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Additive structure of difference sets and a theorem of Følner

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Abstract

A theorem of Følner asserts that for any set $A \subset \mathbb{Z}$ of positive upper density there is a Bohr neigbourhood B of 0 such that $B \setminus (A - A)$ has zero density. We use this result to deduce some consequences about the structure of difference sets of sets of integers having a positive upper density.

1 Introduction

This paper is about the structure of the difference set $D(A) := A - A$ of sets of integers having positive density. By density we mean the upper asymptotic density defined by

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [-n, n]|}{2n + 1} > 0.$$

For sets $X, Y \subseteq \mathbb{Z}$ we mean

$$X + Y = \{x + y : x \in X; y \in Y\}$$

and

$$X \cdot Y = \{x \cdot y : x \in X; y \in Y\}.$$

When $X = \{x\}$ we write $x \cdot Y$.

We define a *Bohr set* as a set of the form

$$B(S,\varepsilon) = \{m \in \mathbb{Z} : \max_{s\in S} \|sm\| < \varepsilon\}, \tag{1.1}$$

where S is a finite set of real numbers. Here $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$, the absolute fractional part.

Recall that every Bohr set has positive density, and for every pair of sets S, S' and for every k, $0 < k \cdot \varepsilon' \leq \varepsilon$, we have

$$k \cdot B(S,\varepsilon') \subseteq B(S,\varepsilon), \tag{1.2}$$

and

$$B(S \cup S',\varepsilon) = B(S,\varepsilon) \cap B(S',\varepsilon) \tag{1.3}$$

(see e.g. [9] p. 165).

These sets are just the basic neighbourhoods of 0 in the Bohr topology. We say that $B(S,\varepsilon)$ is a k,ε-neighbourhood if $|S| = k$ (or a k-neighbourhood if ε is unimportant).

Bogolyubov [4] proved in the case of integers, and Følner [5], [6] generalized for general commutative groups, that the second difference set $D(D(A)) = A-A+A-A$ of a set having positive upper Banach density is always a Bohr neighborhood of 0.

In Bogolyubov's theorem four copies of A are used. Three suffice with a small change. If r, s, t are nonzero integers satisfying $r+s+t = 0$ and A is a set of integers having positive Banach density, then $S = rA+sA+tA$ is a Bohr neighbourhood of 0, see [3]. Here $rA = \{ra : a \in A\}$. The condition $r + s + t = 0$ is necessary to exclude trivial counterexamples; so there is no really "symmetric" result here. (A further comment on this is given in Section 3). The case $r = s = 1$, $t = -2$ immediately generalizes Bogolyubov's theorem.

On the other hand, a theorem of Kříž ([8]) implies that there is a set A with positive upper density whose difference set contains no Bohr set.

In the positive direction in [4] Følner proved that there is a Bohr set which is almost a subset of $A - A$; the exceptional set has zero density.

In this paper we give some applications of Følner's theorem. We investigate $A+A+A$ and $A+A-A$ and Bohr sets. In [1] Bergelson investigated the additive structure of $D(A)$. He also proved that for every k there exists an infinite set B of integers for which $A - A \supseteq B + B + \cdots + B$, k times, provided A has positive upper density. His proof of this theorem is based on an ergodic theorem, namely the Fürstenberg correspondence theorem In [7] the first author gave a purely combinatorial proof of this result. Here we give a third proof of it using Følner's theorem. See also Theorem 2.5 in [2].

2 Structure of sum-differences

We have already mentioned in the introduction that $D(D(A))$ always contains a Bohr set, while the set $D(A)$ does not necessary contain a Bohr set. Now we investigate the three-fold sum-differences of A. This generalizes Bogolyubov's theorem in a different direction.

Theorem 2.1 *There is a symmetric set A of integers such that $0 \in A$, $\overline{d}(A) > 0$ and the set $A + A + A$ does not contain a Bohr set.*

On the other hand we prove that $A + A - A$ is always a Bohr neighborhood of many $a \in A$.

Theorem 2.2 *Assume that $\overline{d}(A) > 0$. There exists a subset A' of A, such that $d(A \setminus A') = 0$ and for every $a' \in A'$, the set $A + A - A - a'$ is a Bohr neighbourhood of 0.*

We remark that the arguments used in our proof (see Section 3) actually yield the following property. For every A and X, with $\overline{d}(A) > 0, \overline{d}(X) > 0$, there exists a subset X' of X such that $d(X \setminus X') = 0$, and for every $x' \in X'$, the set $X + A - A - x'$ is a Bohr neighbourhood of 0. We leave the details of this to the interested reader.

Let $f : \mathbb{N}_+ \to \mathbb{N}_+$ be any function and $C \subseteq \mathbb{N}$; $C \neq \emptyset$. We will use the following notation:

$$FS_f(C) := \Big\{ \sum_{c_i \in X} w_i c_i : X \subseteq C,\ |X| < \infty;\ w_i \in [1, f(i)] \cap \mathbb{N} \Big\}.$$

Let the sum be zero when X is the empty set.

Furthermore write

$$FP(C) := \Big\{ \prod_{c_i \in X} c_i : X \subseteq C;\ X \neq \emptyset,\ |X| < \infty \Big\}.$$

Clearly we have

$$FS_f(\{c_1, c_2, \dots c_n\}) = FS_f(\{c_1, c_2, \dots c_{n-1}\}) + \{0, c_n, \dots, f(n)c_n\}, \tag{2.1}$$

and

$$FP(\{c_1, c_2, \dots c_n\}) = FP(\{c_1, c_2, \dots c_{n-1}\}) \cdot \{1, c_n\}, \tag{2.2}$$

for every $\{c_1, c_2, \dots c_n\} \subseteq \mathbb{N}$; $n \geq 2$; or equivalently,

$$FP(\{c_1, c_2, \dots c_n\}) = FP(\{c_1, c_2, \dots c_{n-1}\}) \cup c_n \cdot FP(\{c_1, c_2, \dots c_{n-1}\}).$$

Theorem 2.3 *Let A be a set of integers $\overline{d}(A) > 0$. Let $f : \mathbb{N}_+ \to \mathbb{N}_+$ be any function. There exists an infinite set C of integers, such that*

$$A - A \supseteq FS_f(C) \cup FP(C).$$

This will give a third proof of Bergelson's theorem (see [1]). In fact we can conclude that $A - A$ contains both an additive and a multiplicative structure.

3 Proofs

Proof of Theorem 2.1.

By the theorem of Kříž [8] we know the existence of a set X of positive integers for which $\overline{d}(X) > 0$, and the set $X - X$ does not contain a Bohr set. Let

$$Y = \{4x + 1 : x \in X\},$$

and

$$A = Y \cup -Y \cup \{0\}.$$

Since $\overline{d}(Y) = \frac{1}{4}\overline{d}(X) > 0$, we have $\overline{d}(A) > 0$ and the set A is symmetric and contains 0.

Now we prove that $A+A+A$ does not contain a Bohr set. Assume to the contrary that there is a $B(S, \varrho) \subseteq A + A + A$. Then by (1.2), $4 \cdot B(S, \varrho/4) \subseteq A + A + A$. Now notice that $4k \in A + A + A$ if and only if $4k \in Y - Y = 4(X - X)$. So we conclude that $B(S, \varrho/4) \subseteq X - X$ which contradicts the fact that $X - X$ does not contain a Bohr set. □

Proof of Theorem 2.2.

Let $B = B(S, \varepsilon)$ be a Bohr set for which

$$d(B(S, \varepsilon) \setminus (A - A)) = 0,$$

the existence of which is given by Følner's theorem. Since $\{B(S, \varepsilon) + x : x \in \mathbb{Z}\}$ is an open covering of $\mathbb{Z}$ in the (compact) Bohr topology, there is a finite set T for which

$$B(S, \varepsilon) + T = \mathbb{Z}.$$

For $t \in T$ write $A_t = A \cap (B + t)$. Some of these sets have positive upper density; let A' be the union of such sets A_t. Clearly $A \setminus A'$ is contained in the union of finitely many A_t of density 0, so it has density 0 itself.

Put $B' = B(S, \varepsilon/3)$. We now show $A + A - A \supset A' + B'$. This is equivalent to $A + A - A \supset A_t + B'$ whenever $\overline{d}(A_t) > 0$.

Take arbitrary $a \in A_t$, $b \in B'$. Consider the set $a + b - A_t$. This has positive upper density and

$$a + b - A_t \subset A_t - A_t + B' \subset (B' + t) - (B' + t) + B' = B' + B' - B' \subset B.$$

Hence $a + b - A_t$ is contained, up to a subset of density 0, in $A - A$, so we can find $a' \in A_t$ such that $a + b - a' \in A - A$, and consequently $a + b \in a' + A - A \subset A + A - A$ as wanted. □

Proof of Theorem 2.3.

We start our proof by quoting Følner's theorem again. We have a Bohr set for which the exceptional set has density zero, i.e. for some $B = B(S, \varepsilon)$, $E := B(S, \varepsilon) \setminus (A - A)$, $d(E) = 0$.

We will prove the existence of the infinite set C inductively.

Let $K_1 := f(1)$. Since any Bohr set has positive density and the exceptional set has zero density, and also using (1.2), it follows that one can find an element c_0 from $B(S, \varepsilon/K_1)$ such that $ic_1 \notin E$, for $i = 1, 2, \dots K_1$. So we have

$$FS_f(\{c_1\}) \cup FP(\{c_1\}) = \{0, c_1, \dots, K_1 c_1\} \subseteq B \setminus E \subseteq A - A.$$

Assume now that the elements $c_1 < c_2 < \cdots < c_n$ have been defined with the property

$$\mathcal{F}_n := FS_f(\{c_1, c_2, \dots, c_n\}) \cup FP(\{c_1, c_2, \dots, c_n\}) \subseteq B \setminus E \subseteq A - A.$$

Write $FP(\{c_1, c_2, \dots, c_n\}) = \{p_1 < p_2 < \cdots < p_m\}$, and let $K := \max\{f(n+1), p_m\}$. Define

$$\varepsilon_1 = \frac{1}{K} \min\{\varepsilon - \|xs\| : x \in FS_f(\{c_1, c_2, \dots, c_n\}); s \in S\}, \tag{3.1}$$

and let $B_1 := B(S, \varepsilon_1)$. Note that $B(S, \varepsilon_1) \subseteq B = B(S, \varepsilon)$.

By (3.1) we have that for every non-negative integer $i \leq K$, for every $u \in FS_f(\{c_1, c_2, \dots, c_n\})$, for every $c \in B_1$ and $s \in S$,

$$\|s(u + ic)\| < \varepsilon$$

holds; hence

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B.$$

Now we claim that there exists an element $c \in B_1$, with $c > c_1$, for which

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B \setminus E \subseteq A - A$$

also holds.

Assume to the contrary that for every $c \in B_1$ with $c > c_1$ there is at least one element $x \in FS_f(\{c_1, c_2, \dots c_n\})$ and one integer $j \in [1, \dots, K]$ for which $x + jc \in E$. Since $d(B_1 \setminus [1, c_n]) > 0$, by the pigeonhole principle there is then an $x_0 \in FS_f(\{c_1, c_2, \dots c_n\})$, $j_0 \in [1, \dots, K]$ and a $B_1' \subseteq B_1$, such that $\underline{d}(B_1) > 0$ and $x_0 + j_0 B_1' \subseteq E$, contradicting the fact that $d(E) = 0$ and $\underline{d}(x_0 + j_0 B_1') > 0$.

Let c_{n+1} be any such c. Since $K \geq p_m$ and $0 \in FS_f(\{c_1, c_2, \dots, c_n\})$ we have

$$c_{n+1} \cdot FP(\{c_1, c_2, \dots, c_n\}) \subseteq \{0, c_{n+1}, 2c_{n+1}, \dots, K \cdot c_{n+1}\} \subseteq B \setminus E.$$

Then by $(2, 2)$ and by the inductive hypothesis, $FP(\{c_1, c_2, \dots, c_n, c_{n+1}\}) \subseteq B \setminus E$. Moreover $K > f(n+1)$,

$$\begin{aligned} FS_f(\{c_1, c_2, \dots c_n, c_{n+1}\}) &\subseteq FS_f(\{c_1, c_2, \dots c_n\}) \\ &\qquad + \{0, c_{n+1}, 2c_{n+1}, \dots, K \cdot c_{n+1}\} \\ &\subseteq B \setminus E. \end{aligned}$$

Thus we have that

$$\mathcal{F}_{n+1} \subseteq B \setminus E \subseteq A - A,$$

as we wanted.

So our desired set is

$$C := \{c_1 < c_2 < \dots c_n < c_{n+1} < \dots\}.$$

□

4 Further problems and results

We mention some open problems and announce some new results without proofs.

Bogolyubov's proof is effective: given the density of A one can specify k, η so that $A + A - A - A$ contains a Bohr k, η-set. Følner's proof is not effective, and the reason is that an effective version does not hold:

For every $\alpha < 1/2$, $k \in \mathbb{N}$ and $\eta > 0$ there is an $A \subset \mathbb{Z}$, $\overline{d}(A) > \alpha$ such that $\overline{d}(V \setminus (A - A)) > 0$ for every Bohr k, η-set V.

Our proof of Theorem 2.2 about $A + A - A$ used Følner's theorem, and so it is not effective. We cannot decide whether an effective version holds. However, we can solve positively a related finite question. The result is as follows:

Let $\alpha > \varepsilon > 0$ be given. There exist k, η depending on α and ε with the following property. For every $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$, the set $S = A + A - A - a$ contains a Bohr k, η-set for all but εm elements $a \in A$.

Here $\mathbb{Z}_m$ is the group of residues modulo m and Bohr sets are defined as in (1.1) with the modification that only rational numbers for $s \in S$ of the form k/m can be used.

Assume $\overline{d}(A) > 0$. Is $A - A$ a Bohr neighbourhood of *something*? We know it may not be a neighbourhood of 0, and 0 is the most natural difference. For the analogous finite question we can give a negative answer, which is as follows:

Let $\alpha < 1/2$, k, η be given. For all large m there is an $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$, such that $A - A - x$ does not contain a Bohr k, η-set for any $x \in \mathbb{Z}_m$.

We close by posing the following open question.

Is $A - A$ a Bohr neighbourhood of 0 under the stronger assumption that A has positive lower Banach density? (In this case A is syndetic, that is, has bounded gaps).

Here we cannot solve the related finite problem either, and do not have any heuristic reasoning in any direction.

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References

[1] V. Bergelson, Sets of recurrence of $\mathbb{Z}^m$-actions and properties of sets of differences, *J. London Math. Soc.* (2) 31 (1985), 295–304.

[2] V. Bergelson, P. Erdős, N. Hindman and T. Łuczak, Dense difference sets and their combinatorial structure, (English summary), *The mathematics of Paul Erdős, I*, 165-175, *Algorithms Combin.* 13, Springer, Berlin, 1997.

[3] V. Bergelson and I.Z. Ruzsa, Sumsets in difference sets, *Israel J. Math.*, 174 (2009), 1–18.

[4] N. N. Bogolyubov, Some algebraical properties of almost periods, (in Russian), *Zapiski katedry matematichnoi fiziji* (Kiev) 4 (1939), 185–194.

[5] E. Følner, Generalization of a theorem of Bogoliuboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups, *Math. Scand.* 2 (1954), 5–18.

[6] E. Følner, Note on a generalization of a theorem of Bogoliuboff, *Math. Scand.* 2 (1954), 224–226.

[7] N. Hegyvári, Note on difference sets in $\mathbb{Z}^n$, *Period. Math. Hung.* 44 (2), 2002, 183–185.

[8] I. Kříž, Large independent sets in shift-invariant graphs: solution of Bergelson's problem, *Graphs Combin.* 3 (1987), 145–158.

[9] T. Tao and V. H. Vu, *Additive Combinatorics*, 526 pp., Cambridge University Press, 2006.

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