When the vertex coloring of a graph is an edge coloring of its line graph — a rare coincidence

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Abstract

The *3-consecutive vertex coloring number* $\psi_{3c}(G)$ of a graph G is the maximum number of colors permitted in a coloring of the vertices of G such that the middle vertex of any path $P_3 \subset G$ has the same color as one of the ends of that P_3. This coloring constraint exactly means that no P_3 subgraph of G is properly colored in the classical sense.

The *3-consecutive edge coloring number* $\psi'_{3c}(G)$ is the maximum number of colors permitted in a coloring of the edges of G such that the middle edge of any sequence of three edges (in a path P_4 or cycle C_3) has the same color as one of the other two edges.

For graphs G of minimum degree at least 2, denoting by $L(G)$ the line graph of G, we prove that there is a bijection between the 3-consecutive *vertex* colorings of G and the 3-consecutive *edge* colorings of $L(G)$, which keeps the number of colors unchanged, too. This implies that $\psi_{3c}(G) = \psi'_{3c}(L(G))$; i.e., the situation is just the opposite of what one would expect for first sight.

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1 Introduction

Since the vertices of a line graph[1] $L(G)$ correspond to the edges of graph G, it follows directly from the definitions that the edge colorings of G are in one-to-one correspondence with the vertex colorings of $L(G)$. This bijection preserves lots of properties (the coloring to be proper, equitable, having at least a given distance between any two elements of the same color or at most a given diameter in each component of every color class, excluding alternately bi-colored cycles, etc.), and hence the corresponding versions of the chromatic index of G are equal to those of the chromatic number of $L(G)$. Equalities of this kind are automatic by definition in most cases.

In this short note, however, we present a rare example where the situation is just the opposite: a certain type of vertex colorings of a graph are in one-to-one correspondence with the analogous edge colorings of the line graph. The following notions will be investigated:

- Three distinct vertices v_1, v_2, v_3 are *consecutive* in a graph $G = (V, E)$ if they form a path in this order; i.e., if $v_1v_2, v_2v_3 \in E$. A mapping $\varphi : V \to \mathbb{N}$ is a *3-consecutive vertex coloring* if at least one of $\varphi(v_1) = \varphi(v_2)$ and $\varphi(v_2) = \varphi(v_3)$ is valid whenever v_1, v_2, v_3 are consecutive. Equivalently, a *3*-consecutive vertex coloring of G means a partition $V_1, V_2, \ldots, V_n$ of V satisfying that for every three consecutive vertices v_1, v_2 and v_3 there exists a class V_i such that at least one of the relations $\{v_1, v_2\} \subseteq V_i$ and $\{v_2, v_3\} \subseteq V_i$ holds.

- Three distinct edges e_1, e_2, e_3 are *consecutive* in $G = (V, E)$ if, in this order, they form a path or cycle of length 3. A mapping $\phi : E \to \mathbb{N}$ is a *3-consecutive edge coloring* if at least one of $\phi(e_1) = \phi(e_2)$ and $\phi(e_2) = \phi(e_3)$ is valid whenever e_1, e_2, e_3 are consecutive. Equivalently, a *3*-consecutive edge coloring of G is a partition $E_1, E_2, \ldots, E_n$ of E satisfying the following condition: if e_1, e_2 and e_3 are *3*-consecutive edges in G, then for some i, $(1 \le i \le n)$ at least one of the relations $\{e_1, e_2\} \subseteq E_i$ and $\{e_2, e_3\} \subseteq E_i$ holds. Note that if e_1, e_2 and e_3 are three edges of a triangle, then they are considered three consecutive edges in the orderings $e_1e_2e_3$, $e_2e_3e_1$ and $e_3e_1e_2$, as well. Hence, in a 3-consecutive edge coloring every K_3 subgraph must be monochromatic.

A 3-consecutive vertex (edge) coloring is obtained by making the entire vertex (edge) set monochromatic. Moreover, identifying any two color classes in a 3-consecutive φ or ϕ we again obtain a 3-consecutive coloring. Hence, the graph invariants which really matter in this context are:

[1]The line graph $L(G)$ of a graph G has the edges of G as its vertices, and two distinct edges of G are adjacent in $L(G)$ if and only if they share a vertex in G.

- $\psi_{3c}(G)$, *3-consecutive vertex coloring number*: the maximum number of colors in a 3-consecutive vertex coloring of G. (In [13], this is called *3-consecutive achromatic number.*) Alternately, $\psi_{3c}(G)$ is the maximum number of classes in a vertex partition of G such that each vertex appears together with all but at most one of its neighbors in the same partition class.

- $\psi'_{3c}(G)$, *3-consecutive edge coloring number*: the maximum number of colors in a 3-consecutive edge coloring of G. Alternately, $\psi'_{3c}(G)$ is the maximum number of classes in an edge partition of G such that at least one endpoint of each edge is a monochromatic star.

These parameters were introduced and related results presented in [4, 5, 11, 13], with various observations on structure, extremal values, and algorithmic complexity. Colorings of similar kinds are investigated in [3, 6, 12].

Motivation. 3-consecutive colorings studied here are antipodal to proper vertex and edge colorings in the following sense. For connected graphs of order at least 3, a vertex coloring is a proper coloring if and only if *every* P_3 subgraph is colored properly; whilst it is a 3-consecutive vertex coloring if and only if *no* P_3 subgraph is colored properly.

We say that three consecutive edges $e_1e_2e_3$ are colored properly if e_1 and e_2 and similarly, e_2 and e_3 have different colors. For connected graphs not isomorphic to a star, an edge coloring is proper if and only if *every* three consecutive edges are colored properly; whilst it is a 3-consecutive edge coloring if and only if *no* three consecutive edges are colored properly.

Applications. As formulated in Lemmas 4 and 5, quoted from [4] and [5], respectively, the kinds of coloring studied here are intimately related with vertex- and edge-separators which, in turn, play substantial role in the design of efficient algorithms for a great variety of problems [1, 2, 7, 9, 10]. More explicit examples for applications of 3-consecutive vertex and edge colorings are described in [4, 5]. To mention just one, ψ_{3c} is equal to the possible maximum number of components in a communication network after the failure of at most one link at each node.

Our results. In this note we focus on the *relationship* between ψ_{3c} and ψ'_{3c}. Concerning the classic notions of chromatic index (edge chromatic number) of G and vertex chromatic number of its line graph $L(G)$, the equality $\chi'(G) = \chi(L(G))$ is straightforward to prove. But the corresponding equality is *not valid* for $\psi'_{3c}(G)$ and $\psi_{3c}(L(G))$, in general. For example, if $G = K_{2,n}$ then $\psi'_{3c}(G) = n$ and $\psi_{3c}(L(G)) = 2$ for every $n \geq 2$. Or even simpler, a star S_n with $n \geq 3$ edges has the following values: $\psi_{3c}(S_n) = 2$ and $\psi'_{3c}(S_n) = n$, moreover $\psi_{3c}(L(S_n)) = \psi'_{3c}(L(S_n)) = 1$.

Restricting attention to graphs of minimum degree at least 2, however, a surprising correspondence of opposite nature will be verified, namely in Section 2 we prove that $\psi_{3c}(G) = \psi'_{3c}(L(G))$ always holds. This appears to be a very rare kind of coincidence.

In Section 3, we recall two characterization theorems from [4] and [5]. Then, applying the result of Section 2, we point out a correspondence between the maximum number of components which can be obtained when certain types of edge sets and vertex sets are deleted from G and $L(G)$, respectively. In the last section we give tight upper bounds on the sums $\psi_{3c}(G) + \psi'_{3c}(G)$ and $\psi'_{3c}(G) + \psi'_{3c}(L(G))$.

Definitions and notation. We use the following notation from [8].

- $\alpha_0(G)$, *vertex covering number, transversal number*: the minimum cardinality of a vertex set meeting all edges of G.
- $\beta_0(G)$, *vertex independence number, stability number*: the largest number of mutually nonadjacent vertices in G.
- $\beta_1(G)$, *edge independence number, matching number*: the largest number of mutually vertex-disjoint edges in G.

By these definitions, the Gallai-type identity $\alpha_0(G) + \beta_0(G) = |V|$ follows for every graph G.

2 Bijection between vertex and edge colorings

The goal of this section is to prove the following theorem which immediately implies the equality of $\psi_{3c}(G)$ and $\psi'_{3c}(L(G))$.

Theorem 1. *For every graph G with minimum degree at least 2 and for every positive integer k, the 3-consecutive vertex colorings of G with exactly k colors and the 3-consecutive edge colorings of its line graph $L(G)$ with exactly k colors are in one-to-one correspondence.*

Proof. Let us introduce the following notation. For an edge e of $G = (V, E)$ the corresponding vertex in $L(G)$ will be denoted by e^*. We apply this convention in the other direction, too; i.e., writing $e^* \in V(L(G))$ we mean that e^* corresponds to $e \in E(G)$.

Consider first the vertex colorings of G. To any 3-consecutive vertex coloring $\varphi : V \to \{1, 2, \dots, k\}$, we associate an edge coloring $\phi : E(L(G)) \to \{1, 2, \dots, k\}$ of the line graph where ϕ is defined by the rule

$$\phi(e^* f^*) = \varphi(v) \qquad \text{where} \qquad v = e \cap f. \tag{$\star$}$$

Observe that the definition assigns a unique color to each edge $e^* f^* \in E(L(G))$. We are going to prove that ϕ is a 3-consecutive edge coloring of $L(G)$.

Let $e^* f^*$, $f^* g^*$, $g^* h^*$ be three consecutive edges in $L(G)$, where $e^* = h^*$ is also possible, but otherwise the vertices must be mutually different. The corresponding edges in G will be $e = xv_1$, $f = v_1 v_2$, $g = v_2 v_3$ and $h = v_3 y$. The coincidences $x = y$, $x = v_3$ or $y = v_1$ might hold, but since f and g are distinct non-loop edges, v_1, v_2 and v_3 are different 3-consecutive vertices of G. Consequently, $\phi(f^* g^*) = \varphi(v_2)$ is the same color as at least one of $\phi(e^* f^*) = \varphi(v_1)$ and $\phi(g^* h^*) = \varphi(v_3)$. Thus, ϕ is a 3-consecutive edge coloring of $L(G)$. Moreover, due to the degree condition, every vertex of G is the intersection of at least two edges, hence every color used in φ occurs in the coloring ϕ, as well.

Because of the same reason, different 3-consecutive vertex colorings of G are associated with different edge colorings of $L(G)$. We will prove that the correspondence $\varphi \mapsto \phi$ is a bijection, invertible in a natural way starting from ϕ.

Let ϕ be a 3-consecutive edge coloring of the line graph $L(G)$, which uses exactly k colors. Recall that every triangle of $L(G)$ is monochromatic in ϕ. By the degree condition, each vertex $v \in V$ is incident with some $\ell = \deg(v) \geq 2$ edges. Hence in the line graph, v corresponds to the edges of a complete subgraph K_ℓ. These edges necessarily have the same color in ϕ and therefore, the following definition determines a unique color for every $v \in V$:

$$\varphi(v) = \phi(e^* f^*) \qquad \text{where} \qquad e \cap f = v. \tag{$\star'$}$$

To prove that φ is a 3-consecutive vertex coloring, assume three consecutive vertices v_1, v_2 and v_3 in G. There exist some neighbor x of v_1 and y of v_3 such that $x \neq v_2 \neq y$. The vertices $(xv_1)^*$, $(v_1 v_2)^*$, $(v_2 v_3)^*$ and $(v_3 y)^*$ induce either a path P_4 or a cycle C_4 or a $K_4 - e$ or just a K_3 in $L(G)$. Since ϕ is a 3-consecutive edge coloring, the color $\phi((v_1 v_2)^* (v_2 v_3)^*)$ coincides with $\phi((xv_1)^* (v_1 v_2)^*)$ or $\phi((v_2 v_3)^* (v_3 y)^*)$. According to the definition of φ, this means that $\varphi(v_2)$ is the same as $\varphi(v_1)$ or $\varphi(v_3)$, which proves that φ is a 3-consecutive vertex coloring of G. Clearly, φ uses all the k colors of ϕ. Moreover, one can check that the correspondences $\varphi \mapsto \phi$ and $\phi \mapsto \varphi$ induced by the rules $(\star)$ and $(\star')$ are exactly the inverses of each other.

By the above facts, ($\star$) establishes a bijection between the 3-consecutive vertex colorings of G with k colors and the 3-consecutive edge colorings of its line graph $L(G)$ with k colors. $\square$

As an immediate consequence we obtain

Theorem 2. *Let G be a graph with minimum degree at least 2. For the 3-consecutive vertex coloring number of G and the 3-consecutive edge coloring number of its line graph $L(G)$, the following equality holds:*

$$\psi_{3c}(G) = \psi'_{3c}(L(G)).$$

Remark 3. Similarly to the case of $\delta(G) \geq 2$, also for $\delta(G) = 1$, a 3-consecutive edge coloring ϕ of $L(G)$ can be obtained from each 3-consecutive vertex coloring φ of G by applying the rule ($\star$). But if u is a vertex of degree 1 then color $\varphi(u)$ possibly does not occur in ϕ. In the other direction, for any 3-consecutive edge coloring ϕ of $L(G)$, we can construct a 3-consecutive vertex coloring φ of G with the same number of colors, if each pendant vertex in G is assigned with the color of its unique neighbor and otherwise the rule ($\star'$) is applied. This proves that $\psi_{3c}(G) \geq \psi'_{3c}(L(G))$ holds if $\delta(G) = 1$. Moreover, the presence of isolated vertices does not change $L(G)$. Therefore, the inequality $\psi_{3c}(G) \geq \psi'_{3c}(L(G))$ is valid for every G.

3 Implication for vertex and edge separators

In this section we quote two results from the papers [4] and [5], which characterize 3-consecutive colorings in terms of cutsets. We need the following two definitions.

- By *cut*-$(k, 1)$ *subgraph* of a (not necessarily connected) graph $G = (V, E)$ we mean a *matching* $F \subseteq E$ whose deletion results in a graph with at least k components.

- By *stable k-separator* of a (not necessarily connected) graph $G = (V, E)$ we mean an independent vertex set $S \subset V$ for which $G - S$ has at least k components.

The next lemma follows from a more general characterization theorem of the paper [4].

Lemma 4. *For every integer $k \geq 2$ and for every graph G, the relation $\psi_{3c}(G) \geq k$ holds if and only if G has a cut-$(k, 1)$ subgraph.*

The following necessary and sufficient condition was proved in [5].

Lemma 5. *Let G be a graph without isolated vertices. Then, for every integer $k \geq 2$, the 3-consecutive edge coloring number of G is at least k if and only if G has a stable k-separator.*

Based on these characterizations, Theorem 2 immediately implies:

Theorem 6. *Let G be a graph with minimum degree at least 2. Then, for each positive integer k, G has a cut-$(k, 1)$ subgraph if and only if its line graph $L(G)$ has a stable k-separator.*

For the particular case of connected G and $k = 1$, this statement has been proved by Brandstädt et al. in [1].

4 Upper bounds on sums

In this section we consider ψ_{3c} and ψ'_{3c} together, also investigating their behavior when both G and its line graph $L(G)$ are involved. We begin with a general estimate on $\psi_{3c} + \psi'_{3c}$, which will be improved later for graphs without pendant vertices. As an auxiliary tool, let us recall the following upper bounds from [11] and [5], respectively.

Lemma 7. *For every connected graph G, the inequality $\psi_{3c}(G) \leq \alpha_0(G)+1$ is valid.*

Lemma 8. *For every graph G, the inequality $\psi'_{3c}(G) \leq \beta_0(G)$ is valid.*

Now, the following general upper bounds can be derived.

Theorem 9. *Let G be a graph of order p.*

(*i*) *If G is connected, then the inequality*

$$\psi_{3c}(G) + \psi'_{3c}(G) \leq p + 1$$

holds and the bound is tight for all $p \geq 2$.

(*ii*) *For G and its line graph $L(G)$ we have*

$$\psi'_{3c}(G) + \psi'_{3c}(L(G)) \leq p$$

and the bound is tight for all $p \geq 3$.

(*iii*) *If G has minimum degree at least 2, then (i) can be strengthened to*

$$\psi_{3c}(G) + \psi'_{3c}(G) \leq p$$

and the bound is tight for all even $p \geq 4$.

Proof. The upper bounds can be verified as follows.

(i) Applying the identity $\alpha_0(G)+\beta_0(G)=p$ for the sum of the inequalities in Lemmas 7 and 8, we obtain

$$\psi_{3c}(G)+\psi'_{3c}(G)\leq\alpha_0(G)+1+\beta_0(G)\leq p+1.$$

(ii) The edge independence number $\beta_1(G)$ is equal to the independence number $\beta_0(L(G))$ of the line graph. Hence, by Lemma 8 we obtain

$$\psi'_{3c}(G)+\psi'_{3c}(L(G))\leq\beta_0(G)+\beta_1(G)\leq\beta_0(G)+\alpha_0(G)=p.$$

(iii) The assertion follows from (ii) and Theorem 2.

Moreover, as noted in [5] and [13], the following equalities are obvious. The path P_n on $n\geq 2$ vertices has $\psi_{3c}(P_n)=\lfloor n/2\rfloor+1$ and $\psi'_{3c}(P_n)=\lceil n/2\rceil$; and the cycle C_n on $n\geq 3$ vertices has $\psi_{3c}(C_n)=\psi'_{3c}(C_n)=\lfloor n/2\rfloor$. These also imply $\psi'_{3c}(L(P_n))=\psi'_{3c}(L(C_n))=\lfloor n/2\rfloor$. Hence, tightness is witnessed for (i) by all paths on at least two vertices, for (ii) by all even cycles and all paths on at least three vertices, and for (iii) by all even cycles. Further examples for (i) and (ii) are the stars. Indeed, for $p\geq 3$ we have $\psi_{3c}(K_{1,p-1})=2$ and $\psi'_{3c}(K_{1,p-1})=p-1$, while $\psi'_{3c}(L(K_{1,p-1}))=1$ since $L(K_{1,p-1})\cong K_{p-1}$. $\square$

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