A note on tilted Sperner families with patterns

Dániel Gerbner* Máté Vizer†

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Abstract

Let p and q be two nonnegative integers with $p+q>0$ and $n>0$. We call $\mathcal{F} \subset \mathcal{P}([n])$ a *(p,q)-tilted Sperner family with patterns* on [n] if there are no distinct $F, G \in \mathcal{F}$ with:

$$(i) \;\; p|F \setminus G| = q|G \setminus F|, \text{ and}$$

$$(ii) \;\; f > g \text{ for all } f \in F \setminus G \text{ and } g \in G \setminus F.$$

E. Long in [10] proved that the cardinality of a (1,2)-tilted Sperner family with patterns on $[n]$ is

$$O(e^{120\sqrt{\log n}} \frac{2^n}{\sqrt{n}}).$$

We improve and generalize this result, and prove that the cardinality of every (p,q)-tilted Sperner family with patterns on $[n]$ is

$$O(\sqrt{\log n} \frac{2^n}{\sqrt{n}}).$$

Keywords: Sperner family, tilted Sperner family, permutation method

1 Introduction

A family $\mathcal{F}$ of subsets of $[n]$ (where for $n>0$ we will use the $[n]$ notation for $\{1,2,...,n\}$ and $\mathcal{P}([n])$ for the power set) is called a *Sperner family* if $F \not\subset G$ for all distinct $F, G \in \mathcal{F}$. A classic result in extremal combinatorics is Sperner's theorem [12], which states that the maximal cardinality of a Sperner family is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. This result has a huge impact on combinatorics and has many generalizations (see e.g. [2]).

*MTA Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Email: gerbner.daniel@renyi.mta.hu Research supported by OTKA grant PD-109537.

†MTA Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Email: vizermate@gmail.com Research supported by OTKA grant SNN–116095.

Recently Sperner's theorem played some role in the Polymath project to discover a new proof of the density Hales-Jewett theorem [11]. Motivated by its role in the proof Kalai asked whether one can achieve 'Sperner-like theorems' for 'Sperner like families' [8].

One direction to generalize the notion of Sperner families is the so called *tilted Sperner families* (see Definition 1.1). As written in [8]: Kalai noted that the 'no containment' condition can be rephrased as follows: $\mathcal{F}$ does not contain two sets F and G such that, in the unique subcube of $\mathcal{P}([n])$ spanned by F and G, the bottom point is F and G is the top point. He asked: what happens if we forbid F and G to be at a different position in this subcube? In particular, he asked how large $\mathcal{F} \subset \mathcal{P}([n])$ can be if we forbid F and G to be at a fixed ratio $p : q$ in this subcube. That is, we forbid F to be $p/(p+q)$ of the way up this subcube and G to be $q/(p+q)$ of the way up this subcube. Equivalently we can say:

Definition 1.1. Let p, q be two nonnegative integers. We call $\mathcal{F} \subseteq \mathcal{P}([n])$ a *(p,q)-tilted Sperner family* if for all distinct $F, G \in \mathcal{F}$ we have

$$p|F \setminus G| \neq q|G \setminus F|.$$

Note that we can restrict ourselves to coprime p and q. Also note the a Sperner family is just a $(1,0)$-tilted Sperner family. In [8] Leader and Long proved the following theorem, which gives an asymptotically tight answer for the maximal cardinality of a *(p,q)*-tilted Sperner family:

Theorem 1.2. *Let p, q be coprime nonnegative integers with $q \geq p$. Suppose $\mathcal{F} \subset \mathcal{P}([n])$ is a (p,q)-tilted Sperner family. Then*

$$|\mathcal{F}| \leq (q - p + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Note that up to the $o(1)$ term, this is the best possible, since the union of $p - q$ consecutive levels is a (p,q)-tilted Sperner family.

In [10] Long started to investigate the cardinality of *tilted Sperner families with patterns* (see Definition 1.3), which was also asked by Kalai ([9]).

Definition 1.3. Let p and q be nonnegative integers with $p + q > 0$. We call $\mathcal{F}$ a *(p,q)-tilted Sperner family with patterns*, if there are no distinct $F, G \in \mathcal{F}$ with:

(*i*) $p|F \setminus G| = q|G \setminus F|$, and

(*ii*) $f > g$ for all $f \in F \setminus G$ and $g \in G \setminus F$.

In [10] he gave an upper bound on the cardinality of a (1,2)-tilted Sperner family with patterns:

Theorem 1.4. *([10], Theorem 1.3) Let $\mathcal{F} \subset \mathcal{P}([n])$ be a (1,2)-tilted Sperner family with patterns. Then*

$$|\mathcal{F}| \leq O(e^{120\sqrt{\log n}} \frac{2^n}{\sqrt{n}}).$$

Actually in [10] he gives a proof of a weaker result with the density Hales-Jewett theorem, and proves Theorem 1.4 with a randomized generalization of Katona's cycle method (see [5]).

In this note we generalize and improve his result by applying another generalization of Katona's cycle method, the so called permutation method. We will apply the permutation method in a somewhat similar way like the authors of [3] and prove the following:

Theorem 1.5. *Let p and q be non negative integers with $p+q>0$ and let $\mathcal{F}$ be a (p,q)-tilted Sperner family with patterns. Then*

$$|\mathcal{F}| \leq O(\sqrt{\log n}\,\frac{2^n}{\sqrt{n}}).$$

The paper is organized as follows: in Section 2 we prove our main theorem and in Section 3 we pose some questions.

2 Proof of Theorem 1.5

Proof. If either p or q is zero, then we get back the usual Sperner family for which we know that the statement is true. In the following we fix $p,q>0$ and furthermore we assume that $p \leq q$. The proof works similarly in case $p>q$.

2.1 The (p,q)-cut point

First we introduce a notion that will have crucial role in the proof.

Definition 2.1. We say that $x \in [n]$ is a *(p,q)-cut point of* $A \subseteq [n]$, if

$$0 \leq \frac{n-x-|([n]\setminus[x])\cap A|}{q} - \frac{|A\cap[x]|}{p} < \frac{1}{p}. \tag{1}$$

We remark that x is a (p,q)-cut point means that $\frac{p}{q}$ times the number of points of A less than x is 'approximately' equal to the number of points not belonging to A that are larger than x.

Lemma 2.2. *Every $A \subseteq [n]$ has a (p,q)-cut point.*

Proof. Let us introduce the following functions: for $u \in \{0\}\cup[n]$ and $A \subseteq [n]$ let

$$f(A,u) := \frac{|A\cap[u]|}{p} \quad \text{and} \quad g(A,u) := \frac{n-u-|([n]\setminus[u])\cap A|}{q},$$

with $|A\cap[0]|=0$. Observe that if $|A|\neq 0$, then we have

$$0 = f(A,0) < g(A,0) = \frac{n-|A|}{q} \quad \text{and} \quad \frac{|A|}{p} = f(A,n) > g(A,n) = 0. \tag{2}$$

Also note that for all $i \in [n]$ if

$\bullet_1$ $i \in A$, then

$$f(A,i-1)+\frac{1}{p}=f(A,i) \quad \text{and} \quad g(A,i-1)=g(A,i)$$

$\bullet_2$ $i \notin A$, then

$$f(A,i-1)=f(A,i) \quad \text{and} \quad g(A,i-1)-\frac{1}{q}=g(A,i).$$

By $\bullet_1$,$\bullet_2$ and (2) we have $f(A,0)<g(A,0)$ and going towards n, f is increasing, g is decreasing, but both of them changes with at most $\frac{1}{p}$ and we have $f(A,n)>g(A,n)$.

We are done with the proof of Lemma 2.2.

□

2.2 Using the permutation method

Let us introduce two pieces of notation:

1) for all $F \in \mathcal{F}$ choose a (p,q)-cut point x_F (we can do it by Lemma 2.2), and let

$$\mathcal{F}_x := \{F \in \mathcal{F} : x = x_F\} \quad \text{for} \quad x \in [n],$$

2) for $x+k \leq n$ let $j(x,k) := \lfloor \frac{p}{q}(n-x-k) \rfloor$.

Note that if x is a (p,q)-cut point for $A \subseteq [n]$, then

$$|A \cap [x]| = j(x, |([n] \setminus [x]) \cap A|).$$

In this section we will prove an upper bound on $|\mathcal{F}_x|$ using the permutation method.

Let us consider the following permutation group of $[n]$: for any $x \in [n]$ let us denote by S_x the symmetric group on x elements, and let $\Pi_x := S_x \times S_{n-x}$, the direct product of S_x and S_{n-x} (for definition of direct product of groups see e.g. [7]). An element $(\pi_1,\pi_2)=\pi \in \Pi_x$ acts on $[n]$ the following way:

$$\pi(i) = \begin{cases} \pi_1(i) & \text{if} \quad i \leq x, \\ \pi_2(i-x)+x & \text{if} \quad i > x. \end{cases}$$

For $A \subseteq [n]$ and $\pi \in \Pi_x$ we will use the notation $\pi(A)$ for $\{\pi(a) : a \in A\}$.

Let us define the following families of sets for $x \in [n]$, $0 \leq k \leq n-x$ if $j(x,k) < x$:

$$C(x,k) := \{1,2,...,j(x,k),x+1,x+2,...,x+k\}.$$

Observe two things:

$\circ_1$ For any $x \in [n]$ and $r < q$ we have

$$|\{C(x, tq + r) : 0 \leq t \leq \frac{n}{q}\} \cap \mathcal{F}| \leq 1$$

by the assumptions that $\mathcal{F}$ is a (p, q)-tilted Sperner family with patterns and two such sets for different t's are forbidden. Note here that $C(x, tq + r)$ does not even exist for some t. We also have that for all $\pi \in \Pi_x$

$$|\{\pi(C(x, tq + r)) : 0 \leq t \leq \frac{n}{q}\} \cap \mathcal{F}_x| \leq 1.$$

Indeed, if F and G are both in this family, it is easy to calculate that $p|F \setminus G| = q|G \setminus F|$, and elements of $F \setminus G$ are smaller than x while elements of $G \setminus F$ are larger than x.

$\circ_2$ For any $F \in \mathcal{F}_x$ there are $k \leq n - x$ and $\pi \in \Pi_x$ with

$$F = \pi(C(x, k)).$$

Now let us do the following computation: fix $x \in [n]$. Using $\circ_1$ we have the following

$$\sum_{\pi \in \Pi_x} \sum_{r=0}^{q-1} \sum_{t=0}^{\lfloor \frac{n}{q} \rfloor} |\pi(C(x, tq + r)) \cap \mathcal{F}_x| \leq q(n - x)!x!.$$

After changing the order summations using $\circ_2$ we get

$$\sum_{F \in \mathcal{F}_x} |F \cap [x]|!(x - |F \cap [x]|)!(|F \setminus [x]|)!(n - x - |F \setminus [x]|)! \leq q(n - x)!x!,$$

and finally, dividing both sides by $(n - x)!x!$ we have

$$\sum_{F \in \mathcal{F}_x} \frac{1}{\binom{x}{|F \cap [x]|}\binom{n-x}{|F \setminus [x]|}} \leq q. \tag{3}$$

Using the fact that $\binom{x}{i} \leq 2^x/\sqrt{x}$, from (3) we have that for all $x \in [n]$:

$$|\mathcal{F}_x| \leq O(\frac{2^n}{\sqrt{x(n - x)}}). \tag{4}$$

2.3 Finishing the proof of Theorem 1.5

We finish the proof of Theorem 1.5 by a standard application of the Chernoff-Hoeffding bound ([1], [4]):

Chernoff-Hoeffding bound: Let X_i be independent random variables in the $[0,1]$ interval and let

$$X(n) := \sum_{i=1}^{n} X_i.$$

Then for $t \leq \mathbb{E}[X(n)]$ we have

$$\mathbb{P}(|X(n) - \mathbb{E}[X(n)]| \geq t) \leq 2\exp(-\frac{2t^2}{n}).$$

The next lemma is probably well known, however for the sake of completeness we present a proof here. Let

$$\mathcal{G} := \{G \subseteq [n] : \text{there is } x \in [n] \text{ with } \big||[x] \cap G| - \frac{x}{2}\big| > \sqrt{n \log n}\}.$$

Lemma 2.3. *We have*

$$|\mathcal{G}| \leq O(\frac{2^n}{n}).$$

Proof. Note that $\mathcal{G} = \cup_{x\in[n]}\mathcal{G}_x$, where

$$\mathcal{G}_x := \{G \in \mathcal{G} : \Big||[x] \cap G| - \frac{x}{2}\Big| > \sqrt{n \log n}\}.$$

Observe that

$$|\mathcal{G}_x|\frac{1}{2^n} \leq \Big(\sum_{y=0}^{\lfloor \frac{x}{2} - \sqrt{n\log n}\rfloor} \binom{x}{y} + \sum_{y=\lceil \frac{x}{2}+\sqrt{n \log n}\rceil}^{x} \binom{x}{y}\Big)\frac{1}{2^x} \tag{5}$$

Applying the Chernoff-Hoeffding bound on the right hand side of (5) with $t = \sqrt{n \log n}$ (which is less than $\frac{n}{2}$ for $n \geq 10$) we have

$$|\mathcal{G}_x|\frac{1}{2^n} \leq 2exp(-\frac{2n \log n}{x}). \tag{6}$$

Using $x \leq n$ on the right hand side of (6), we have

$$|\mathcal{G}_x| \leq O(\frac{2^n}{n^2}),$$

which easily implies the statement of the lemma.

□

Let $\mathcal{F}' := \mathcal{F} \setminus \mathcal{G}$.

Using Lemma 2.3 we prove that a (p, q)-cut point of any $F \in \mathcal{F}'$ is in a $O(\sqrt{n \log n})$ neighborhood of $\frac{p}{p+q}n$.

Lemma 2.4. *For $n \geq 2$ and all $F \in \mathcal{F}'$ we have*

$$|x_F - \frac{p}{p+q}n| \leq 8\sqrt{n \log n}.$$

Proof. By the fact that $F \in \mathcal{F}'$ we have both

$$\Big||[x_F] \cap F| - \lfloor \frac{x_F}{2} \rfloor\Big| \leq \sqrt{n \log n} \tag{7}$$

and

$$\Big||[n] \cap F| - \lfloor \frac{n}{2} \rfloor\Big| \leq \sqrt{n \log n}. \tag{8}$$

By (7) and (8) we have (loosing at most 1 in putting together two inequalities and using that $1 \leq \sqrt{n \log n}$ for $n \geq 2$.)

$$\Big||([n] \setminus [x_F]) \cap F| - \lfloor \frac{n - x_F}{2} \rfloor\Big| \leq 4\sqrt{n \log n}. \tag{9}$$

However x_F is a (p, q)-cut point for F, so by (7), (8) and (9) we have

$$\Big|(n - x_F - \lfloor \frac{n - x_F}{2} \rfloor)\frac{1}{q} - \lfloor \frac{x_F}{2} \rfloor \frac{1}{p}\Big| \leq 8\sqrt{n \log n},$$

and we are done with Lemma 2.4.

□

By (4) and Lemma 2.4 we have

$$|\mathcal{F}'| \leq O(\sqrt{n \log n}\ \frac{2^n}{n}),$$

and by Lemma 2.3 we are done with the proof of Theorem 1.5.

□

3 Concluding remarks

We proved in Theorem 1.5 that the cardinality of a (p, q)-tilted Sperner family with patterns on $[n]$ is $O(\sqrt{\log n}\ \frac{2^n}{\sqrt{n}})$, however we do not have much better constructions than the ones in [8]. We conjecture that for different p and q the order of a maximal size (p, q)-tilted Sperner family with patterns on $[n]$ is $\Theta(\frac{2^n}{\sqrt{n}})$.

For $p = q$ we are not able to give really good constructions, we only know that the $(0, 0)$-tilted Sperner family with patterns on $[n]$ (which we define just with property *(ii)* in Definition 1.3) is $O(\frac{2^n}{n})$, and we do not know what should be the right order.

It is worth mentioning that the whole topic from a more general viewpoint is investigated in the recent paper [6].

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