On the Schneider-Vigneras functor for principal series

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Abstract

We study the Schneider-Vigneras functor attaching a module over the Iwasawa algebra $\Lambda(N_0)$ to a B-representation for irreducible modulo π principal series of the group $\mathrm{GL}_n(F)$ for any finite field extension $F|\mathbb{Q}_p$.

Keywords: p-Adic Langlands programme; Smooth modulo p representations; Principal series; Schneider-Vigneras functor;

1 Introduction

Let $\mathbb{Q}_p$ be the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ its algebraic closure, $F, K \leq \overline{\mathbb{Q}}_p$ finite extensions of $\mathbb{Q}_p$. Let o_F, respectively o_K be the rings of integers in F, respectively in K, $\pi_F \in o_F$ and $\pi_K \in o_K$ uniformizers, ν_F and ν_K the standard valuations and $k_F = o_F/\pi_F o_F$, $k_K = o_K/\pi_K o_K$ the residue fields.

The Langlands philosophy predicts a natural correspondence between certain admissible unitary representations of $\mathrm{GL}_n(F)$ over Banach K-vector spaces and certain n-dimensional K-representations of the Galois-group $\mathrm{Gal}(\overline{\mathbb{Q}}_p|F)$.

Colmez proved the existence of such a correspondence in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, but for any other group even the conjectural picture is not developed yet. It turned out, that Fontaine's theory of (φ, Γ)-modules is a fundamental intermediary between the representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Schneider and Vigneras managed to generalize parts of Colmez's work to reductive groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

Our aim is to understand the construction of Schneider and Vigneras, attaching a generalized (φ, Γ)-module to a smooth torsion o_K-representation of G, for principal series representations V in the case $G = \mathrm{GL}_n(F)$. Originally this functor (which we denote by D) is defined only for $F = \mathbb{Q}_p$, but our considerations work for any finite extension $F|\mathbb{Q}_p$ and the analogous definitions.

In order to that, we need to understand the B_+-module structure of the principal series, where B_+ is a certain submodule of a Borel subgroup B in G. In section 3 we decompose G to open N_0-invariant subsets (where N_0 is a totally decomposed compact open subgroup in the unipotent radical of B), indexed by the Weyl group.

With the help of this in section 4 we prove that there exists a minimal element M_0 in the set of generating B_+-subrepresentations of V.

Now we have that $D(V) = M_0^*$ - the dual of this minimal B_+-subrepresentation. We do not know whether it is finitely generated or it has rank 1 as a module over $\Omega(N_0) = \Lambda(N_0)/\pi_K\Lambda(N_0)$ (where $\Lambda(N_0)$ is the Iwasawa algebra of N_0). However, we show that in some sense only a rank 1 quotient of $D(V)$ is relevant if we want to get an étale (φ, Γ)-module.

In the last section we point out some properties of M_0, which sheds some light on why the picture is more difficult for principal series than in the case of subquotients defined by the Bruhat filtration.

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2 Notations

Let G be the F-points of a F-split connected reductive group over $\mathbb{Q}_p$. Let $B \leq G$ be a fixed Borel subgroup, with maximal torus T and unipotent radical N. Let $W \simeq N_G(T)/C_G(T)$ be the Weyl group of G, Φ^+ the set of positive roots with respect to B, and N_α denote the root subgroup for each $\alpha \in \Phi^+$. A subgroup $N_0 \leq N$ is called totally decomposed if for any total ordering of Φ^+ we have $N_0 = \prod_{\alpha\in\Phi^+}(N_0 \cap N_\alpha)$.

As an o_K-representation of G we mean a pair $V = (V, \rho)$, where V is a torsion o_K-module, $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. V is smooth if ρ is locally constant ($\forall v \in V\ \exists U \subset G$ open, such that $\forall u \in U : \rho(u)v = v$). V is admissible if for any $U \leq G$ open subgroup, the vector space $k_K \otimes_{o_K} V^U$ is finite dimensional.

For an o_K-representation V let $V^* = \mathrm{Hom}_{o_K}(V, K/o_K)$ be the Pontrjagin dual of V. Pontrjagin duality sets up an anti-equivalence between the category of torsion o_K-modules and the category of all compact linear-topological o_K-modules.

Let $G_0 \leq G$ be a compact open subgroup and $\Lambda(G_0)$ denote the completed group ring of the profinite group G_0 over o_K. Any smooth o_K-representation V is the union of its finite G_0-subrepresentations, therefore V^* is a left $\Lambda(G_0)$-module (through the inversion map on G_0).

Let $\Omega(G_0) = \Lambda(G_0)/\pi_K\Lambda(G_0)$. $\Omega(N_0)$ is noetherian and has no zero divisors, so it has a fraction (skew) field. If M is a $\Omega(N_0)$-module, by the rank of M we mean $\dim_{k_K}(\mathrm{Frac}(\Omega(N_0)) \otimes_{\Omega(N_0)} M)$.

From now on fix $n \in \mathbb{N}$, and let $G = \mathrm{GL}_n(F)$, and $G_0 = \mathrm{GL}_n(o_F)$.

Let B be the set of upper triangular matrices in G, T the set of diagonal matrices, N the set of upper triangular unipotent matrices. Let N^- be the lower unipotent matrices - the opposite of N - and $N_0 = N \cap G_0$ - a totally decomposed compact open subgroup of N - those matrices wich has coefficients in o_F, define the following submonoid of T:

$$T_+ = \{t \in T | tN_0t^{-1} \subset N_0\} = \{\mathrm{diag}(x_1, x_2, \ldots, x_n) | i > j : \nu_F(x_i) \geq \nu_F(x_j)\}.$$

We have the following partial ordering on T_+: $t \leq t'$ if there exists $t'' \in T_+$ such that $tt'' = t'$. Let $B_+ = N_0T_+$, this is a submonoid of B.

By the abuse of notation let $w \in W$ denote also the permutation matrices - representatives of W in G (with $w_{ij} = 1$ if $w(j) = i$, and $w_{ij} = 0$ otherwise), and also the corresponding permutation of the set $\{1, 2, \dots, n\}$. For $w \in W$ denote length of w - the length of the shortest word representing w in the terms of the standard generators of W - by $l(w)$.

Let the kernel of the projection $pr : G_0 \to \mathrm{GL}_n(k_F)$ be $U^{(1)}$. This is a compact open pro-p normal subgroup of G_0. We have $G = G_0 B$ and $U^{(1)} \subset (N^- \cap U^{(1)})B$.

Let $C^\infty(G)$ (respectively $C_c^\infty(G)$) denote the set of locally constant $G \to k_K$ functions (respectively locally constant functions with compact support), with the group G acting by left multiplication ($gf : x \mapsto f(g^{-1}x)$ for $f \in C^\infty(G)$ and $g, x \in G$). Let

$$\chi = \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n : T \to k_K^*$$

be a locally constant character of T with $\chi_i : F^* \to k_K^*$ multiplicative. Note that then for all i $\chi_i(1 + \pi_F o_F) = 1$ and $\chi_i(o_F^*) \subset k_F^* \cap k_K^* \leq \overline{\mathbb{F}_p}^*$. Since $T \simeq B/[B, B]$, also denote the correspondig $B \to k_K^*$ character by χ. Let

$$V = \mathrm{Ind}_B^G(\chi) = \{f \in C^\infty(G) | \forall g \in G, b \in B : f(gb) = \chi^{-1}(b)f(g)\}$$

V is called a principal series representation of G. V is irreducible exactly when for all i we have $\chi_i \neq \chi_{i+1}$ ([5], theorem 4). For any open right B-invariant subset $X \subset G$ we write $\mathrm{Ind}_B^X = \{F \in \mathrm{Ind}_B^G(\chi) | F|_{G\setminus X} \equiv 0\}$.

We can understand the stucture of V better (see [8], section 4.), by the Bruhat decomposition $G = \bigcup_{w \in W} BwB$. Let $\prec$ denote the strong Bruhat ordering (see [4] II. 13.7): we say $w' \prec w$ for $w \neq w' \in W$ if there exist transpositions $w_1, w_2, \dots, w_i \in W$ such that $w' = ww_1w_2\dots w_i$ and $l(w) > l(ww_1) > l(ww_1w_2) > \cdots > l(ww_1w_2\dots w_i)$. Fix a total ordering $\prec_T$ refining the Bruhat ordering $\prec$ of W, and let

$$w_1 = \mathrm{id}_W \prec_T w_2 \prec_T w_3 \prec_T \cdots \prec_T w_{n!} = w_0.$$

Let us denote by $G_m = \bigcup_{1 \leq l \leq m} Bw_lB$ - a closed subset of G. We obtain a descending B-invariant filtration of V by

$$V_m = \mathrm{Ind}_B^{G\setminus G_m}(\chi) = \{F \in \mathrm{Ind}_B^G(\chi) | F|_{G_m} \equiv 0\} \qquad (0 < m \leq n!),$$

with quotients V_{m-1}/V_m via $f \mapsto f(\cdot w_m)$ isomorphic to $V(w_m, \chi) = C_c^\infty(N/N'_{w_m})$ (see [6], section 12), where $N'_{w_m} = N \cap w_m N w_m^{-1}$, with N acting by left translations and T acting via

$$(t\phi)(n) = \chi(w_m^{-1} t w_m)\phi(t^{-1}nt).$$

For any $w \in W$ put

$$N_w = \{n \in N | \forall i < j, w^{-1}(i) < w^{-1}(j) : n_{ij} = 0\} = N \cap wN^- w^{-1} \leq N,$$

and $N_{0,w} = N_0 \cap N_w$. Then we have the following form of the Bruhat decomposition $G = \coprod_{w \in W} N_w w B$.

3 The action of B_+ on G

The first goal is to partition G to N_0-invariant open subsets $\{U_w | w \in W\}$ indexed by the Weyl-group, which are respected by the B_+-action in the sense that if $x \in U_w$ $b \in B_+$ then there exists $w' \preceq w$ in W such that $b^{-1}x \in U_{w'}$.

Definition Let for any $w \in W$ $r_w : N^- \cap G_0 \to G(k_F), n^- \mapsto pr(wn^- w^{-1})$, $R_w = wr_w^{-1}(N_0(k_F))$, $R = \cup_{w \in W} R_w$.

We have that

$$R_w = \left\{ (a_{ij}) \in G | \forall i, j : a_{ij} \begin{cases} = 1, & \text{if } w^{-1}(i) = j \\ = 0, & \text{if } w^{-1}(i) < j \\ \in o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) > i \\ \in \pi_F o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) < i \end{cases} \right\}$$

For $n = 3$ in details (with $o = o_F$ and $\pi = \pi_F$):

w	R_w	w	R_w
$\mathrm{id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \pi o & 1 & 0 \\ \pi o & \pi o & 1 \end{pmatrix}$	$(23) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \pi o & o & 1 \\ \pi o & 1 & 0 \end{pmatrix}$
$(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} o & 1 & 0 \\ 1 & 0 & 0 \\ \pi o & \pi o & 1 \end{pmatrix}$	$(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} o & o & 1 \\ 1 & 0 & 0 \\ \pi o & 1 & 0 \end{pmatrix}$
$(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} o & 1 & 0 \\ o & \pi o & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} o & o & 1 \\ o & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Let $N(k_F)$ be the k_F-points of N (the upper triangular unipotent matrices with coefficients in k_F). k_F has canonical (multiplicative) injection to $o_F \subset F$, hence any subgroup $H(k_F) \leq N(k_F)$ is mapped injectively to N_0 (however this is not a group homomorphism). We denote this subset of N_0 by $\widetilde{H(k_F)}$.

Proposition 3.1 *A set of double coset representatives of* $U^{(1)} \setminus G/B$ *is* $\cup_{w\in W}\widetilde{N_w(k_F)}w$. *Every element of* G *can be written uniquely in the form* rb *with* $r \in R$ *and* $b \in B$.

Proof By the Bruhat decomposition of $G(k_F)$ a set of double coset representatives of $U^{(1)} \setminus G_0/(B \cap G_0)$ is the set as above. Since $G = G_0B$, we have the first part of proposition.

Let $g = unwb \in G$ with $u \in U^{(1)}$, $w \in W$, $n \in \widetilde{N_w(k_F)}$ and $b \in B$. Then $g = w(w^{-1}nw)u'b$ with $u' = w^{-1}n^{-1}unw \in U^{(1)}$. But then there exist $n' \in N^- \cap U^{(1)}$ and $b' \in B$ such that $u' = n'b'$. Then $g = w(w^{-1}nwn')(b'b)$, where $w^{-1}nwn' \in r_w^{-1}(N_0(k_F))$ because of the definition of N_w.

For any $w \in W$ we clearly have $U^{(1)}\widetilde{N_w(k_F)}wB = R_wB$. Hence the uniqueness follows: if $rb = r'b'$ then there exists $w \in W$ such that $r, r' \in R_w$ and $b'b^{-1} = (r'^{-1}w^{-1})(wr) \in B \cap N^- = \{\mathrm{id}\}$. □

Definition For any $w \in W$ let $U_w = U^{(1)}\widetilde{N_w(k_F)}wB$. This way we partitioned G into open subsets indexed by the Weyl group. We obviously have $U_w = R_wB$.

Corollary 3.2 *For any* $w \in W$ *we have that* U_w *is (left)* N_0*-invariant.*

Proof Let $n' \in N_0$ and $x = unwb \in U^{(1)}\widetilde{N_w(k_F)}wB$. We have $N_0 = N_{0,w}(N'_w \cap N_0)$, thus $n'n = mm'$ for some $m \in N_{0,w}$ and $m' \in N'_w \cap N_0$, moreover we can write $m = m_1m_0 \in (N_w \cap U^{(1)})\widetilde{N_w(k_F)}$. By the definition of N'_w

$$n'x = (n'un'^{-1}m_1)m_0w(w^{-1}m'wb) \in U^{(1)}\widetilde{N_w(k_F)}wB,$$

meaning that U_w is N_0-invariant. □

Proposition 3.3 *Let* $y \in U_w = R_wB$, $nt \in B_+ = N_0T_+$, *and* $x = t^{-1}n^{-1}y \in U_{w'} = R_{w'}B$. *Then* $w' \preceq w$.

Proof Let $y = rb$ with $r \in R_w$ and $b \in B$. By the previous proposition we may assume that $n = \mathrm{id}$. If $t = \mathrm{diag}(t_1, t_2, \dots, t_n) \in G_0$, then

$$x = w(w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw)(w^{-1}t^{-1}wb),$$

where $w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw \in r_w^{-1}(N_0(k_F))$, because it is in N^- and the coefficients under the diagonal have the same valuation as those in $w^{-1}r$.

T_+ as a monoid is generated by $T \cap G_0$, the center $Z(G)$ and the elements with the form $(\pi_F, \pi_F, \ldots, \pi_F, 1, 1, \ldots, 1)$, hence it is enough to prove the proposition for such t-s.

So fix $t = (t_1 = \pi_F, t_2 = \pi_F, \ldots, t_l = \pi_F, t_{l+1} = 1, t_{l+2} = 1, \ldots, t_n = 1)$, $r = (r_{ij})$ and try to write x in the form as in Proposition 3.1. For all $j = 0, 1, 2, \ldots, n$ we construct inductively a decomposition $x = (t^{(j)})^{-1} r^{(j)} b^{(j)}$ together with $w^{(j)} \in W$, where

- $w^{(j+1)} \preceq w^{(j)}$ for $j < n$ and such that the first j columns of $w^{(j)}$ are the same as the first j columns of $w^{(j+1)}$,
- $t^{(j)} = \mathrm{diag}(t_i^{(j)}) \in T$ with

$$t_i^{(j)} = \begin{cases} 1, & \text{if } (w^{(j)})^{-1}(i) \leq j \\ t_i, & \text{if } (w^{(j)})^{-1}(i) > j \end{cases},$$

- $r^{(j)} \in R_{w^{(j)}}$, and if we change the first j columns of $r^{(j)}$ to the first j columns of $(t^{(j)})^{-1} r^{(j)}$ it is still in $R_{w^{(j)}}$ (by de definition of $t^{(j)}$ it is enough to verify the condition for $(t^{(j)})^{-1} r^{(j)}$),
- $b^{(j)} \in B$.

Then $w^{(n)} \preceq w^{(n-1)} \preceq w^{(n-2)} \preceq \cdots \preceq w^{(1)} = w$. However for $j = n$ we have $t^{(n)} = \mathrm{id}$, hence $w^{(n)} = w'$ by disjointness of the sets $R_v B$ for $v \in W$, so we have the proposition.

For $j = 0$ we have $t^{(0)} = t, r^{(0)} = r, b^{(0)} = b$ and $w^{(0)} = w$. From j to $j + 1$:

- If $w^{(j)}(j+1) \leq l$, then let $w^{(j+1)} = w^{(j)}$, so $t^{(j+1)} = e^{-1}_{w^{(j)}(j+1)} t^{(j)}$, where for $1 \leq k \leq n$ we denote $e_k = e_k(\pi)$ the diagonal matrix with π_F in the k-th row and 1 everywhere else. We can choose $r^{(j+1)} = e^{-1}_{w^{(j)}(j+1)} r^{(j)} e_{j+1}$, and $b^{(j+1)} = e^{-1}_{j+1} b^{(j)}$.

  Then the first j columns of $(t^{(j+1)})^{-1} r^{(j+1)}$ are equal of those of $(t^{(j)})^{-1} r^{(j)}$, and the entries at place $(i, j+1)$ with $i \neq w^{(j+1)}(j+1)$ are multiplied by π_F. Because of the conditions for $r^{(j)}$, this is in $R_{w^{(j+1)}}$. The other conditions for $w^{(j+1)}, t^{(j+1)}, r^{(j+1)}$ and $b^{(j+1)}$ obviously hold.

- If $w^{(j)}(j+1) > l$ and if $\nu_F(r^{(j)}_{i,j+1}) \geq 1$ for all $i \leq l$, then it suffices to choose $w^{(j+1)} = w^{(j)}, t^{(j+1)} = t^{(j)}, r^{(j+1)} = r^{(j)}$ and $b^{(j+1)} = b^{(j)}$.

- Assume that $w^{(j)}(j+1) > l$ and that there exists $i \leq l$ such that $\nu_F(r^{(j)}_{i,j+1}) = 0$. Let i_0 be the maximal such i. Then choose $w^{(j+1)}(j+1) = i_0$, and $t^{(j+1)} = e_{i_0}^{-1} t^{(j)}$.

  Let $r' = e_{i_0}^{-1} r^{(j)} e_{j+1}((r^{(j)}_{i_0,j+1})^{-1} \cdot \pi)$, where $e_j(\alpha)$ is the diagonal matrix with $\alpha \in F$ in the j-th row and 1 everywhere else. Note that $r'_{i_0,j+1} = 1$ and r' differs from $r^{(j)}$ only in the i_0-th row and the $j+1$-st column. But $(t^{(j+1)})^{-1} r'$ is not in $\mathrm{GL}_n(o_F)$ - for example $\nu_F(r'_{i_0,(w^{(j)})^{-1}(i_0)}) = -1$, and there might be some other elements of r' in the i_0-th row and columns between the $j+2$-nd and $j' = (w^{(j)})^{-1}(i_0)$-th.

  To see this note first that $w^{(j)}(j+1) > l \geq i_0$, so $(w^{(j)})^{-1}(i_0) \neq j+1$. In particular the right multiplication with e_{j+1} does not change the entry at place $(i_0, (w^{(j)})^{-1}(i_0))$. Since $r^{(j)} \in R_{w^{(j)}}$, the defining conditions of $R_{w^{(j)}}$ and that $(w^{(j)})^{-1}(i_0) \neq j+1$ imply $(w^{(j)})^{-1}(i_0) > j+1$. Thus $(t^{(j)}_{i_0})^{-1} = (t_{i_0})^{-1} = \pi_F^{-1}$, since $i_0 \leq l$. By the definition of $R_{w^{(j)}}$ we have $r^{(j)}_{i_0,(w^{(j)})^{-1}(i_0)} = 1$. Therefore $r'_{i_0,(w^{(j)})^{-1}(i_0)} = \pi^{-1}$ which has valuation -1.

  But note, that in the $j+1$-st column of r' the i_0-th element is 1, all the other has valuation at least 1. Thus the first $j+1$ columns of $(t^{(j+1)})^{-1} r'$ satisfy the condition for the first $j+1$ columns of $(t^{(j+1)})^{-1} r^{(j+1)}$ - this is meaningful, because we already fixed the first $j+1$ columns of $w^{(j+1)}$.

  So we want to find $r^{(j+1)} = r'b'$ with $b' \in B$ such that the first $j+1$ columns of b' is those of the identity matrix, and $(t^{(j+1)})^{-1} r^{(j+1)} \in R_{w^{(j+1)}}$ with $w^{(j)} \preceq w^{(j+1)}$.

  Let $j_0 = j+1$, and if $j_i < j'$ then

  $$j_{i+1} = \min\{h | j+1 < h, r'_{i_0,h} \notin o_F, w^{(j)}(j_i) > w^{(j)}(h)\}.$$

  We claim that the set on the right hand side contains j' if $j_i < j'$. We prove it by induction on i. For $i = 0$ we already verified it. Assume by contradiction that $w^{(j)}(j_i) < i_0 = w^{(j)}(j')$. Since $j' > j_i$ we get $r^{(j)}_{i_0,j_i} \in \pi_F o_F$, because $r^{(j)} \in R_{w^{(j)}}$. But then $r'_{i_0,j_i} \in o_F$, because $r' \in e_{i_0}^{-1} r^{(j)} \cdot \mathrm{Mat}(o_F)$, contradicting the defining conditions of j_i. Thus we have $w^{(j)}(j_i) \geq i_0 = w^{(j)}(j')$.

  Let s be minimal such that $j_s = j'$ and set $j_{s+1} = n+1$. We claim that $r^{(j+1)}$ will be in $R_{w^{(j+1)}}$ with $w^{(j+1)} = w^{(j)}(j_{s-1}, j_s)(j_{s-2}, j_{s-1}) \dots (j_0, j_1)$.

Then the condition $w^{(j+1)} \prec w^{(j)}$ holds, because the multiplication from right with each transposition (j_i, j_{i+1}) decreases the inversion number and the length respectively, by the definition of j_{i+1}.

For the existence of a $b' \in B$ such that $r'b' \in R_{w^{(j+1)}}$ we prove the following statements inductively:

Lemma 3.4 *For all $j+1 \le k \le n$ there exist*

- *$b'^{(k)} \in B$ such that the first k column of $r'^{(k)} = r'b'^{(k)}$ satisfy the defining condition for the first k column in $R_{w^{(j+1)}}$, and if we have $k < n$ then $r'^{(k)}$ and $r'^{(k+1)}$ differ only in the $k+1$-st column.*
- *a linear combination $s^{(k)}$ of the columns $j+1, j+2, \ldots, k$ in $r'^{(k)}$ for which we have*

$$s_i^{(k)} = \begin{cases} 1, & \text{if } i = i_0 \\ 0, & \text{if } (w^{(j+1)})^{-1}(i) \le k, \text{ and } i \ne i_0 \\ \pi_F x, & \text{for some } x \in o_F \text{ otherwise} \end{cases}$$

and the maximal i such that $\nu_F(s_i^{(k)}) = 1$ is $w^{(j)}(j_{i'})$, where i' is so, that $j_{i'} \le k < j_{i'+1}$.

Proof This holds for $k = j+1$ with $b'^{(j+1)} = \mathrm{id}$, $r'^{(j+1)} = r'$ and $s^{(j+1)}$ the $j+1$-st column of r'. To verify the condition for $s^{(j+1)}$ note that $r'_{(w^{(j)}(j+1),j+1)} = \pi$ and if $i > j+1$, then by the definition of $R_{w^{(j)}}$ we have that $r^{(j)}_{i,j+1}$ has valuation at least 1 and $r'_{(i,j+1)} = \pi_F (r^{(j)}_{i_0,j+1})^{-1} r^{(j)}_{i,j+1}$ has valuation at least 2.

Assume that we have $r'^{(k)}$, $b'^{(k)}$ and $s^{(k)}$. Let i' be so that $j_{i'} \le k < j_{i'+1}$ and s' be the $k+1$-st column of $r'^{(k)}$ (which is equal with the $k+1$-st column of r', thus for $i \ne i_0$ we have $s'_i = r^{(j)}_{i,k+1}$) and $s'' = s' - r'^{(k)}_{(i_0,k+1)} s^{(k)}$. Then by the conditions on s' we can change the $k+1$-st column of $r'^{(k)}$ to s'' with multiplication from right by an element $b'' \in B$. Moreover $s''_{i_0} = 0$, and the element in s'' with minimal valuation and biggest row index is the $w^{(j+1)}(k+1)$-st:

- If $\nu_F(r'^{(k)}_{(i_0,k+1)}) \ge 0$ then for $i \ne i_0$ we have $s'_i \equiv s''_i = s'_i - r'^{(k)}_{(i_0,k+1)} s_i^{(k)} \mod \pi_F$, hence the element with minimal valuation is in the row $w^{(j+1)}(k+1) = w^{(j)}(k+1)$ (because $r^{(j)} \in R_{w^{(j)}}$ and $j_{i'+1} \ne k+1$).

– If $\nu_F(r'^{(k)}_{(i_0,k+1)}) < 0$ then it is -1 and for $i \neq i_0$ we have $s''_i = r^{(j)}_{(i,k+1)} - r'^{(k)}_{(i_0,k+1)} \cdot s^{(k)}_i$. Where on the right hand side the first term has positive valuation for $i > w^{(j)}(k+1)$ and 0 valuation for $i = w^{(j)}(k+1)$ (because $r^{(j)} \in R_{w^{(j)}}$), and the second has valuation 0=-1+1 for $i = w^{(j)}(j_{i'})$ and at least 1 for $i > w^{(j)}(j_{i'})$ (by the induction hypothesis on $s^{(k)}$). Moreover $j_{i'} \neq k+1$, because $j_{i'} \leq k$, hence $w^{(j)}(j_{i'}) \neq w^{(j)}(k+1)$.
  If $w^{(j)}(j_{i'}) < w^{(j)}(k+1)$ then $j_{i'+1} \neq k+1$ and $w^{(j)}(k+1) = w^{(j+1)}(k+1)$. If $w^{(j)}(j_{i'}) > w^{(j)}(k+1)$ then $j_{i'+1} = k+1$ and $w^{(j+1)}(k+1) = w^{(j+1)}(j_{i'+1}) = w^{(j)}(j_{i'})$.

By multiplying this column with $(s''_{w^{(j+1)}(k+1)})^{-1}$ we get the element $r'^{(k+1)}$ (we also have to multiply the $k+1$-st row of b'' with $s''_{w^{(j+1)}(k+1)}$, this is $b'^{(k+1)}$). This satisfies the condition for the $k+1$-st row of $R_{w^{(j+1)}}$ because the defining conditions for $r^{(j)} \in R_{w^{(j)}}$, $s^{(k)}$ and the equality

$$\{i|(w^{(j+1)})^{-1}(i) < k+1\} = \{i|(w^{(j)})^{-1}(i) < k+1\} \setminus \{w^{(j)}(j_{i'})\} \cup \{i_0\}.$$

The last thing to verify is the existence of an appropriate linear combination $s^{(k+1)}$. Let $s^{(k+1)} = s^{(k)} - s^{(k)}_{w^{(j+1)}(k+1)}(s''_{w^{(j+1)}(k+1)})^{-1} \cdot s''$. Since $\nu_F(s^{(k)}_{w^{(j+1)}(k+1)}) > 0$, we have $\nu_F(s^{(k+1)}_i) > 0$ if $i \neq i_0$, and by the previous argument also $s^{(k+1)}_{w^{(j+1)}(j')} = 0$ for $j' \leq k+1$ and $j' \neq j+1$.

If $w^{(j+1)}(k+1) > w^{(j)}(j_{i'})$, then $s^{(k)}_{w^{(j+1)}(k+1)} > 1$ and $s^{(k+1)} \equiv s^{(k)} \mod \pi_F^2$. If $w^{(j+1)}(k+1) < w^{(j)}(j_{i'})$ then by the definition of $R_{w^{(j+1)}}$ for all $i > w^{(j+1)}(k+1)$ we have $\nu(s''_i) > 1$ and again $s^{(k+1)}_i \equiv s^{(k)}_i \mod \pi_F^2$. If $w^{(j+1)}(k+1) = w^{(j)}(j_{i'})$, then by the definition of $R_{w^{(j)}}$ we have $s'_{w^{(j)}(j_{i'})} = r'_{(w^{(j)}(j_{i'}),k+1)} = 0$, $s''_{w^{(j+1)}(k+1)} = 0 - r'^{(k)}_{(i_0,k+1)} s^{(k)}_{w^{(j)}(j_{i'})}$ and $s^{(k+1)} =$

$$= s^{(k)} - s^{(k)}_{w^{(j)}(j_{i'})}(-r'^{(k)}_{(i_0,k+1)} s^{(k)}_{w^{(j)}(j_{i'})})^{-1} \cdot \left(s' - r'^{(k)}_{(i_0,k+1)} s^{(k)}\right) = (r'^{(k)}_{(i_0,k+1)})^{-1} s',$$

which satisfies the condition because s' is the $j_{i'+1} = k+1$-st column of $r'^{(k)}$ and because of the definition of $R_{w^{(j)}}$. □

To finish the proof we set $b' = b'^{(n)}$, $r^{(j+1)} = r' b'^{(n)} \in R_{w^{(j+1)}}$ and $b^{(j+1)} = (b'^{(n)})^{-1}(r^{(j)}_{i_0,j+1} \cdot e^{-1}_{j+1}) b^{(j)} \in B$.

□

Corollary 3.5 *For any $w \in W$ we have $BwB = N_w wB \subset \cup_{w' \preceq w} U_{w'}$. In particular for any $0 < m_0 \leq n!$ we have that*

$$\bigcup_{m \geq m_0} U_{w_m} \subset G \setminus G_{m_0 - 1} = \bigcup_{m \geq m_0} B w_m B.$$

Proof Let $x = n_w w b \in N_w wB$. Then there exists $t \in T_+$ such that $n' = t n_w t^{-1} \in N_0$. Thus $x = t^{-1} n' w (w^{-1} t w) b = t^{-1} n' w b''$ with $b'' \in B$. By the previous proposition for $w = w \cdot \mathrm{id} \in R_w B$ and $(n')^{-1} t \in B_+$, there exist $w' \prec w$, $r_{w'} \in R_{w'}$ and $b' \in B$ such that $t^{-1} n' w = r_{w'} b'$, hence $x = r_{w'}(b' b'') \in U_{w'}$. The second assertion follows from that:

$$\bigcup_{m \geq m_0} U_{w_m} = G \setminus \bigcup_{1 \leq m < m_0} U_{w_m} \subset G \setminus \bigcup_{1 \leq m < m_0} B w_m B = G \setminus G_{m_0 - 1}.$$

□

Remark We can achieve the results of this section not only for GL_n, but different groups: let G' be such that

- G' is isomorphic to a closed subgroup in G which we also denote by G',
- In G' a maximal torus is $T' = T \cap G'$, a Borel subgroup $B' = B \cap G'$ with unipotent radical $N' = N \cap G'$, such that $N_{G'}(T') = N_G(T) \cap G'$ and hence $W' \leq W$ with $w_0 \in W'$, with representatives w' of W' in $G'_0 \leq G_0$ such that the representatives w of W in G can be written in the form $w = w't$ such that $t \in T \cap G_0$.
- $G'_0 = G_0 \cap G'$ with $G' = G'_0 B'$ and
- $U'^{(1)} = U^{(1)} \cap G'$ such that $U'^{(1)} \subset (N'^{-} \cap U'^{(1)}) B'$ for $N'^{-} = w_0 N' w_0$.

For example these condititons are satisfied for the group SL_n.

The proof of the first proposition works for such G', and from a decomposition $x = r'b' \in R'_w B' \subset G'$ we get some $r \in R_w$ and $b \in B$ such that $x = rb \in G$. Hence the B'_+-action on G' respects the restriction of $\prec$ to W' in the sense that if $x \in R_{w'} B'$ and $b' \in B'$ then there exists $w'' \preceq w'$ in W' such that $b'^{-1} x \in R'_{w''} B'$.

4 Generating B_+-subrepresentations

For any torsion o_K-module X with o_K-linear B-action denote the (partially ordered) set of generating B_+-subrepresentations of X (those B_+-submodules M of X for which $BM = X$) by $\mathcal{B}_+(X)$.

For example $\mathrm{Ind}_B^{U_{w_0}}(\chi) \simeq C^\infty(N_0)$ is the minimal generating B_+-subrepresentation of the Steinberg representation $V_{n!-1} = \mathrm{Ind}_B^{Bw_0B}(\chi) \simeq C_c^\infty(N)$. (cf [6], Lemma 2.6)

Proposition 4.1 *Let X be a smooth admissible and irreducible torsion o_K-representation of G. Then $M_0 = B_+X^{U^{(1)}}$ is a generating B_+-subrepresentation of X. For any $M \in \mathcal{B}_+(X)$ there exists a $t_+ \in T_+$ such that $t_+M_0 \subset M$.*

Proof X is a π_K vectorspace as well, because $\pi_K X \leq X$, hence by the irreducibility it is either 0 or X, and since X is torsion $\pi_K X = X$ gives $X = 0$.

BM_0 is a B-subrepresentation, and also a G_0-subrepresentation (because $U^{(1)} \triangleleft G_0$). $G_0B = BG_0 = G$, so BM_0 is a G-subrepresentation of X. M_0 is not $\{0\}$, since $U^{(1)}$ is pro-p and since X is irreducible $BM_0 = X$, hence M_0 is generating. And M_0 is clearly a B_+-submodule of X.

X is admissible, hence $X^{U^{(1)}}$ has a finite generating set, say R. Let M be as in the proposition. For any $r \in R$ there exists an element $t_r \in T_+$ such that $t_r r \in M$ ([6], Lemma 2.1). The cardinality of R is finite, hence for $t_+ = \prod_{r\in R} t_r$ we have $t_r^{-1}t_+ \in T_+$ for all $r \in R$, and then $t_+M_0 \subset M$. $\square$

From now on let $V = \mathrm{Ind}_B^G(\chi)$ as before and $M_0 = B_+V^{U^{(1)}}$. Then $V^{U^{(1)}}$ (as a vector space) is generated by

$$f_r : \begin{cases} urb & \mapsto \quad \chi^{-1}(b) \\ y \neq urb & \mapsto \quad 0 \end{cases} \qquad \left(r \in U^{(1)} \setminus G/B = \bigcup_{w\in W} \widetilde{N_w(k_F)w} \right).$$

If we denote the coset $U^{(1)}wB$ also with w, then $V^{U^{(1)}}$ is generated by $\{f_w | w \in W\}$ as an N_0-module. Hence any $f \in M_0$ can be written in the form $\sum_{i=1}^s \lambda_i n_i t_i f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0, t_i \in T_+$ and $w_i \in W$.

Proposition 4.2 *M_0 is minimal in $\mathcal{B}_+(V)$.*

Remark In [6] section 12 Schneider and Vigneras treated the case of the subquotients V_{m-1}/V_m. Unfortunately M_0 does not generally give the minimal generating B_+-subrepresentation of V_{m-1}/V_m on this subqoutient, since that their method does not work on the whole V. It is not true even for $\mathrm{GL}_3(\mathbb{Q}_p)$: an explicit example is shown in Corollary 6.2.

Proof By the previous proposition, it is enough to show, that for any $t' \in T_+$ we have $M_0 \subset B_+t'M_0$.

If $t' \in G_0$, then $t'^{-1} \in T_+$ thus we have $B_+t' = B_+$, and $B_+t'M_0 = B_+M_0 = M_0$. The same is true for central elements $t' \in Z(G)$. So it is enough to prove for $t' = (\pi_F, \pi_F, \ldots, \pi_F, 1, 1, \ldots, 1)$ that $M_0 \subset B_+t'M_0$.

Let $j_0 \in \mathbb{N}$ be such that $t'_{j_0} = \pi_F$ and $t'_{j_0+1} = 1$. We need to show, that for all $w \in W$ we have $f_w \in B_+t'M_0$. We prove it by descending induction on w with respect to $\prec$.

Let us denote $N_{j_0}^{(1)} = \{n \in N \cap U^{(1)} | \forall i < j, (j_0 - i)(j - j_0) < 0 : n_{ij} = 0\}$, $N_{w,j_0} = N_w \cap N_{j_0}^{(1)}$ and

$$\Theta_{w,j_0} = \{\text{a set of representatives of } N_{w,j_0}/t'N_{w,j_0}t'^{-1}\} \subset N_0 \cap U^{(1)}.$$

It is enough to prove the following:

Lemma 4.3 *Let* $g = \sum_{m \in \Theta_{w,j_0}} mt'f_w$. *Then* $\chi(w^{-1}t'w)f_w - g$ *is in* $\sum_{w' : w \prec w'} N_0 f_{w'}$.

We claim that for $r \in R_w$ we have

$$t'f_w(r) = \begin{cases} \chi(w^{-1}t'w), & \text{if } \forall i \le j_0 < j, w^{-1}(i) > w^{-1}(j) : r_{ij} \in \pi_F^2 o_F, \\ 0, & \text{otherwise.} \end{cases}$$

$t'f_w(r) = f(t'^{-1}r)$ is nonzero if and only if $t'^{-1}r \in U^{(1)}wB$. Following the proof of Proposition 3.3, it is equivalent to that for all $1 \le j \le n$ we have $w = w^{(j)}$ and that the first j column of $(t^{(j)})^{-1}r^{(j)}$ is as the first j column of $U^{(1)}w$. This holds if and only if $r_{ij} \in \pi_F^2 o_F$ for all i and j as above. Then we have $r^{(n)} = t'^{-1}rw^{-1}t'w$ and $b^{(n)} = w^{-1}(t')^{-1}w$, hence our claim.

Therefore $\chi(w^{-1}t'w)f_w|_{U_w} = \sum_{m \in \Theta_{w,j_0}} mt'f_w|_{U_w}$. Hence by the induction hypothesis and Proposition 3.3 it suffices to prove that g is $U^{(1)}$-invariant.

To do that, first notice that since f_w is $U^{(1)}$-invariant, we have that $t'f_w$ is $t'U^{(1)}t'^{-1}$-invariant. Moreover, since for all $m \in \Theta_{w,j_0}$ we have

$m \in N_0 \cap U^{(1)} \subseteq t'N_0t'^{-1}$, m normalizes $t'U^{(1)}t'^{-1}$, $mt'f_w$ is also $t'U^{(1)}t'^{-1}$-invariant, and so is g.

On the other hand, we can write

$$g = \sum_{m\in\Theta_{w,j_0}} mt'f_w = \sum_{m\in\Theta_{w,j_0}} t'(t'^{-1}mt')f_w = t'\left(\sum_{n\in t'^{-1}N_{w,j_0}t'/N_{w,j_0}} nf_w\right),$$

where the sum in the bracket on the right hand side is obviously $t'^{-1}N_{w,j_0}t'$-invariant, hence g is N_{w,j_0}-invariant.

Denote $N'_{w,j_0} = N'_w \cap N^{(1)}_{j_0}$. Then N_{w,j_0} centralizes $t'^{-1}N'_{w,j_0}t'$: let $n_0 = \mathrm{id} + m_0 \in t'^{-1}N'_{w,j_0}t'$, $n \in N_{w,j_0}$,

$$(n^{-1}n_0n - n_0)_{xy} = (n^{-1}m_0n - m_0)_{xy} = \sum_{x\le s\le t\le y}(n^{-1})_{xs}(m_0)_{st}n_{ty} - (m_0)_{xy},$$

and by the definition $N^{(1)}_{j_0}$, $(m_0)_{st}$ is 0, unless $s \le j_0 \le t$ and hence $(n^{-1})_{xs}m_{st}n_{ty} = 0$, unless $x = s$ and $y = t$.

By the definiton of N'_w we have $w^{-1}N'_{w,j_0}w \subset B$, so for any $u \in U^{(1)}$ and $n_0 \in t'^{-1}N'_{w,j_0}t' \subset G_0$ we have $n_0uw = (n_0un_0^{-1})w(w^{-1}n_0w) \in U^{(1)}wB$, and hence f_w is $t'^{-1}N'_{w,j_0}t'$-invariant.

Altogether for any representative $n \in \Theta_{w,j_0}$

$$nf_w(n_0x) = f_w(n^{-1}n_0x) = f_w(n_0n^{-1}x) = f_w(n^{-1}x) = nf_w(x),$$

meaning that nf_w is $t'^{-1}N'_{w,j_0}t'$-invariant, and $t'nf_w$ is N'_{w,j_0}-invariant. So g is also N'_{w,j_0}-invariant.

$U^{(1)}$ is contained in $\langle t'U^{(1)}t'^{-1}, N_{w,j_0}, N'_{w,j_0}\rangle$, so g is $U^{(1)}$-invariant, and we are done. □

Corollary 4.4 *For any $f \in M_0$ there exists $t \in T_+$ such that f can be written in form $\sum_{i=1}^{s}\lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0$ and $w_i \in W$.*

Define the $k_K[B_+]$-submodules $M_{0,m} = \sum_{m'>m} B_+ f_{w_{m'}} \le \mathrm{Ind}_B^{G_m}(\chi)$. We obtain a descending filtration $M_0 = M_{0,0} \ge M_{0,1} \ge \cdots \ge M_{0,n!} = 0$. Then $M_{0,n!-1} = \mathrm{Ind}_B^{U_{w_0}}(\chi)$ is the minimal generating subrepresentation of $V_{n!-1}$.

Proposition 4.5 *Let $1 < m \leq n!$, $w = w_{m-1}$ and $n' \in N'_{0,w} = N'_w \cap N_0$ and $t \in T_+$. Then $g = n'tf_w - nf_w \in M_{0,m}$.*

Proof For $w' \prec w$ we have $tf_w|_{U_{w'}} = n'tf_w|_{U_{w'}} = 0$ and following the proof of Proposition 3.3 we get $n'tf_w|_{U_w} = tf_w|_{U_w}$. Moreover g is $tU^{(1)}t^{-1}$-invariant, thus it is contained in $\sum_{m'>m-1} tf_{w_{m'}} \subset M_{0,m}$. $\square$

Corollary 4.6 *For any $f \in M_0$ there exists $t \in T_+$ such that f can be written in form $\sum_{i=1}^s \lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K$, $w_i \in W$ and $n_i \in N_{0,w_i}$.*

Remarks 1. V is the modulo π_K reduction of the p-adic principal series representation. This can be done with any $l \in \mathbb{N}$ for the modulo π_K^l reduction. Then the π_K-torsion part of the minimal generating B_+-representation is exactly M_0.

2. This can be carried out in the same way for groups G' as in the previous section satisfying moreover $N_0 \subset G'$. For example $G' = \mathrm{SL}_n$ has this property (but its center is not connected), or $G' = P$ for arbitary $P \leq G$ parabolic subgroup has also (but these are not reduvtive).

5 The Schneider-Vigneras functor

Following Schneider and Vigneras ([6], section 2) we introduce the functor D from torsion o_K-modules to modules over the Iwasawa algebra of N_0.

Let us denote the completed group ring of N_0 over o_K by $\Lambda(N_0)$, and define

$$D(X) = \varinjlim_{M \in \mathcal{B}_+(X)} M^*,$$

as an $\Lambda(N_0)$-module, equipped with the natural T_+^{-1}-action ψ.

On $D(V)$ the action of π_K is 0, hence we can view it as a $\Omega(N_0) = \Lambda(N_0)/\pi_K\Lambda(N_0)$-module.

By Proposition 4.2 we have

Proposition 5.1 *The $\Omega(N_0)$-module $D(V)$ is equal to M_0^*.*

Remarks 1. We do not now whether $D(V)$ is finitely generated or it has rank 1 as an $\Omega(N_0)$-module.

2. On M_0 we have an action of $U^{(1)}$: if $x \in U^{(1)}$, $n \in N_0, t \in T_+$ and $w \in W$ then we can write $n^{-1}xn = n_1 n_2 \in U^{(1)}$ with $n_1 \in N_0$ and $n_2 \in B^- T \cap U^{(1)}$ (with $B^- = N^- T$), thus

$$xntf_w = n(n^{-1}xn)tf_w = (nn_1)t(t^{-1}n_2 t)f_w = (nn_1)tf_w \in M_0,$$

since $t^{-1}n_2 t \in U^{(1)}$ and f_w is $U^{(1)}$-invariant. Thus on $D(V)$ there is an action of $\Lambda(U^{(1)})$, therefore an action of $\Lambda(I)$ (with I denoting the Iwahori subgroup).

Till this point we considered only the $\Lambda(N_0)$-module structure of $D(V)$. Now we shall examine the ψ-action as well. We need to get an étale module from $D(V)$, thus we examine the ψ-invariant images of $D(V)$ in an étale module.

Let D be a topologically étale (see [7] the first lines of Section 4) (φ, Γ)-module over $\Omega(N_0)$, with the following properties:

- D is torsion-free as an $\Omega(N_0)$-module,
- on D the topology is Hausdorff,
- D has a basis of neighborhoods of 0, containing φ-invariant $\Omega(N_0)$-submodules ($O \leq D$ open such that $\varphi_t(O) \subseteq O$ for all $t \in T_+$).

Theorem 5.2 *Let D be as above and $F : D(V) \to D$ a continuous ψ-invariant map (where ψ is the canonical left inverse of φ on D). Then F factors through the natural map $F_0 : D(V) \to D(V_{n!-1})$: there exists a continuous ψ-invariant map $G : D(V_{n!-1}) \to D$ such that $F = F_0 \circ G$.*

Proof $\overline{D(V) - tors}$ is in the kernel of F (the torsion submodules exist, because the rings are Ore rings).

In $M_0/(M_0 \cap V_{n!-1})$ there are no nontrivial $k_K[N_0]$-divisible elements, because if $f \in M_0$ the image of it in $M_0/(M_0 \cap V_{n!-1})$ is $f' = f|_{G \setminus Bw_0 B}$. Assume by contradiction that f' is $k_K[N_0]$-divisible. If it is nontrivial, then there exists $bw_m b \in G$ such that $f(bw_m b) \neq 0$ with some $m < n!$ Let $n' \in N'_{0,w_m} = N_0 \cap w_m N_0 w_m^{-1}$ with $n' \neq \mathrm{id}$, and $[n'] - [\mathrm{id}] \in k_K[N_0]$. Then for any $g \in M_0$ we have

$$([n'] - [\mathrm{id}])g(w_m) = g(n'^{-1}w_m) - g(w_m) = g(w_m(w_m^{-1}n'^{-1}w_m)) - g(w_m) = 0,$$

because $w_m^{-1}n'^{-1}w_m \in N$. Thus f' is not divisible by $[n'] - [\mathrm{id}]$.

It follows that F factors through $(M_0 \cap V_{n!-1})^*$: The fact that there are no nontrivial divisible submodules in $M_0/(M_0 \cap V_{n!-1})$ implies that for any (closed) submodule the maps $f \mapsto \lambda f$ are not surjective for all $\lambda \in k_K[N_0]^*$. Hence dual maps are not injective for all λ - the dual has no torsionfree quotient arising as a dual of a submodule of $M_0/(M_0 \cap V_{n!-1})$, thus $(M_0/(M_0 \cap V_{n!-1}))^* \leq \overline{D(V) - tors}$. Now consider the exact sequence

$$0 \to M_0 \cap V_{n!-1} \to M_0 \to M_0/(M_0 \cap V_{n!-1}) \to 0.$$

We claim that F factors through $M^*_{0,n!-1}$ as well. If $f \in (M_0 \cap V_{n!-1})^*$ such that $f|_{M_{0,n!-1}} \equiv 0$, then $\psi_t(u^{-1}f)|_{t^{-1}M_{0,n!-1}} \equiv 0$ for all $t \in T_+$ and $u \in N_0$:

The ψ-action on $D(V)$ comes from the T_+-action on V, hence $\psi_t(u^{-1}f)(t^{-1}x) = (u^{-1}f)(tt^{-1}x) = f(ux) = 0$ if $x \in M_{0,n!-1}$.

For all $O \subseteq D$ open subset there exists $t \in T_+$ such that $\mathrm{Ker}(f \mapsto f|_{t^{-1}M_{0,n!-1}}) \subset F^{-1}(O)$, since F is continuous and $\bigcup_{t \in T_+} t^{-1}M_{0,n!-1} = V_{0,n!-1}$. If O is φ and N_0-invariant as well, then

$$F(f) = \sum_{u \in N_0/tN_0t^{-1}} u\varphi_t(F(\psi_t(u^{-1}f)) \subseteq O.$$

Then $F(f) = 0$ by the Hausdorff property.

By [6], Proposition 12.1, we have $D(V_{n!-1}) = M^*_{0,n!-1}$, which completes the proof. □

Remarks

1. For this we do not need the Γ-action of D, the statement is true for D étale φ-modules with continuous N_0 and φ-action.

2. Let D' be the maximal quotient of $D(V)$, which is torsionfree, Haussdorff and on which the action of ψ is nondegenerate in the following sense: for all $d \in D' \setminus \{0\}$ and $t \in T_+$ there exists $u \in N_0$ such that $\psi_t(ud) \neq 0$. Then the natural map from D' to $D(V_{n!-1})$ is bijective.

3. By [9] section 4 if $F = \mathbb{Q}_p$, we have that $D^0(V_{n!-1}) = D(V_{n!-1})$ and $D^i(V_{n!-1}) = 0$ for $i > 0$.

Following [6] we choose a surjective homomorphism $\ell : N_0 \to \mathbb{Q}_p$. Then we can get (φ, Γ)-modules from $D(V)$: Let $\Lambda_\ell(N_0)$ denote the ring $\Lambda_{N_1}(N_0)$ of [6] with $N_1 = \mathrm{Ker}(\ell)$, with maximal ideal $\mathcal{M}_\ell(N_0)$,

$\Omega_\ell(N_0) = \Lambda_\ell(N_0)/\pi_K\Lambda_\ell(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(N_1)[[X]]$ of skew Taylor series over $\Lambda(N_1)$ in the variable $X = [u] - 1$ where $u \in N_0$ and (u) is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $n \in \mathbb{Z} a_n X^n$ over $\Lambda(N_1)$ in the variable X with $\lim_{n\to-\infty} a_n = 0$ for the compact topology of $\Lambda(N_1)$.

Let $D_\ell(V) = \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D(V)$.

Corollary 5.3 *Let D be a finitely generated topologically étale (φ, Γ)-module over $\Omega_\ell(N_0)$, and $F' : D_\ell(V) \to D$ a continuous map. Then F' factors through the natural map $F'_0 : D_\ell(V) \to D_\ell(V_{n!-1})$.*

Proof If D is a finitely generated topologically étale (φ, Γ)-module over $\Omega_\ell(N_0)$, then it automatically satisfies the conditions above:

D is étale, hence $\Omega_\ell(N_0)$-torsion free (Theorem 8.20 in [7]), thus $\Omega(N_0)$-torsion free as well. It is Hausdorff, since finitely generated and the weak topology is Haussdorff on $\Omega_\ell(N_0)$ (Lemma 8.2.iii in [6]).

Finally we need to verify the condition for the neighborhoods. The sets $\mathcal{M}_\ell(N_0)^k D + \Omega(N_0) \otimes_{k[[X]]} X^n \ell(D)^{++}$ (where $\ell(D)$ is the étale (φ, Γ)-module attached to D at the category equivalence [7] Theorem 8.20) are open φ-invariant $\Omega(N_0)$ submodules and form a basis of neighborhoods of 0 in the weak topology of D.

Thus $D(V) \to D_\ell(V) \to D$ factors through $D(V) \to D(V_{n!-1})$, hence the corollary. □

6 Some properties of M_0

In this section we point out some properties of M_0, which make the picture more difficult than the known case of subqoutients V_{m-1}/V_m. Recall ([6] section 12) that $V_{m-1}/V_m \simeq V(w_m, \chi)$, which has a minimal generating B_+-subrepresentation

$$M(w_m, \chi) = C^\infty(N_0/N'_{w_m} \cap N_0) \in \mathcal{B}_+(V(w_m, \chi)).$$

Proposition 6.1 *Let $n = 3$, $F = \mathbb{Q}_p$, then $M_0 \cap V_{n!-1} \supsetneq M_{0,n!-1}$.*

Corollary 6.2 *Thus $M_0 \cap V_{n!-1}$ is not equal to the minimal generating B_+-subrepresentation of $V_{n!-1}$, which is $C^\infty(N_0) = M_{0,n!-1}$ ([6] section 12).*

Proof Assume that $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 : T \to k_K^*$ is a character, such that neither χ_1/χ_2, nor χ_2/χ_3 is trivial on o_K^*. Similar construction can be carried out in the other cases.

Let $\prec_T$ be the following total ordering of the Weyl group of G refining the Bruhat ordering:

$$w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \prec_T$$

$$\prec_T w_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \prec_T w_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \prec_T w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = w_0.$$

And let

$$h = \sum_{a=0}^{p^2-1} \sum_{b=0}^{p^2-1} \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_2} \in M_0,$$

$$f = h - \frac{1}{\chi_3(p^2)} \sum_{a=0}^{p^3-1} \sum_{b=0}^{p^3-1} h\left(\begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_5}.$$

Then it is easy to verify that $f \in M_0 \cap V_5$, and that $f(z) \neq 0$ for

$$z = \begin{pmatrix} p^2 & 0 & 1 \\ 1 & 0 & 0 \\ p & 1 & 0 \end{pmatrix} \in Bw_0B \setminus N_0w_0B.$$

Thus $f \notin M_{0,5} = B_+ f_6 \subseteq \{f \in V | \mathrm{supp}(f) \leq N_0 w_0 B\}$. □

However, if $f \in M_0 \cap V_5$ then $\mathrm{supp}(f)$ is contained in $Bw_0B \cap \bigcup_{i>3} R_i B$: A straightforward computation shows that for any $n \in N_0$, $t \in T_+$, $w \in W$ and

- for any $r \in R_{w_1}$ we have $ntf_w(r) = ntf_w(w_1)$. Let $r' = w_1 \in G_5$,
- for any $r \in R_{w_2}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & 0 & 1 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & \gamma' & 1 \end{pmatrix},$$

- for any $r \in R_{w_3}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' - \beta\gamma' & \gamma & 1 \\ 0 & 1 & 0 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' & \gamma & 1 \\ \beta' & 1 & 0 \end{pmatrix}.$$

Thus if $i < 4$ and $r \in R_{w_i}$, then since $r' \notin Bw_0B$ we have $f(r) = f(r') = 0$.

Proposition 6.3 *The quotients $M_{0,m-1}/M_{0,m-1} \cap V_m$ via $f \mapsto f(\cdot w_m)$ are isomorphic to $M(w_m, \chi)$.*

Proof It is obvious, that $f(\cdot w_m) \equiv 0$ implies $f|_{G_m \setminus G_{m-1}} \equiv 0$ and $f \in M_{0,m-1} \cap V_m$. Hence the map $M_{0,m-1}/M_{0,m-1} \cap V_m \to M(w_m, \chi)$, $f \mapsto f(\cdot w_m)$ is injective.

Let $t_0 = \mathrm{diag}(\pi_F^{n-1}, \pi_F^{n-2}, \dots, \pi_F, 1) \in T_+$, and for any $l \in \mathbb{N}$ let $U^{(l)} = \mathrm{Ker}(G_0 \to G(o_F/\pi_F^l o_F))$. For $x = rb \in R_{w_m}B$ we have

$$\sum_{n \in (N_0 \cap U^{(l)})/t_0^l N_0 t_0^{-l}} nt_0^l f_{w_m}(rb) = \begin{cases} \chi^{-1}(b), & \text{if } r \in U^{(l)} w_m, \\ 0, & \text{if not.} \end{cases}$$

The image of these generate $M(w_m, \chi)$ as an N_0-module, so $f \mapsto f(\cdot w_m)$ is surjective. □

Since $M_{0,m} \leq V_m$, $M(w_m, \chi)$ is naturally a quotient of $M_{0,m-1}/M_{0,m}$, we have $D(V_{m-1}/V_m) \leq (M_{0,m-1}/M_{0,m})^*$.

Proposition 6.4 *For $m = 1$ and $m = n! - n + 1, n! - n + 2, \dots, n!$ $(M_{0,m-1}/M_{0,m})^* = D(V_{m-1}/V_m)$. For other m-s it is not true, for example if $n = 3$, $F = \mathbb{Q}_p$ and $m = 2, 3$.*

Proof By the previous proposition it is enough to show that $M_{0,m} = M_{0,m-1} \cap V_m$ for $m = 1$ and $m > n! - n$.

For $m = 1$ the quotient is obviously k_K, for $m > n! - n$ we have $w \prec w_m$ implies $w = w_{n!}$, so if $f \in B_+ f_{w_m} \cap V_{m-1} = B_+ f_{w_m} \cap V_{n!-1}$, then $\mathrm{supp}(f) \subset U^{(1)} R^{(1)}_{w_{n!-1}} B$. But

$$M_{0,n!-1} \simeq C^\infty(N_0) \simeq \{f \in V_{n!-1} | \mathrm{supp}(f) \subset U^{(1)} R_{w_{n!-1}} B\}.$$

The fuction f constructed in the beginning of this section is in $M_{0,1} \cap V_2 \setminus M_{0,2}$. The same can be done for $m = 3$. □

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