PATHS ON THE SPHERE WITHOUT SMALL ANGLES

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ABSTRACT. It is a recent result that given a finitely many points on $\mathbb{R}^2$, it is possible to arrange them on a polygonal path so that every angle on the polygonal path is at least $\pi/9$. Here we extend this result to finite sets contained in the 2-dimensional sphere.

1. Introduction and results

Let X be a finite set in $\mathbb{R}^2$. An ordering, $x_1, x_2, \dots, x_n$, of the points of X gives rise to a polygonal path $p = x_1x_2 \dots x_n$ on X: its edges are the segments connecting x_i to x_{i+1}. The angle of p at x_i is just $\angle x_{i-1}x_ix_{i+1}$. The path is called α-good if all of its angles are at least α where $\alpha > 0$. Answering a question of Sándor Fekete [3] from 1992, (cf [4] as well) we proved in [1] the following result.

Theorem 1. *If X is a finite set in the plane, then there is an α_0-good path on X with $\alpha_0 = 20° = \pi/9$.*

The aim of this paper is to extend this result to finite sets $X \subset S^2$, the 2-dimensional Euclidean sphere. The definitions are almost the same. Given $a, b \in S^2$ there is a shortest path $\widehat{ab} \subset S^2$ connecting a and b in S^2. This shortest path is an arc of the great circle containing a and b, and is unique unless a and b are antipodal. An ordering, $x_1, x_2, \dots, x_n$, of the points of X is identified with a path $x_1x_2 \dots x_n$ on X consisting of the arcs $\widehat{x_ix_{i+1}}$. The angle of this path at x_i is just the spherical angle at x_i of the spherical triangle with vertices x_{i-1}, x_i, x_{i+1}. The path is called α-good if all of its angles are at least α where $\alpha > 0$.

Theorem 2. *There is $\alpha > 0$ such that for every finite set $X \subset S^2$ there exists an α-good path on the points of X (using every point of X exactly once).*

The proof gives $\alpha = 5°$ via generous computations. Slightly larger value for α can be reached by more careful calculations but we have not tried to find the best possible α. The planar example consisting of the vertices of an equilateral triangle and its center shows that Theorem 1 cannot hold with $\alpha_0 > 30°$. The same applies to the spherical case as shown by a small size

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spherical and equilateral triangle in S^2. Jan Kynčl [2] has proved recently that Theorem 1 holds with $\alpha = 30°$, the best possible bound. Using his results the bound $\alpha = 5°$ can be improved to $\alpha = 7°$.

The same question can be asked on higher dimensional spheres S^d. The methods of this paper work there as well, resulting in a smaller universal α, see Section 6.

We will need a stronger version of Theorem 1 which is proved in [1]. To state it a few additional definitions are needed. The direction $\overline{xy}$ of a pair $x, y \in \mathbb{R}^2$ is the unit vector $(y-x)/|y-x|$, we suppose here that $x \neq y$. So $\overline{xy} \in S^1$, the unit circle.

Given a path $z_1 z_2 \dots z_n$ in the plane the directions $\overline{z_2 z_1}$ and $\overline{z_{n-1} z_n}$ are called the *end directions* of the path. We call a subset R of S^1 a *restriction* if it is the disjoint union of two closed arcs $R_1, R_2 \subset S^1$ such that both have length $4\alpha_0$ and their distance from each other (along the unit circle) is larger than $2\alpha_0$. (Recall that $\alpha_0 = 20°$.) We call the path $z_1 \dots z_n$ *R-avoiding* if the path is α_0-good and the two end directions are not in the same R_i $(i = 1, 2)$.

Theorem 3. *Let X be a finite set of points in the plane. For every restriction R there is an R-avoiding path on all the points of X.*

2. Preparations

In the proofs to come we assume that our finite set $X \subset S^2$ contains no antipodal pair. The general case follows from this by a simple limit argument.

Given $a, b \in S^2$, the length of the arc $\widehat{ab}$ is simply the angle between the vectors a and b, measured in degrees (sometimes in radians). Of course the length of $\widehat{ab}$ can be expressed by the Euclidean distance $|a - b|$. The pair $a, b \in X$ is a diameter of X if it has the largest length among all pairs in X.

For the proof of Theorem 2 we need two auxiliary results. The first one is simpler: it is essentially the planar case, that is Theorem 1 applied on S^2. Precisely, let P be a plane touching S^2 at a point $z \in S^2$ and let $C = C(t)$ be the cap of S^2 defined by

$$C(t) = \{x \in S^2 : z \cdot x \geq t\}$$

where $t \in (0, 1)$.

Theorem 4. *If $X \subset C(t)$ is finite, then there is an $\alpha(t)$-good path on X where $\alpha(t) \in (0, 90°)$ is given by $\sin \alpha(t) = t \sin 20°$.*

The proof is given in Section 4. The following corollary to Theorem 4 will be used in the proof of Theorem 2. Note that $\alpha(1/2) = 9.846..° > 9°$. Set $\alpha_1 = 9°$.

Corollary 1. *If the diameter of X is at most $60°$, then there is an α_1-good path on X.*

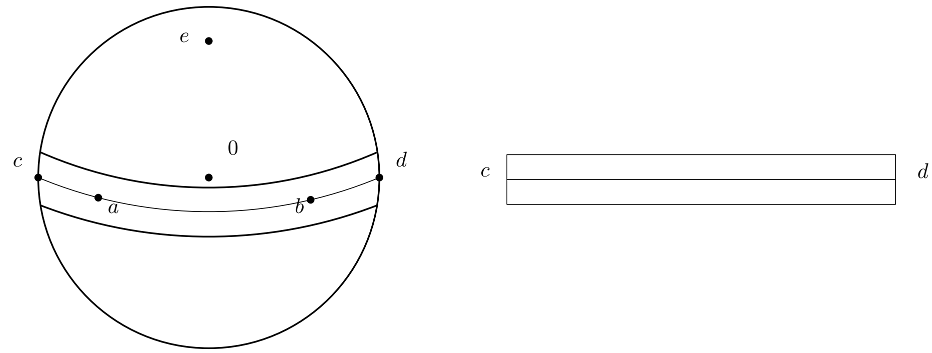

FIGURE 1. The spherical halfslab $Q(a,b)$ and its planar representation

To state the second auxiliary result we need some definitions. Let $a, b \in X$ form a diameter of $X \subset S^2$. Set $c = (a-b)/|a-b|$ so $c \in S^2$. Choose $e \in S^2$ that is orthogonal to both c and $a+b$. Let $\beta = 10°$. We define the halfslab $Q = Q(a,b)$ as

$$Q = \{x \in S^2 : (a+b)\cdot x \geq 0,\ |e \cdot x| \leq \sin\beta\},$$

see Figure 1. Here is the second auxiliary result.

Theorem 5. *If a, b form a diameter of X and $X \subset Q(a,b)$, then there is an α-good path on X (where $\alpha = 5°$).*

We prove this theorem in Section 6 with some preparations in Section 5. The next section contains the proof of Theorem 2. It is essentially an induction argument reducing the problem to two cases: when X lies in a cap $C(t)$ for some t and when X lies in the halfslab $Q(a,b)$. These two cases are covered by Theorems 4 and 5.

3. Proof of Theorem 2

We introduce further terminology and notation before the proof. Given $u, v \in S^2$ with $u \neq \pm v$, let $L(u;v)$ be the half of the great circle connecting u to $-u$ that contains v. The union of $L(u;v)$ and $L(u;w)$ (when $w \notin L(u;v)$) is a closed curve without self-intersection on S^2 so it splits S^2 into two connected components to be called *sectors*. Let $E(u;v,w)$ denote the smaller one of the two. No confusion will arise here since $E(u;v,w)$ will always be much smaller than the other sector.

Note that for $x, y \in E(u;v,w)$ the arc $\widehat{xy} \subset E(u;v,w)$.

Let $L(u;z)$ be the half of the great circle exactly halving $E(u;v,w)$. Let γ be the angle between the planes $L(u;v)$ and $L(u;z)$, we call γ the *angle of the sector* $E(u;v,w)$. Note that this angle is at most $90°$ always.

We will often write $E(u;\gamma,z)$ or simply $E(u;\gamma)$ instead of $E(u;v,w)$ where γ is the angle of this sector, especially when v and w are not important.

Proof of Theorem 2. It goes by induction on $|X|$. Everything is easy when $|X| = 1, 2$ or 3. Suppose now that $|X| > 3$. Assume that there is a spherical triangle $\triangle$ with vertices a, b, c with all of its angles at least 2α which is not contained in any sector $E(z;\alpha)$ when $z \in X$. Then induction works as follows. Find first an α-good path $p = x_1x_2 \dots x_n$ on $X \setminus \{a, b, c\}$. Define $E(x_1;\alpha)$ as $E(x_1;\alpha, x_2)$ if $n > 1$ (that is, $|X| > 4$), and as $E(x_1;\alpha, c)$ if $n = 1$. As some vertex of $\triangle$, say a, is not contained in $E(x_1;\alpha)$, $ax_1x_2 \dots x_n$ is an α-good path. The angle of $\triangle$ at a is at least 2α so either $\angle bax_1$ or $\angle cax_1 \geq \alpha$. Suppose, say, that $\angle bax_1 \geq \alpha$. Then $cbax_1 \dots x_n$ is an α-good path on X, even $\angle cba \geq 2\alpha$.

So we can assume that no such triangle $\triangle$ exists. If the diameter of X is at most $60°$, then Corollary 1 applies and gives an α_1-good path on X (where $\alpha_1 = 9°$). So suppose that the diameter, formed by the pair $a, b \in X$ is at least $60°$.

Observe now that $\widehat{ab}$ is contained in no sector $E(z;\alpha)$ with $z \in X \setminus \{a, b\}$. Indeed, $\widehat{ab}$ is the longest side of the spherical triangle with vertices a, b, z. Then the largest angle of this triangle is at vertex z, and this largest angle is more than $60° > 2\alpha$.

We claim now that no point of X is outside of the set

$$F := E(a; 2\alpha, b) \cup E(b; 2\alpha, a).$$

Assume the contrary and let $c \in X \setminus F$. All angles of the spherical triangle $\triangle$ with vertices a, b, c are larger than 2α: the angle at c is at least $60° > 2\alpha$ as we just saw, while for the angles at a, b this follows from $c \notin F$. The triangle $\triangle$ is not contained in any sector $E(z;\alpha)$ for $z \in X \setminus \{a, b\}$ as $\widehat{ab}$ is not contained in such a sector. Further $\triangle \subset E(a;\alpha)$ is impossible because $c \notin E(a; 2\alpha, b)$, and $\triangle \subset E(b;\alpha)$ cannot hold for the same reason. Thus $\triangle$ is not contained in any sector $E(z, \alpha)$, $z \in X$, contradicting our previous assumption.

Consequently

$$X \subset F \cap \{x \in S^2 : |x - a|, |x - b| \leq |a - b|\}.$$

We observe that the set $F \cap \{x \in S^2 : |x-a|, |x-b| \leq |a-b|\}$ is contained in the halfslab $Q(a, b)$. Then Theorem 5 applies and finishes the proof. □

4. Proof of Theorem 4

For $x \in C(t)$ let x^* denote its radial projection (from the origin which is the center of S^2) to P. Then X^*, the radial projection of X, is a finite set in the plane P. So by Theorem 1 there is a polygonal path $p^* = x_1^* \dots x_n^*$ on X^* with all of its angles at least $20°$. The next lemma implies that the path $p = x_1 \dots x_n$ on X is $\alpha(t)$-good.

Lemma 1. *Assume $a, b, c \in C(t)$. Let the angle of the spherical triangle abc at c be $\phi < 90°$ and that of the (planar) triangle $a^*b^*c^*$ at c^* be ϕ^*. Then $\sin\phi \geq t \sin\phi^*$ if $\phi^* \leq 90°$ and $\sin\phi \geq t$ if $\phi^* > 90°$.*

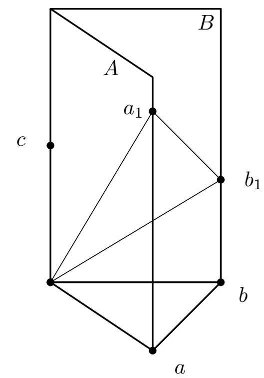

FIGURE 2. Proof of Lemma 1

Proof. Let $K \subset \mathbb{R}^3$ be the cone consisting of all the points of the form $\alpha a + \beta b + \gamma c$ where $\alpha, \beta \geq 0$ and $\gamma \in \mathbb{R}$. Its boundary consists of two halfplanes $A = \{\alpha a + \gamma c : \alpha \geq 0\}$ and $B = \{\beta b + \gamma c : \beta \geq 0\}$. The angle of this cone is $\phi \in (0, 180^\circ)$, which is the same as the angle between the two halfplanes A, B. The plane P that is tangent to S^2 at z intersects K in a 2-dimensional cone with angle ϕ^*. Translate P by $-z$. The translated copy P_1 contains the origin and intersects K in a 2-dimensional cone whose angle is also ϕ^*. We assume first that $\phi^* \leq 90^\circ$.

The condition $c \in C(t)$ implies that $z \cdot c \geq t$.

Let S be the unit circle centered at the origin in the plane orthogonal to c. We can assume that $a = S \cap A$ and $b = S \cap B$ as the angle ϕ remains the same. The plane P_1 intersects the lines $\{a + \lambda c : \lambda \in \mathbb{R}\}$ resp. $\{b + \lambda c : \lambda \in \mathbb{R}\}$ in points a_1 and b_1. Let T resp. T_1 be the triangle with vertices $0, a, b$ and $0, a_1, b_1$, see Figure 2.

Then Area $T = \frac{1}{2} \sin \phi$, and Area $T_1 = \frac{1}{2}|a_1| \cdot |b_1| \sin \phi^* \geq \frac{1}{2} \sin \phi^*$ since $|a_1|, |b_1| \geq 1$. As T is the orthogonal projection of T_1 to the plane orthogonal to c, Area $T = \cos \gamma$Area T_1 where γ is the angle of the planes containing T and T_1. Here $\cos \gamma = c \cdot z$ so we have

$$\sin \phi \geq c \cdot z \sin \phi^* \geq t \sin \phi^*$$

finishing the proof when $\phi^* \leq 90^\circ$.

In the case $\phi^* > \frac{\pi}{2}$ fix c and b and rotate a towards b around the line through 0 and c. The angles ϕ and ϕ^* will continuously decrease. Rotate a till $\phi_1^* = 90^\circ$. Now $\sin \phi \geq \sin \phi_1 \geq t \sin \phi_1^* = t$ which finishes the proof. □

5. DECREASING PATHS

Some preparations are needed before the proof of Theorem 5. We assume that S^2 is centered at the origin. For $A \subset \mathbb{R}^3$ we let lin A denote the linear hull of A. We call the 2-dimensional plane $H = \text{lin}\,\{a, b\}$ *the horizontal*

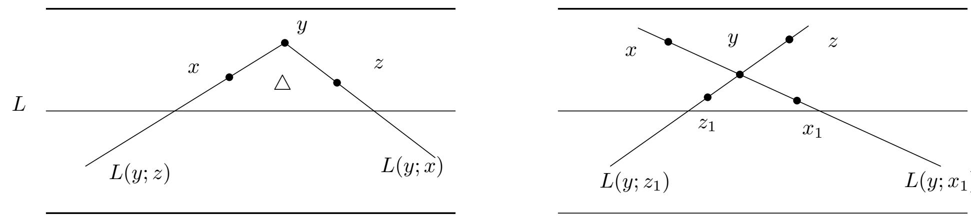

FIGURE 3. The two cases in Proposition 1

plane. H intersects the halfslab $Q = Q(a,b)$ in the halfcircle $L = L(c;a)$ whose endpoints are c and $d = -c$. Let e be the unit normal vector of H.

The *slope* of a pair $x, y \in X$ is the angle between H and the 2-plane $\text{lin}\{x,y\}$. We denote this angle by $\sigma(x,y)$. Note that $\sigma(x,y) \in [0, 90°]$ always. We call a pair $x, y \in X$ *steep* if $\sigma(x,y) \geq 40^0$.

If there is no steep pair in X, then one can construct an α_2-good path on X with $\alpha_2 = 100°$ very easily. For $x \in Q$ let $h(x) = e \cdot x$ (the height of x) and let $\tau(x)$ be the angle between c and the midpoint of the half great circle $L(e;x)$. (Thus for instance, $\tau(c) = 0$ and $\tau(d) = 180°$). Order the points of X by decreasing $\tau(x)$ and call the resulting path the *decreasing path* of X. The following proposition shows that all angles of the decreasing path are at least $180° - 2 \cdot 40° = 100°$.

Proposition 1. *Assume $x, y, z \in Q$ and let γ be the angle of the spherical triangle with vertices x, y, z at vertex y. If $\tau(x) \leq \tau(y) \leq \tau(z)$, then $\gamma \geq 180° - \sigma(x,y) - \sigma(y,z)$.*

Proof. We may assume by symmetry that $h(y) \geq 0$. To simplify the proof we also assume that $\tau(x) < \tau(y) < \tau(z)$ and $h(y) > 0$. The general case follows from this by a simple limit argument.

Observe next that x can be replaced by any point (distinct from y) on the arc $\widehat{xy}$. The same applies to z. So we assume that x and z are close to y, in particular, $h(x), h(z) > 0$.

The first and basic case is when z lies below the plane $\text{lin}\{x,y\}$. Then the half-circles $L(y;x), L(y;z)$ and the great circle $H \cap S^2$ delimit a spherical triangle $\triangle$, see Figure 3 left. The angle of $\triangle$ at y coincides with $\angle xyz$, and its other two angles are $\sigma(x,y)$ and $\sigma(y,z)$. Thus $\angle xyz \geq 180° - \sigma(x,y) - \sigma(y,z)$, indeed.

The second case is when z is above the plane $\text{lin}\{x,y\}$. Choose points x_1 and z_1 in S^2 close to, but distinct from, y so that $y \in \widehat{xx_1}$ and $y \in \widehat{zz_1}$, see Figure 3 right. Then $\tau(z_1) < \tau(y) < \tau(x_1)$, and $h(z_1), h(y), h(x_1)$ are all positive, and x_1 lies below the plane $\text{lin}\{z_1, y\}$. The previous basic case applies now to z_1, y, x_1 in place of x, y, z. Thus $\angle z_1yx_1 \geq 180° - \sigma(z_1,y) - \sigma(y,x_1)$. Here $\angle z_1yx_1 = \angle xyz$ and $\sigma(z_1,y) = \sigma(y,z)$ and $\sigma(y,x_1) = \sigma(x,y)$. Consequently $\angle xyz \geq 180° - \sigma(x,y) - \sigma(y,z)$ again. □

We remark that the decreasing path method is applicable to any subset, say Y, of X that contains no steep pair. In that case the decreasing path on Y is α_2-good.

6. Proof of Theorem 5

For $u, v \in L$ with $\tau(u) < \tau(v)$ we define

$$T(u, v) = \{x \in Q : \tau(u) \le \tau(x) \le \tau(v)\}.$$

Now let $u, v \in L$ be two points with $\tau(v) - \tau(u) = 30°$. Thus c, u, v, d come on L in this order.

Proposition 2. *If $x, y \in X$ with $x \in T(c, u)$ and $y \in T(v, d)$, then $\sigma(x, y) < 35°$.*

Proof. The spherical cotangent formula (see spherical trigonometry on wikipedia, for instance) says that $\cos c \cos B = \cot a \sin c - \cot A \sin B$ where a, b, c are the sides, and A, B, C the opposite angles of the spherical triangle. With $b = 10°$, $c = 15°$, $B = 90°$ this shows that the angle in question is at most $\text{arccot}(\cot 10° \sin 15°) = 34.2656..$, indeed smaller than $35°$. □

Define now $t = \sin 15° \cos 10° = 0.25488..$ and set $\alpha_3 \in (0, 90°)$ by

$$\sin \alpha_3 = t \sin 20° = 0.087176..$$

and $\alpha_3 > 5°$ follows.

Lemma 2. *Assume again that $u, v \in L$ with $\tau(v) - \tau(u) = 30°$, and that $\tau(u) \in [90°, 120°]$ and further that there is no steep pair from X in $T(u, v)$. Set $Y = X \cap T(c, v)$. Then there is an α_3-good path $y_1 \dots y_m$ on Y such that $\angle xy_1y_2 > 5°$ for every $x \in T(v, d)$.*

Proof. The conditions imply that $\tau(v) \le 150°$. Then a simple computation shows that Y is contained in a cap $C(t)$ with center $z \in L$ where $t = \sin 15° \cos 10°$. This value for $t = \cos b$ comes from the spherical cosine theorem $\cos b = \cos c \cos a + \sin c \sin b \cos B$ with $B = 90°$, $c = 75°$, $a = 10°$. Project Y radially to the plane P that touches S^2 at z. We get a finite set Y^* in P. The unit circle $S \subset P$ is centered at z. Let $R = R_1 \cup R_2 \subset S$ be the restriction in P where the line $H \cap P$ halves both R_1 and R_2. The radial projection c^* of c lies in P and we choose the names so that $c^* \in R_1$.

According to Theorem 3, there is a $20°$-good path $y_1^* y_2^* \dots y_m^*$ on Y^* which is R-avoiding, that is, not both end directions are in the same R_i. Here we choose the names so that $\overline{y_2^* y_1^*} \notin R_1$. Theorem 4 implies that $y_1 \dots y_m$ is an α_3-good path on $Y \subset S^2$.

We have to check that $\angle xy_1y_2 > 5°$ for every $x \in T(v, d)$. We distinguish two cases.

Case 1. When y_1, y_2 is not a steep pair. Then, as is easy to check, the angle between the line spanned by y_1^*, y_2^* and the line $H \cap P$ is smaller than

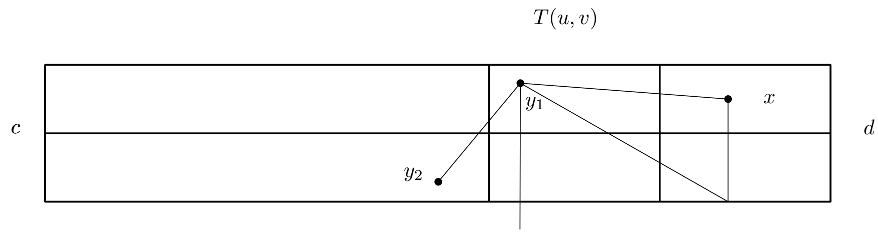

FIGURE 4. Case 2a

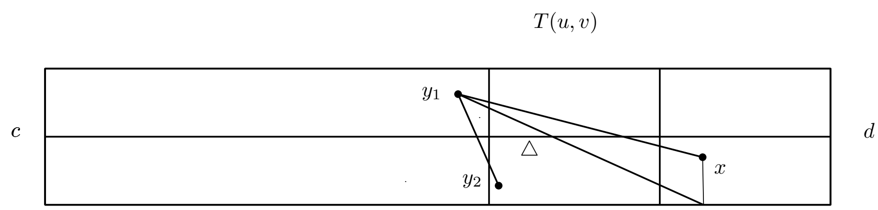

FIGURE 5. Case 2b

$40°$, so $\overline{y_2^* y_1^*} \in R_2$. Then $\tau(y_2) < \tau(y_1) \leq \tau(x)$. Proposition 1 shows that $\angle xy_1y_2 \geq 180° - 90° - 40° > 5°$.

Case 2. When y_1, y_2 is a steep pair. Then at least one of y_1 and y_2 is in $T(c, u)$. We assume by symmetry that $h(y_1) \geq h(y_2)$. Clearly $\tau(x) > \tau(y_1), \tau(y_2)$. There are two subcases.

Case 2a. When $\tau(y_2) \leq \tau(y_1)$. Then $y_2 \in T(c, u)$ and the angle in question decreases if x is pushed down to $h(x) = -\sin 10°$ while keeping $\tau(x)$ the same. The halfcircle $L(e; y_1)$ cuts the angle $\angle xy_1y_2$ into two parts, see Figure 4. Assume that $\angle xy_1y_2 \leq 5°$, then both parts are at most $5°$. The spherical cosine theorem implies then that $\tau(y_1) - \tau(y_2) \leq 5°$ and $\tau(x) - \tau(y_1) \leq 5°$, contradicting $\tau(x) - \tau(y_2) \geq 30°$.

Case 2b. When $\tau(y_1) \leq \tau(y_2)$. Then $y_1 \in T(c, u)$. The angle in question decreases again if x is pushed down to $h(x) = -\sin 10°$ while keeping $\tau(x)$ the same. Note that while x is pushed down, y_1, y_2 and x do not become coplanar as otherwise y_1, x would become a steep pair contradicting Proposition 2. Let $\triangle$ be the spherical triangle delimited by $L, L(x; y_1), L(y_2; y_1)$, see Figure 5. The angle of $\triangle$ at vertex y_1 equals $\angle y_2y_1x$. The other two angles of $\triangle$ are $180° - \sigma(y_1, y_2) \leq 140°$ because y_1, y_2 is a steep pair, and $\sigma(y_1, x) < 35°$ by Proposition 2. Thus $\angle y_2y_1x > 180° - (140° + 35°) = 5°$. □

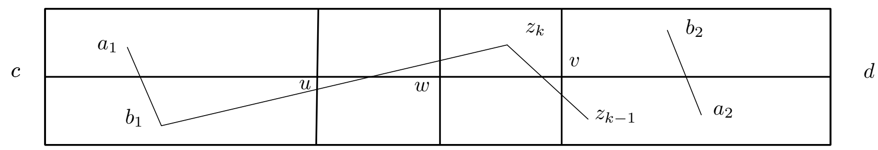

FIGURE 6. Part of the constructed path

Proof of Theorem 5. We have to consider two cases.

Case 1. There is no steep pair in $T(u,d)$ where $\tau(u) = 120°$. We can apply Lemma 2 to $T(u,v)$: setting $Y = X \cap T(c,v)$ there is no steep pair from Y in $T(u,v)$. We get an α_3-good path $y_1 \dots y_m$ on Y. Let $x_1 \dots x_k$ (where $m+k=n$) be the decreasing path on the points of $X \setminus Y$ which is an α_2-good path on $X \setminus Y$ ($\alpha_2 = 100°$). We claim that $x_1 \dots x_k y_1 \dots y_m$ is an α-good path on X. We only have to check its angles at x_k and y_1. The angle at y_1 is at least α by Lemma 2. The pair $x_{k-1}x_k$ is not steep and $\tau(y_1) \le \tau(x_k) < \tau(x_{k-1})$. Then Proposition 1 shows that the angle at x_k is at least $180° - 90° - 40° > 5°$.

The same method works when there is no steep pair in $T(c,v)$ where $\tau(v) = 60°$.

Define now $u, v, w \in L$ by $\tau(u) = 60°$, $\tau(w) = 90°$, and $\tau(v) = 120°$. We are left with the following case.

Case 2. There is a steep pair $a_1, b_1 \in X \cap T(c,u)$ and a steep pair $a_2, b_2 \in X \cap T(v,d))$. By swapping names if necessary we may assume that $\tau(a_1) \le \tau(b_1)$ and that $\tau(b_2) \le \tau(a_2)$. Set $Y = T(c,w) \cap X \setminus \{a_1, b_1\}$ and $Z = T(w,d) \cap X \setminus \{a_2, b_2\}$. Lemma 2 applies to $T(w,v)$ and Y because there is no steep pair from Y in $T(w,v)$ (actually, no point of Y there at all). We get an α_3-good path $y_1 \dots y_m$ on Y such that $\angle b_2 y_1 y_2 > 5°$. The same lemma applies to Z and $T(u,w)$ giving an α_3-good path $z_1 \dots z_k$ on Z with $\angle b_1 z_k z_{k-1} > 5°$. Here $m + 4 + k = n$, and the case when either Y or Z is empty or singleton is easy.

We claim finally that $z_1 \dots z_k b_1 a_1 a_2 b_2 y_1 \dots y_m$ is an α-good path, see Figure 6. We have to check the angles at a_1, b_1 and also at a_2, b_2 but the latter would follow by symmetry. Observe that $\tau(a_1) < \tau(b_1) \le \tau(z_k)$ and $\sigma(z_k, b_1) < 35°$. Then Proposition 1 shows that the angle at b_1 is at least $180° - 90° - 35° > 5°$. Finally, the pair a_1, b_1 is steep and $\tau(a_1) \le \tau(b_1)$, and $\tau(a_2) - \tau(a_1) \ge 60° > 30°$. The spherical triangle with vertices b_1, a_1, a_2 satisfies the same conditions as the triangle y_2, y_1, x in Case 2b in the proof of Lemma 2. The same argument shows then that the angle at a_1 is larger than $5°$. □

7. Higher dimensions

In the paper [1] we proved the higher dimension analogue of Theorem 1 in the following form.

Theorem 6. *For every $d \geq 2$ there is a positive α_d such that for every finite set of points $X \subset \mathbb{R}^d$ there exists an α_d-good path on X.*

Here the value of α_d is $\pi/80$ (for $d > 2$), see [1]. The proof of Theorem 2 goes through in higher dimensions without any real difficulty, and gives the following result.

Theorem 7. *There exists a constant $\alpha > 0$ such that for every $d \geq 2$ and for every finite set of points $X \subset S^d$ there exists an α-good path on X.*

We omit the details.

8. Open problems

The same question comes up in more general settings. For instance when X is a finite subset of the boundary of a convex body (compact convex set with nonempty interior) $K \subset \mathbb{R}^3$ (and $\mathbb{R}^d$, $d \geq 2$). Again there is a shortest path $\widehat{ab}$, the geodesic connecting a, b in ∂K. So an ordering $x_1, \ldots, x_n$ of the elements of a finite set $X \subset \partial K$ gives rise to a path on ∂K. The angle at x_i is defined in the usual way. Extending Theorem 2 would mean that there is $\alpha > 0$ such that for every convex body $K \subset \mathbb{R}^3$ and for every finite $X \subset \partial K$ there is an ordering such that every angle of the corresponding path is at least α. We suspect that such a universal α exists.

The same problem can be considered on a smooth or piecewise linear manifold. We remark however that in the hyperbolic plane there are triangles with all three angles very small. The same thing occurs on other 2-dimensional manifolds for instance when they have three long "tentacles".

Here comes an abstract or combinatorial version of the same problem. Let X be a finite set. For every three elements a, b, c in X the *combinatorial angle* is a real number $\angle abc \in [0, 1]$ satisfying the following conditions:

- $\angle abc = \angle cba$ for all $a, b, c \in X$, (symmetry),
- $\angle abc + \angle cbd \geq \angle abd$ for all $a, b, c, d \in X$, (triangle inequality),
- $\angle abc + \angle bca + \angle cab \geq 1$ for all $a, b, c \in X$, (no small triangle).

The question is now whether there exists an $\varepsilon > 0$ such that for every finite set X with angles satisfying these three conditions there is an ordering $x_1, \ldots, x_n$ of the elements of X such that $\angle x_{i-1}x_ix_{i+1} \geq \varepsilon$ for every $2 \leq i \leq n-1$.

It turns out that for every n there exists a largest number $\varepsilon = \varepsilon(n)$ such that if $|X| = n$ there exist an ε-good path on X. In Lemma 3 below we show that if $\varepsilon(n)$ is not zero, then $\varepsilon(n) = \frac{1}{k}$ for some integer k. One can check the case $n = 4$ directly and show that $\varepsilon(4) = \frac{1}{6}$.

Let X be a finite set and let S be a subset of the combinatorial angles of X. We say that S is *blocking* if any path on X has an angle in S. Let $\angle_S abc$ be the smallest number t such that there are $b_0, \dots, b_t \in X$, where $b_0 = a, b_t = c$ and all the combinatorial angles $\angle b_i b b_{i+1}$ for $i = 0, \dots, t-1$ are in S. It is possible that $\angle_S abc = \infty$. Let $\alpha(S) = \min_{a,b,c \in X} (\angle_S abc + \angle_S bca + \angle_S cab)$. It is possible that $\alpha(S) = \infty$. Define $\alpha(n) = \max_{|X|=n, S \text{ is blocking}} \alpha(S)$. Clearly $\alpha(n)$ is an integer, or ∞.

Lemma 3. *If $\alpha(n)$ is an integer, then $\varepsilon(n) = \frac{1}{\alpha(n)}$.*

Proof. Let S be the blocking set where $\alpha(S) = \alpha(n)$. Then the abstract combinatorial geomery with $\angle abc = \frac{\angle_S abc}{\alpha(S)}$ shows that $\varepsilon(n) \le \frac{1}{\alpha(S)}$ since all the angles in S have that size, and S is blocking each path.

Assume that $\varepsilon(n) < \frac{1}{\alpha(n)}$. Then for some abstract geometry X every path contains an angle smaller than $\frac{1}{\alpha(n)}$. Let S be the set of all angles smaller than $\frac{1}{\alpha(n)}$. By definition S is blocking. By the triangle inequality $\angle abc < \frac{\angle_S abc}{\alpha(n)}$ for any angle. Let abc be the triangle where $\alpha(S) = \angle_S abc + \angle_S bca + \angle_S cab$. Obviously $\angle abc + \angle bca + \angle cab < \frac{\alpha(S)}{\alpha(n)} \le 1$ which is a contradiction. $\square$

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