arXiv:1312.7351v1 [math.PR] 27 Dec 2013

Borel Liftings of Graph Limits

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Abstract

The cut pseudo-metric on the space of graph limits induces an equivalence relation. The quotient space obtained by collapsing each equivalence class to a point is a metric space with appealing analytic properties. We show that the equivalence relation admits a Borel lifting: There exists a Borel-measurable mapping which maps each equivalence class to one of its elements.

Let $(\Omega, \mathcal{B}(\Omega), P)$ be an atomless Borel probability space and $L_1(\Omega^2)$ the Banach space of integrable functions on $\Omega \times \Omega$, equipped with the L_1-metric d_1. Let $\mathbf{W} \subset L_1(\Omega^2)$ be the subspace of symmetric integrable functions $\Omega^2 \to [0,1]$. Define a pseudo-norm on $\mathbf{W}$ by

$$\|w\|_{\square} = \sup_{S,T \in \mathcal{B}(\Omega)} \int_{S \times T} w(s,t) dP(s) dP(t) \;. \tag{1}$$

Following [1], we use $\|\,.\,\|_{\square}$ to define a pseudo-metric on $\mathbf{W}$ as

$$\delta_{\square}(w,w') := \inf_{\psi} \|w^{\psi} - w'\|_{\square} \qquad \text{where} \qquad w^{\psi}(x,y) = w(\psi(x),\psi(y)) \;. \tag{2}$$

The infimum is taken over all invertible measure-preserving transformations of Ω, i.e. all invertible measurable mappings $\psi : \Omega \to \Omega$ satisfying $\psi P = P$. The pseudo-metric induces an equivalence relation on $\mathbf{W}$, given by $w \equiv w' \Leftrightarrow \delta_{\square}(w,w') = 0$. The relation $w \equiv w'$ is also known as *weak isomorphy* of w and w' [6]. Denote the equivalence class of $w \in \mathbf{W}$ by $[w]_{\square}$, and the quotient space of all equivalence classes by $\widehat{\mathbf{W}}$. On the quotient space, $\delta_{\square}$ is a metric, and the metric space $(\mathbf{W}, \delta_{\square})$ is known to be compact [8]. For each $\widehat{w} \in \widehat{\mathbf{W}}$, we write $[\widehat{w}]_{\square} \subset \mathbf{W}$ for the corresponding equivalence class of elements of $\mathbf{W}$.

Theorem 1 below shows that weak isomorphy admits a Borel lifting, i.e. there exists a Borel-measurable mapping $\xi : (\widehat{\mathbf{W}}, \delta_{\square}) \to (\mathbf{W}, d_1)$ such that

$$\xi(\widehat{w}) \in [\widehat{w}]_{\square} \qquad\qquad \text{for all } \widehat{w} \in \widehat{\mathbf{W}} \;. \tag{3}$$

The lifting is not unique. More precisely:

Theorem 1. *There is a sequence (ξ_n) of measurable mappings $\xi_n : (\widehat{\mathbf{W}}, \delta_{\square}) \to (\mathbf{W}, d_1)$ such that, for every $\widehat{w} \in \widehat{\mathbf{W}}$, the set $\{\xi_n(\widehat{w}) \,|\, n \in \mathbb{N}\}$ is a dense subset of $[\widehat{w}]_{\square}$.*

Proof. Theorem 1 can be stated equivalently by defining a set-valued mapping

$$\phi_{\square} : \widehat{\mathbf{W}} \to 2^{\mathbf{W}} \qquad \text{with} \qquad \phi_{\square}(\widehat{w}) := [\widehat{w}]_{\square} \ . \tag{4}$$

We then have to show that there are measurable mappings ξ_n with

$$\overline{\{\xi_n(\widehat{w}) \,|\, n \in \mathbb{N}\}} = \phi_{\square}(\widehat{w}) \qquad \text{for all } \widehat{w} \in \widehat{\mathbf{W}} \ , \tag{5}$$

where $\overline{A}$ denotes the closure of A.

Liftings of set-valued maps are a well-studied topic in analysis, and we use a result of Kuratowski and Ryll-Nardzewski [5] on the existence of liftings, and a generalization by Castaing [2] (see e.g. [3, Theorem 12.16] and [4, Theorem 14.4.1] for textbook statements). For our purposes, these results can be summarized as follows:

Theorem 2. *Let $\mathbf{X}$ be a measurable space, $\mathbf{Y}$ a Polish space, and $\phi : \mathbf{X} \to 2^{\mathbf{Y}}$ a set-valued mapping. Require $\phi(x)$ to be non-empty and closed for all $x \in \mathbf{X}$, and that*

$$\phi^{-1}(A) := \{x \in \mathbf{X} \,|\, \phi(x) \cap A \neq \emptyset\} \tag{6}$$

is a measurable set in $\mathbf{X}$ for each open set A in $\mathbf{Y}$. Then there exists a sequence of measurable mappings $\xi_n : \mathbf{X} \to \mathbf{Y}$ such that $\overline{\{\xi_n(x) \,|\, n \in \mathbb{N}\}} = \phi(x)$ for all $x \in \mathbf{X}$.

For $\phi_{\square}$ as defined in (4) and any subset $A \subset \mathbf{W}$, the set $\phi_{\square}^{-1}(A)$ in (6) simply consists of all $\widehat{w} \in \widehat{\mathbf{W}}$ for which A contains at least one element of the equivalence class $[\widehat{w}]_{\square}$. If A is in particular an open d_1-ball in $\mathbf{W}$, this set has the following property:

Lemma 3. *Denote by $U_\varepsilon(v)$ the open d_1-ball of radius ε centered at $v \in \mathbf{W}$. If $\varepsilon < \delta$,*

$$\overline{\phi_{\square}^{-1}(U_\varepsilon(v))} \ \subseteq \ \phi_{\square}^{-1}(U_\delta(v)) \tag{7}$$

for all $v \in \mathbf{W}$.

Proof of Lemma 3. Let $(\widehat{w}_1, \widehat{w}_2, \ldots)$ be a sequence in $\phi_{\square}^{-1}(U_\varepsilon(v))$ with $\widehat{w}_i \xrightarrow{\delta_{\square}} \widehat{w}$. We have to show that $\widehat{w} \in \phi_{\square}^{-1}(U_\delta(v))$. By definition of $\phi_{\square}^{-1}$, the sets $\phi_{\square}(\widehat{w}_i) \cap U_\varepsilon(v)$ are non-empty. Suppose (w_i) is a sequence with $w_i \in \phi_{\square}(\widehat{w}_i) \cap U_\varepsilon(v)$ for each $i \in \mathbb{N}$. By [9, Lemma 2.11], convergence of $(\widehat{w}_i)$ to $\widehat{w}$ then implies

$$\varepsilon \geq \liminf \delta_1(w_i, v) \geq \delta_1(w, v) = \inf_{\psi} d_1(w^{\psi}, v) \tag{8}$$

for any $w \in \phi_{\square}(\widehat{w})$. Since $\varepsilon < \delta$, there is hence a measure-preserving transformation ψ such that $d_1(w^{\psi}, v) < \delta$, that is, $w^{\psi} \in U_\delta(v)$. Because w^{ψ} and w are weakly isomorphic, we also have $w^{\psi} \in \phi_{\square}(\widehat{w})$, and therefore

$$\widehat{w} \ \in \ \phi_{\square}^{-1}(\phi_{\square}(\widehat{w}) \cap U_\delta(v)) \ \subset \ \phi_{\square}^{-1}(U_\delta(v)) \ . \tag{9}$$

□

Proof of Theorem 1. The space (W, d_1) is a closed subspace of the separable Banach space $L_1(\Omega^2)$ and hence Polish. The sets $\phi_{\square}(\widehat{w})$ are non-empty, by definition of the space $\widehat{\mathbf{W}}$ as a quotient. We will show that, additionally:

i. The sets $\phi_{\square}(\widehat{w})$ are closed.

ii. For all open sets A in $\mathbf{W}$, the set $\phi_{\square}^{-1}(A)$ is Borel in $\widehat{\mathbf{W}}$.

The mapping $\phi_\square$ therefore satisfies the hypothesis of Theorem 2, and Theorem 1 follows.

(i) Denote by $t_F : \mathbf{W} \to [0,1]$ the homomorphism density indexed by a finite graph F [7]. Two elements of $\mathbf{W}$ are weakly isomorphic if and only if their homomorphism densities coincide for all finite graphs F. Let $\widehat{w} \in \widehat{\mathbf{W}}$, and let $(w_1, w_2, \ldots)$ be a sequence in the set $\phi(\widehat{w})$ with limit w in $(\mathbf{W}, d_1)$. The homomorphism densities are $\delta_\square$-continuous and hence d_1-continuous. Therefore,

$$\lim t_F(w_i) = t_F(w) \qquad \text{for all } F \;, \tag{10}$$

and since the w_i are weakly isomorphic, $t_F(w_i) = t_F(w)$ for all i and all F. Thus, $w \in \phi(\widehat{w})$, and the set is closed.

(ii) Let $U_\delta(v)$ denote the open ball of radius δ centered at $v \in \mathbf{W}$. Since W is Polish, the open balls form a base of the topology and it is sufficient to consider sets of the form $A = U_\delta(v)$. Let $\delta_i \in \mathbb{R}_+$ be an increasing sequence $\delta_i \to \delta$. Then, by Lemma 3,

$$\phi_\square^{-1}(U_\delta(v)) \;=\; \bigcup_i \phi_\square^{-1}(U_{\delta_i}(v)) \;\subseteq\; \bigcup_i \overline{\phi_\square^{-1}(U_{\delta_i}(v))} \;\overset{(7)}{\subseteq}\; \bigcup_i \phi_\square^{-1}(U_\delta(v)) \;=\; \phi_\square^{-1}(U_\delta(v)) \;.$$

In particular, $\phi_\square^{-1}(U_\delta(v))$ is a countable union of the closed sets $\overline{\phi_\square^{-1}(U_{\delta_i}(v))}$, and hence Borel. □

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