$, and $\overline{T}^\gamma$ has a fixed point in $\mathcal{P}$ if and only if u is a non-negative solution to problem (3.8). Moreover, there exists a positive constant R such that $\overline{T}^\gamma(u) < R$ for all $u \in \mathcal{P}$ and $\Gamma \subset B_R$, where B_R is an open ball with center θ and radius R in X. Applying Theorem 2.4 with $\mathcal{O} = B_R$, $i(\overline{T}^\gamma, B_R \cap \mathcal{P}, \mathcal{P}) = 1$. Since all fixed points of $\overline{T}^\gamma$ are contained in Γ, by the excision property, $i(\overline{T}^\gamma, \Gamma \cap \mathcal{P}, \mathcal{P}) = i(\overline{T}^\gamma, B_R \cap \mathcal{P}, \mathcal{P}) = 1$. Since problem $(P_{\lambda,\mu})$ with $\lambda = \lambda_0$ is equivalent to problem (3.8) on $\Gamma \cap \mathcal{P}$, $(P_{\lambda,\mu})$ has a positive solution in $\Gamma \cap \mathcal{P}$ for $\lambda = \lambda_0 \in (0, \lambda^*(\mu))$. Assume that $T_\mu(\lambda_0,\cdot)$ has no fixed points in $\partial\Gamma\cap\mathcal{P}$, otherwise we get second positive solution to problem ($P_{\lambda,\mu}$) with $\lambda=\lambda_0$ and the proof is done. In that case,

$$i(T_\mu(\lambda_0,\cdot),\Gamma\cap\mathcal{P},\mathcal{P})=i(\overline{T}^\gamma,\Gamma\cap\mathcal{P},\mathcal{P})=1. \tag{3.13}$$

On the other hand, by Lemma 3.3, there is $\lambda_1(>\lambda_0)$ such that ($P_{\lambda,\mu}$) has no positive solution at $\lambda=\lambda_1$. This implies that $T_\mu(\lambda_1,\cdot)$ has no fixed point in $\mathcal{P}$. Thus for any open set $\mathcal{U}$ in X, we have

$$i(T_\mu(\lambda_1,\cdot),\mathcal{U}\cap\mathcal{P},\mathcal{P})=0. \tag{3.14}$$

By Lemma 3.1, we may choose $R_1>0$ such that $\overline{\Gamma}\subset B_{R_1}$ and all possible solutions u of ($P_{\lambda,\mu}$) for any $\lambda\in[\lambda_0,\lambda_1]$ satisfy $u\in B_{R_1}$. Here, B_{R_1} is an open ball with center θ and radius R_1 in X. Define a homotopy $g:[0,1]\times(\overline{B_{R_1}}\cap\mathcal{P})\to\mathcal{P}$ by $g(\tau,u)=T_\mu(\tau\lambda_1+(1-\tau)\lambda_0,u)$. Then g is completely continuous on $[0,1]\times\mathcal{P}$. Furthermore, by Lemma 3.1, $g(\tau,u)\neq u$ for all $(\tau,u)\in[0,1]\times(\partial B_{R_1}\cap\mathcal{P})$. Thus by homotopy invariance property and (3.14), we have

$$i(T_\mu(\lambda_0,\cdot),B_{R_1}\cap\mathcal{P},\mathcal{P})=i(T_\mu(\lambda_1,\cdot),B_{R_1}\cap\mathcal{P},\mathcal{P})=0. \tag{3.15}$$

Combining (3.13) and (3.15) with the additive property, we have

$$i(T_\mu(\lambda_0,\cdot),(B_{R_1}\setminus\overline{\Gamma})\cap\mathcal{P},\mathcal{P})=-1.$$

Consequently ($P_{\lambda,\mu}$) has another positive solution in $(B_{R_1}\setminus\overline{\Gamma})\cap\mathcal{P}$ for any $\lambda=\lambda_0\in(0,\lambda^*(\mu))$, and this completes the proof. □

4 Applications

In this section, we investigate the existence of infinitely many positive radial solutions to problems (1.3) + (1.4) or (1.3) + (1.5).

We assume the following assumptions hold in this section.

(A_1) For $i\in\{1,2\}$, there exists $\alpha>p-1$ such that $\int_{r_0}^\infty r^\alpha K_i(r)dr<\infty$.

(A_2) $0\le\max_{i\in\{1,2\}}\left\{\int_\Omega[l_{1i}(|x|)+l_{2i}(|x|)]dx\right\}<1$.

($\hat{F}_1$) For fixed $(r,u)\in[r_0,\infty)\times\mathbb{R}_+$, $\hat{f}_1=\hat{f}_1(r,u,v)$ is quasi-monotone nondecreasing with respect to v, i.e., $\hat{f}_1(r,u,v_1)\le\hat{f}_1(r,u,v_2)$ whenever $0\le v_1\le v_2$. For fixed $(r,v)\in[r_0,\infty)\times\mathbb{R}_+$, $\hat{f}_2=\hat{f}_2(r,u,v)$ is quasi-monotone nondecreasing with respect to u.

($\hat{F}_\infty$) For each $i\in\{1,2\}$, there exists a non-degenerate compact interval $J_i\subset(r_0,\infty)$ such that

$$\lim_{|s_1|+|s_2|\to\infty}\frac{\hat{f}_i(r,s_1,s_2)}{s_i^{p-1}}=\infty \quad \text{uniformly in } r\in J_i.$$

By applying consecutive changes of variables, $r=|x|$, $w(r)=z(x)$ and $t=1-\left(\frac{r}{r_0}\right)^{\frac{-N+p}{p-1}}$, $u(t)=w(r)$, problem (1.3) + (1.4) with $z(x)=(z_1(x),z_2(x))\to\mu\in\mathbb{R}^2_+$ as $|x|\to\infty$ is equivalently transformed into problem ($P_{\lambda,\mu}$) with $p_1 = p_2 = p > 1$. Here f_i, h_i and k_{ij} for $i, j \in \{1, 2\}$ are given by

$$f_i(t, s_1, s_2) = \hat{f}_i\left(r_0(1-t)^{\frac{-(p-1)}{N-p}}, s_1, s_2\right),$$
$$h_i(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p (1-t)^{\frac{-p(N-1)}{N-p}} K_i\left(r_0(1-t)^{\frac{-(p-1)}{N-p}}\right),$$
$$k_{ij}(t) = w_N\left(\frac{p-1}{N-p}\right) r_0^N (1-t)^{\frac{-p(N-1)}{N-p}} l_{ij}\left(r_0(1-t)^{\frac{-(p-1)}{N-p}}\right),$$

where w_N is the surface area of unit sphere in $\mathbb{R}^N$.

Assume that $(A_1), (A_2)$, $(\hat{F}_1)$ and $(\hat{F}_\infty)$ hold. Then $(H_1), (H_2''), (H_3)$ and (F_∞) hold. In fact, if K_i $(i = 1, 2)$ satisfies (A_1), then there exists $\beta < p - 1$ such that

$$\int_0^1 (1-s)^\beta h_i(s)ds < \infty,$$

which implies that $h_i \in \mathcal{A}_i$, and thus (H_1) holds. Since

$$\int_\Omega l_{ij}(|x|)dx = w_N \int_{r_0}^\infty l_{ij}(r) r^{N-1} dr = \int_0^1 k_{ij}(s)ds,$$

if we assume (A_2) is satisfied, then (H_2''') is satisfied, and (H_2'') and (H_3) are satisfied by Remark 3.8. If we assume $(\hat{F}_1)$ and $(\hat{F}_\infty)$ are satisfied, then (F_1) and (F_∞) are satisfied.

In a similar manner, by applying consecutive changes of variables, $r = |x|$, $w(r) = z(x)$ and $t = \left(\frac{r}{r_0}\right)^{\frac{-N+p}{p-1}}$, $u(t) = w(r)$, problem (1.3) + (1.5) with $z(x) = (z_1(x), z_2(x)) = \mu \in \mathbb{R}^2_+$ on $|x| = r_0$ is equivalently transformed into problem ($P_{\lambda,\mu}$) with $p_1 = p_2 = p > 1$. Here, f_i, h_i and k_{ij} are given by

$$f_i(t, s_1, s_2) = \hat{f}_i\left(r_0 t^{\frac{-(p-1)}{N-p}}, s_1, s_2\right)$$
$$h_i(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{-p(N-1)}{N-p}} K_i\left(r_0 t^{\frac{-(p-1)}{N-p}}\right),$$
$$k_{ij}(t) = w_N\left(\frac{p-1}{N-p}\right) r_0^N t^{\frac{-p(N-1)}{N-p}} l_{ij}\left(r_0 t^{\frac{-(p-1)}{N-p}}\right),$$

Then if $(A_1), (A_2)$, $(\hat{F}_1)$ and $(\hat{F}_\infty)$ hold, then $(H_1), (H_2''), (H_3)$, (F_1) and (F_∞) hold by similar arguments as above.

By Proposition 3.9, $\lambda^* : \mathbb{R}^2_+ \to (0, \infty)$ is a continuous function satisfying $\lambda^*(\mu^1) \geq \lambda^*(\mu^2)$ for $\mu^1 \leq \mu^2$ and $\lambda^*(\mu) \to 0$ as $|\mu|_{\mathbb{R}^2} \to \infty$ (see, e.g., Figure 3.2). Thus the inverse image $(\lambda^*)^{-1}(\{\lambda\})$ is a nonempty connected component in $\mathbb{R}^2_+$ for all $\lambda \in (0, \lambda^*(\theta))$, and we denote it $E(\lambda)$. Then the connected component $E(\lambda)$ separates $\mathbb{R}^2_+$ into a bounded region $U_1(\lambda)$ and an unbounded region $U_2(\lambda)$ which are open relative to $\mathbb{R}^2_+$, and by Theorem 3.10, ($P_{\lambda,\mu}$) has at least two positive solutions for $\mu \in U_1(\lambda)$, at least one positive solution for $\mu \in C(\lambda)$, and no positive solutions for $\mu \in U_2(\lambda)$ (see, e.g., Figure 4.1). Moreover, $U_1(\lambda_2) \subset U_1(\lambda_1)$ for $0 < \lambda_1 < \lambda_2 < \lambda^*(\theta)$. Thus we have the following corollary.

Corollary 4.1. *Assume that (A_1), (A_2), $(\hat{F}_1)$ and $(\hat{F}_\infty)$ hold. Then the following statements hold.*

(1) *For $\lambda \in (0, \lambda^*(\theta))$, $E(\lambda)$ is a connected component in $\mathbb{R}^2_+$ which separates $\mathbb{R}^2_+$ into a bounded region $U_1(\lambda)$ and an unbounded region $U_2(\lambda)$ which are open relative to $\mathbb{R}^2_+$ such that the followings hold:*

(i) *problem* (1.3) + (1.4) *(resp.* (1.3) + (1.5)*) has infinitely many positive radial solutions* $\bar{z}_\lambda^\mu$, $\underline{z}_\lambda^\mu$ *with* $\bar{z}_\lambda^\mu \neq \underline{z}_\lambda^\mu$ *satisfying*

$$\lim_{|x|\to\infty} \bar{z}_\lambda^\mu(x) = \lim_{|x|\to\infty} \underline{z}_\lambda^\mu = \mu \quad (\textit{resp. } \bar{z}_\lambda^\mu(x) = \underline{z}_\lambda^\mu = \mu \quad \textit{for } |x| = r_0) \quad \textit{for } \mu \in U_1(\lambda);$$

(ii) *problem* (1.3) + (1.4) *(resp.* (1.3) + (1.5)*) has infinitely many positive radial solutions* z_λ^μ *satisfying*

$$\lim_{|x|\to\infty} z_\lambda^\mu(x) = \mu \quad (\textit{resp. } z_\lambda^\mu(x) = \mu \quad \textit{for } |x| = r_0) \quad \textit{for } \mu \in \mathcal{C}_\lambda;$$

(iii) *problem* (1.3) + (1.4) *(resp.* (1.3) + (1.5)*) has no positive radial solutions* z_λ^μ *satisfying*

$$\lim_{|x|\to\infty} z_\lambda^\mu(x) = \mu \quad (\textit{resp. } z_\lambda^\mu(x) = \mu \quad \textit{for } |x| = r_0)) \quad \textit{for } \mu \in U_2(\lambda);$$

(iv) *moreover,* $U_1(\lambda_2) \subset U_1(\lambda_1)$ *for* $0 < \lambda_1 < \lambda_2 < \lambda^*(\theta)$.

(2) *For* $\lambda = \lambda^*(\theta)$, *problem* (1.3) + (1.4) *(resp.* (1.3) + (1.5)*) has at least one positive radial solution* z^* *satisfying*

$$\lim_{|x|\to\infty} z^*(x) = \theta \quad (\textit{resp. } z^*(x) = \theta \quad \textit{for } |x| = r_0).$$

(3) *For* $\lambda > \lambda^*(\theta)$, *problem* (1.3) + (1.4) *(resp.* (1.3) + (1.5)*) has no positive radial solutions* z *satisfying*

$$\lim_{|x|\to\infty} z(x) = \mu \quad (\textit{resp. } z(x) = \mu \quad \textit{for } |x| = r_0) \quad \textit{for any } \mu \in \mathbb{R}_+^2.$$

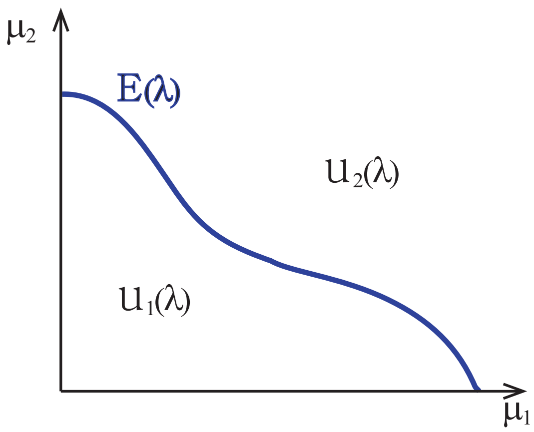

Figure 4.1: A possible picture of $E(\lambda)$, $U_1(\lambda)$ and $U_2(\lambda)$ for $\lambda \in (0, \lambda^*(\theta))$.

Finally, we give the examples satisfying the hypotheses (A_1), (A_2), $(\hat{F}_1)$ and $(\hat{F}_\infty)$ to illustrate Corollary 4.1.

Example 4.2. Let $r_0 = 1$ and $\Omega = \{x \in \mathbb{R}^N : |x| > 1\}$, where $N > p > 1$. For $i \in \{1,2\}$, let $K_i(r) = r^{\beta_i}$ for $r \in [1,\infty)$, where $\beta_i < -p$. Then $p - 1 < -\beta_i - 1$, and there exists α such that $p - 1 < \alpha < \min\{-\beta_1, -\beta_2\} - 1$. Consequently, $\alpha + \beta_i < -1$, and $\int_1^\infty r^\alpha K_i(r)dr = \int_1^\infty r^{\alpha+\beta_i}dr < \infty$. Thus (A_1) holds. For $i,j \in \{1,2\}$, let α_{ij} be constants satisfying $\alpha_{ij} > N$. Let $l_{ij}(r) = C_{ij}r^{-\alpha_{ij}}$ for $r \in [1,\infty)$, where $0 \le C_{ij} < \frac{\bar{\alpha}-N}{2w_N}$ and $\bar{\alpha} = \min_{i,j\in\{1,2\}}\{\alpha_{ij}\}$. Then $0 \le \max_{i\in\{1,2\}}\left\{\int_\Omega [l_{1i}(|x|) + l_{2i}(|x|)]dx\right\} \le C_{ij}w_N \max_{i\in\{1,2\}}\left\{\frac{1}{\alpha_{1i}-N} + \frac{1}{\alpha_{2i}-N}\right\} \le \frac{2C_{ij}w_N}{\bar{\alpha}-N} < 1$, and thus (A_2) holds. Let $\hat{f}_1(r,u,v) = e^{-r}[u^q - u^{p-1} + 1 + v]$ and $\hat{f}_2(r,u,v) = r^{-1}[u^2 + e^{(p-1)v} - v^{p-1}]$ for $(r,u,v) \in [1,\infty) \times \mathbb{R}_+^2$, where $q > p - 1$. Then it can be easily verified that $(\hat{F}_1)$ and $(\hat{F}_\infty)$ hold.

Acknowledgements

The authors thank the referee for his or her helpful remarks. This study was financially supported by the 2015 Post-Doc. Development Program of Pusan National University.

References

[1] R. P. Agarwal, H. Lü, D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **266**(2002), 383–400. MR1880513; url

[2] O. Bolojan-Nica, G. Infante, R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.* **94**(2014), 231–242. MR3120688; url

[3] F. Cianciaruso, P. Pietramala, Multiple positive solutions of a (p_1, p_2)-Laplacian system with nonlinear BCs, *Bound. Value Probl.* **2015**, No. 163, 18 pp. MR3399259; url

[4] X. Cheng, H. Lü, Multiplicity of positive solutions for a (p_1, p_2)-Laplacian system and its applications, *Nonlinear Anal. Real World Appl.* **13**(2012), 2375–2390. MR2911922; url

[5] X. Cheng, Z. Zhang, Positive solutions for a class of multi-parameter elliptic systems, *Nonlinear Anal. Real World Appl.* **14**(2013), 1551–1562. MR3004521; url

[6] R. Cui, P. Li, J. Shi, Y. Wang, Existence, uniqueness and stability of positive solutions for a class of semilinear elliptic systems, *Topol. Methods Nonlinear Anal.* **42**(2013), 91–104. MR3155616

[7] Z. Deng, Y. Huang, Existence of symmetric solutions for singular semilinear elliptic systems with critical Hardy–Sobolev exponents, *Nonlinear Anal. Real World Appl.* **14**(2013), 613–625. MR2969860; url

[8] M. del Pino, M. Elgueta, R. Manásevich, A homotopic deformation along p of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0,\ u(0) = u(T) = 0,\ p > 1$, *J. Differential Equations* **80**(1989), 1–13. MR1003248; url

[9] Y. Deng, Y. Li, On the existence of multiple positive solutions for a semilinear problem in exterior domains, *J. Differential Equations* **181**(2002), 197–229. MR1900465; url

[10] C. S. Goodrich, Coupled systems of boundary value problems with nonlocal boundary conditions, *Appl. Math. Lett.* **41**(2015), 17–22. MR3282393; url

[11] C. S. Goodrich, On a nonlocal BVP with nonlinear boundary conditions, *Results Math.* **63**(2013), 1351–1364. MR3057373; url

[12] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering, Vol. 5, Academic Press Inc., Boston, MA, 1988. MR959889

[13] J. Jaroš, T. Kusano, A Picone type identity for second order half-linear differential equations, *Acta Math. Univ. Comenian. (N.S.)* **68**(1999), 137–151. MR1711081

[14] C.-G. Kim, Existence of positive solutions for multi-point boundary value problem with strong singularity, *Acta Appl. Math.* **112**(2010), 79–90. MR2684975; url

[15] C.-G. Kim, E. K. Lee, Multiple positive solutions for singular multi-point boundary value problems with a positive parameter, *Electron. J. Differential Equations* **2014**, No. 38, 1–13. MR3177547

[16] C.-G. Kim, Y.-H. Lee, Existence of multiple positive solutions for p-Laplacian problems with a general indefinite weight, *Commun. Contemp. Math.* **10**(2008), 337–362. MR2417920; url

[17] C.-G. Kim, Y.-H. Lee, Existence of multiple positive radial solutions for p-Laplacian problems with an L^1-indefinite weight, *Taiwanese J. Math.* **15**(2011), 723–736. MR2810178

[18] C.-G. Kim, Y.-H. Lee, I. Sim, Multiplicity results of positive radial solutions for p-Laplacian problems in exterior domains, *Bound. Value Probl.* **2008**, Art. ID 395080, 18 pp. MR2392920

[19] E. Ko, E. K. Lee, R. Shivaji, Multiplicity results for classes of singular problems on an exterior domain, *Discrete Contin. Dyn. Syst.* **33**(2013), 5153–5166. MR3060831; url

[20] T. Kusano, J. Jaroš, N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, *Nonlinear Anal.* **40**(2000), 381–395. MR1768900; url

[21] E. K. Lee, Y.-H. Lee, A multiplicity result for generalized Laplacian systems with multi-parameters, *Nonlinear Anal.* **71**(2009), e366–e376. MR2671846; url

[22] J. Leray, J. Schauder, Topologie et équations fonctionnelles, *Ann. Sci. École Norm. Sup.* **51**(1934), 45-78. MR1509338

[23] Y. Li, H. Zhang, Solvability for system of nonlinear singular differential equations with integral boundary conditions, *Bound. Value Probl.* **2014**, No. 158, 12 pp. MR3277890; url

[24] R. Ma, Existence theorems for a second order m-point boundary value problem, *J. Math. Anal. Appl.* **211**(1997), 545–555. MR1458512; url

[25] J. M. do Ó, S. Lorca, P. Ubilla, Local superlinearity for elliptic systems involving parameters, *J. Differential Equations* **211**(2005), 1–19. MR2121107; url

[26] J. M. do Ó, S. Lorca, J. Sánchez, P. Ubilla, Positive radial solutions for some quasilinear elliptic systems in exterior domains, *Commun. Pure Appl. Anal.* **5**(2006), 571–581. MR2217602; url

[27] C. V. Pao, Dynamics of reaction-diffusion equations with nonlocal boundary conditions, *Quart. Appl. Math.* **53**(1995), 173–186. MR1315454

[28] C. V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992. MR1212084

[29] C. V. Pao, Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions, *J. Math. Anal. Appl.* **195**(1995), 702–718. MR1356638; url

[30] B. P. Rynne, Linear second-order problems with Sturm–Liouville-type multi-point boundary conditions, *Electron. J. Differential Equations* **2012**, No. 146, 1–21. MR2967211

[31] J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc. (2)* **74**(2006), 673–693. MR2286439; url

[32] J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA Nonlinear Differential Equations Appl.* **15**(2008), 45–67. MR2408344; url

[33] B. Yan, D. O'Regan, R. P. Agarwal, Multiplicity and uniqueness results for the singular nonlocal boundary value problem involving nonlinear integral conditions, *Bound. Value Probl.* **2014**, No. 148, 22 pp. MR3286090; url

[34] Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlinear Anal.* **62**(2005), 1251–1265. MR2154107; url

[35] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, New York, 1985. MR0816732

[36] M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. Differential Equations* **166**(2000), 33–50. MR1779254; url

[37] X. Zhang, M. Feng, Existence of a positive solution for one-dimensional singular p-Laplacian problems and its parameter dependence, *J. Math. Anal. Appl.* **413**(2014), 566–582. MR3159788; url