ON THE DISCRETE LOGARITHMIC MINKOWSKI PROBLEM

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Abstract. A new sufficient condition for the existence of a solution for the logarithmic Minkowski problem is established. This new condition contains the one established by Zhu [69] and the discrete case established by Böröczky, Lutwak, Yang, Zhang [6] as two important special cases.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. If K is a convex body in $\mathbb{R}^n$, then the *surface area measure*, S_K, of K is a Borel measure on the unit sphere, S^{n-1}, defined for a Borel $\omega \subset S^{n-1}$ (see, e.g., Schneider [61]), by

$$S_K(\omega) = \int_{x\in\nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

As one of the cornerstones of the classical Brunn-Minkowski theory, the Minkowski's existence theorem can be stated as follows (see, e.g., Schneider [61]): If μ is not concentrated on a great subsphere of S^{n-1}, then μ is the surface area measure of a convex body if and only if

$$\int_{S^{n-1}} u d\mu(u) = 0.$$

The solution is unique up to translation, and even the regularity of the solution is well investigated, see e.g., Lewy [40], Nirenberg [57], Cheng and Yau [12], Pogorelov [60], and Caffarelli [9].

The surface area measure of a convex body has clear geometric significance. Another important measure that is associated with a convex body and that has clear geometric importance is the cone-volume measure. If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the *cone-volume measure*, V_K, of K is a Borel measure on S^{n-1} defined for each Borel $\omega \subset S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_{x\in\nu_K^{-1}(\omega)} x\cdot\nu_K(x)\, d\mathcal{H}^{n-1}(x).$$

For references regarding cone-volume measure see, e.g., [5–8, 42–44, 55, 56, 58, 62–64, 69].

The Minkowski's existence theorem deals with the question of prescribing the surface area measure. The following problem is prescribing the cone-volume measure.

Logarithmic Minkowski problem: What are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the cone-volume measure of a convex body in $\mathbb{R}^n$?

In [45], Lutwak showed that there is an L_p analogue of the surface area measure (known as the L_p surface area measure). In recent years, the L_p surface area measure appeared in, e.g.,

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[1, 4, 10, 22, 23, 25, 26, 31, 42–44, 47–49, 52, 53, 55, 56, 58, 59, 64]. In [45], Lutwak posed the associated L_p Minkowski problem which extends the classical Minkowski problem for $p \geq 1$. In addition, the L_p Minkowski problem for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990's. The L_p Minkowski problem is the classical Minkowski problem when $p = 1$, while the L_p Minkowski problem is the logarithmic Minkowski problem when $p = 0$. The L_p Minkowski problem is interesting for all real p, and have been studied by, e.g., Lutwak [45], Lutwak and Oliker [46], Chou and Wang [14], Guan and Lin [21], Hug, et al. [35], Böröczky, et al. [6]. Additional references regarding the L_p Minkowski problem and Minkowski-type problems can be found in, e.g., [6, 11, 14, 20–24, 33–35, 38, 39, 41, 45, 46, 51, 54, 62, 63, 70, 71]. Applications of the solutions to the L_p Minkowski problem can be found in, e.g., [2, 3, 13, 15, 16, 27–29, 36, 37, 50, 66, 68].

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if, for every subspace ξ of $\mathbb{R}^n$, such that $0 < \dim \xi < n$,

$$\mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}), \tag{1.2}$$

and if equality holds in (1.2) for some subspace ξ, then there exists a subspace ξ', that is complementary to ξ in $\mathbb{R}^n$, so that also

$$\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}).$$

The measure μ on S^{n-1} is said to satisfy the *strict subspace concentration inequality* if the inequality in (1.2) is strict for each subspace $\xi \subset \mathbb{R}^n$, such that $0 < \dim \xi < n$.

Very recently, Böröczky and Henk [5] proved that if the centroid of a convex body is the origin, then the cone-volume measure of this convex body satisfies the subspace concentration condition. For more references on the progress of the subspace concentration condition, see, e.g., Henk et al. [32], He et al. [30], Xiong [67], Böröczky et al. [8], and Henk and Linke [31].

In [6], Böröczky, et al. established the following necessary and sufficient conditions for the existence of solutions to the even logarithmic Minkowski problem.

Theorem 1.1 (Böröczky,Lutwak,Yang,Zhang)**.** *A non-zero finite even Borel measure on S^{n-1} is the cone-volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.*

The convex hull of a finite set is called a polytope provided that it has positive n-dimensional volume. The convex hull of a subset of these points is called a *facet* of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. If a polytope P contains the origin in its interior and has N facets whose outer unit normals are $u_1, ..., u_N$, and such that if the facet with outer unit normal u_k has $(n-1)$-measure a_k and distance from the origin h_k for all $k \in \{1, ..., N\}$, then

$$V_P = \frac{1}{n} \sum_{k=1}^{N} h_k a_k \delta_{u_k}.$$

where δ_{u_k} denotes the delta measure that is concentrated at the point u_k.

A finite subset U (with no less than n elements) of S^{n-1} is said to be *in general position* if any k elements of U, $1 \leq k \leq n$, are linearly independent.

For a long time, people believed that the data for a cone-volume measure can not be arbitrary. However, Zhu [69] proved that any discrete measure on S^{n-1} whose support is in general position is a cone-volume measure.

Theorem 1.2 (Zhu)**.** *A discrete measure, μ, on the unit sphere S^{n-1} is the cone-volume measure of a polytope whose outer unit normals are in general position if and only if the support of μ is in general position and not concentrated on a closed hemisphere of S^{n-1}.*

A linear subspace ξ ($1 \le \dim \xi \le n-1$) of $\mathbb{R}^n$ is said to be essential with respect to a Borel measure μ on S^{n-1} if $\xi \cap \operatorname{supp}(\mu)$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Definition 1.3. *A finite Borel measure μ on S^{n-1} is said to satisfy the essential subspace concentration condition if, for every essential subspace ξ (with respect to μ) of $\mathbb{R}^n$, such that $0 < \dim \xi < n$,*

$$\mu(\xi \cap S^{n-1}) \le \frac{\dim \xi}{n} \mu(S^{n-1}), \tag{1.3}$$

and if equality holds in (1.3) for some essential subspace ξ (with respect to μ), then there exists a subspace ξ', that is complementary to ξ in $\mathbb{R}^n$, so that

$$\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}). \tag{1.4}$$

Definition 1.4. *The measure μ on S^{n-1} is said to satisfy the strict essential subspace concentration inequality if the inequality in (1.3) is strict for each essential subspace ξ (with respect to μ) of $\mathbb{R}^n$, such that $0 < \dim \xi < n$.*

We would like to note that if μ is a Borel measure on the unit sphere that is not concentrated on a closed hemisphere and satisfies the essential subspace concentration condition, and ξ is an essential subspace (with respect to μ) that reaches the equality in (1.3), then by Lemma 5.2, ξ' (in (1.4)) is an essential subspace with respect to μ.

It is the aim of this paper to establish the following.

Theorem 1.5. *If μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition, then μ is the cone-volume measure of a polytope in $\mathbb{R}^n$ containing the origin in its interior.*

If μ is a non-trivial even Borel measure on S^{n-1}, and ξ is a k-dimensional linear subspace of $\mathbb{R}^n$ spanned by some vectors $v_1, \dots, v_k \in \operatorname{supp}(\mu)$ for $1 \le k \le n-1$, then $-v_1, \dots, -v_k \in \operatorname{supp}(\mu)$, as well, and hence ξ is an essential subspace. In particular, for even discrete measures, Theorem 1.5 is equivalent to the sufficient condition of Theorem 1.1. However, there are non-even discrete measures that satisfy the essential subspace concentration condition, but not the subspace concentration condition. For example, if a k-dimensional subspace ξ, $1 \le k \le n-1$, intersects the support of the measure in $k+1$ unit vectors $u_0, \dots, u_k$ such that $u_1, \dots, u_k$ are independent, and $u_0 = \alpha_1 u_1 + \ldots + \alpha_k u_k$ for $\alpha_1, \dots, \alpha_k > 0$, then there is no condition on the restriction of the measure to $\xi \cap S^{n-1}$. Therefore, for discrete measures, Theorem 1.5 is a generalization of the sufficient condition of Theorem 1.1.

We claim that if the support of a discrete measure μ is in general position, then the set of essential subspaces (with respect to μ) is empty. Otherwise, there exists a subspace ξ with $1 \le \dim \xi \le n-1$ such that $\operatorname{supp}(\mu) \cap \xi$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi$. Then we can choose $\dim \xi + 1$ ($\le n$) vectors from $\operatorname{supp}(\mu) \cap \xi$ that are linearly dependent. But this contradicts the fact that $\operatorname{supp}(\mu)$ is in general position. From our declaration, we have, Theorem 1.5 contains Theorem 1.2 as an important special case.

In $\mathbb{R}^2$, Theorem 1.5 leads to the main result of Stancu ([62], pp. 162), where she applied a different method called the crystalline deformation.

New inequalities for cone-volume measures are established in section 6.

2. Preliminaries

In this section, we collect some basic notations and facts about convex bodies. For general references regarding convex bodies see, e.g., [17–19, 61, 65].

The vectors of this paper are column vectors. For $x, y \in \mathbb{R}^n$, we will write $x \cdot y$ for the standard inner product of x and y, and write $|x|$ for the Euclidean norm of x. We write $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the boundary of the Euclidean unit ball B^n in $\mathbb{R}^n$, and write κ_n for the volume of the unit ball. Let $V_k(M)$ denote the k-dimensional Hausdorff measure of an at most k-dimensional convex set M. In addition, if $k = n-1$, then we also use the notation $|M|$.

Suppose X_1, X_2 are subspaces of $\mathbb{R}^n$, we write $X_1 \perp X_2$ if $x_1 \cdot x_2 = 0$ for all $x \in X_1$ and $x_2 \in X_2$. Suppose X is a subspace of $\mathbb{R}^n$ and S is a subset of $\mathbb{R}^n$, we write $S|_X$ for the orthogonal projection of S on X.

Suppose C is a subset of $\mathbb{R}^n$, the positive hull, $\mathrm{pos}(C)$, of C is the set of all positive combinations of any finitely many elements of C. Let $\mathrm{lin}(C)$ be the smallest linear subspace of $\mathbb{R}^n$ containing C. The diameter of C is defined by

$$d(C) = \sup\{|x - y| : x, y \in C\}.$$

For $K_1, K_2 \subset \mathbb{R}^n$ and $c_1, c_2 \geq 0$, the Minkowski combination, $c_1K_1 + c_2K_2$, is defined by

$$c_1K_1 + c_2K_2 = \{c_1x_1 + c_2x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $c \geq 0$ and $x \in \mathbb{R}^n$, we have

$$h(cK, x) = h(K, cx) = ch(K, x).$$

The *convex hull* of two convex sets K, L in $\mathbb{R}^n$ is defined by

$$[K, L] = \{z : z = \lambda x + (1-\lambda)y, 0 \leq \lambda \leq 1 \text{ and } x, y \in K \cup L\}.$$

The *Hausdorff distance* of two compact sets K, L in $\mathbb{R}^n$ is defined by

$$\delta(K, L) = \inf\{t \geq 0 : K \subset L + tB^n, L \subset K + tB^n\}.$$

It is known that the Hausdorff distance between two convex bodies, K and L, is

$$\delta(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|.$$

We always consider the space of convex bodies as metric space equipped with the Hausdorff distance. It is known that if a sequence $\{K_m\}$ of convex bodies tends to a convex body K in $\mathbb{R}^n$ containing the origin in its interior, then S_{K_m} tends weakly to S_K, and hence V_{K_m} tends weakly to V_K (see Schneider [61]).

For a convex body K in $\mathbb{R}^n$, and $u \in S^{n-1}$, the *support hyperplane* $H(K, u)$ in direction u is defined by

$$H(K, u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\},$$

the *face* $F(K, u)$ in direction u is defined by

$$F(K, u) = K \cap H(K, u).$$

Let $\mathcal{P}$ be the set of all polytopes in $\mathbb{R}^n$. If the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, let $\mathcal{P}(u_1, ..., u_N)$ be the set of all polytopes $P \in \mathcal{P}$ such that the set of outer unit normals of the facets of P is a subset of $\{u_1, ..., u_N\}$, and let $\mathcal{P}_N(u_1, ..., u_N)$ be the the set of all polytopes $P \in \mathcal{P}$ such that the set of outer unit normals of the facets of P is $\{u_1, ..., u_N\}$.

3. An extremal problem related to the logarithmic Minkowski problem

Let us suppose $\gamma_1, ..., \gamma_N \in (0, \infty)$, and the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere. Let

$$\mu = \sum_{i=1}^{N} \gamma_i \delta_{u_i}, \tag{3.0}$$

and for $P \in \mathcal{P}(u_1, ..., u_N)$ define $\Phi_P : \text{Int } (P) \to \mathbb{R}$ by

$$\begin{aligned} \Phi_P(\xi) &= \int_{S^{n-1}} \log\left(h(P,u) - \xi \cdot u\right) d\mu(u) \\ &= \sum_{k=1}^{N} \gamma_k \log\big(h(P,u_k) - \xi \cdot u_k\big), \end{aligned} \tag{3.1}$$

where Int (P) is the interior of P.

In this section, we study the following extremal problem:

$$\inf \left\{ \max_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = |\mu| \right\}, \tag{3.2}$$

where $|\mu| = \sum_{k=1}^{N} \gamma_k$.

We will prove that the solution of problem (3.2) solves the corresponding logarithmic Minkowski problem.

For the case where $u_1, ..., u_N$ are in general position and $Q \in \mathcal{P}_N(u_1, ..., u_N)$, problem (3.2) was studied in [69]. The results and proofs in this section are similar to [69]. However, for convenience of the readers, we give detailed proofs for these results.

Lemma 3.1. *Suppose $\mu = \sum_{k=1}^{N} \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, and $P \in \mathcal{P}(u_1, ..., u_N)$, then there exists a unique point $\xi(P) \in \text{Int } (P)$ such that*

$$\Phi_P(\xi(P)) = \max_{\xi \in \text{Int } (P)} \Phi_P(\xi).$$

Proof. Let $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{Int } (P)$. From the concavity of the logarithmic function,

$$\begin{aligned} \lambda \Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2) &= \lambda \int_{S^{n-1}} \log\left(h(P,u) - \xi_1 \cdot u\right) d\mu(u) \\ &\quad + (1-\lambda) \int_{S^{n-1}} \log\left(h(P,u) - \xi_2 \cdot u\right) d\mu(u) \\ &= \sum_{k=1}^{N} \gamma_k \left[\lambda \log(h(P,u_k) - \xi_1 \cdot u_k) + (1-\lambda)\log(h(P,u_k) - \xi_2 \cdot u_k)\right] \\ &\le \sum_{k=1}^{N} \gamma_k \log\left[h(P,u_k) - (\lambda\xi_1 + (1-\lambda)\xi_2) \cdot u_k\right] \\ &= \Phi_P(\lambda\xi_1 + (1-\lambda)\xi_2), \end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, ..., N$. Since the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, $\mathbb{R}^n$=lin$\{u_1, ..., u_N\}$. Thus, $\xi_1 = \xi_2$. Therefore, Φ_P is strictly concave on Int (P).

Since $P \in \mathcal{P}(u_1, ..., u_N)$, for any $x \in \partial P$, there exists some $i_0 \in \{1, \ldots, N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus, $\Phi_P(\xi) \to -\infty$ whenever $\xi \in \text{Int }(P)$ and $\xi \to x$. Therefore, there exists a unique interior point $\xi(P)$ of P such that

$$\Phi_P(\xi(P)) = \max_{\xi \in \text{Int }(P)} \Phi_P(\xi).$$

□

Obviously, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, ..., u_N)$,

$$\xi(\lambda P) = \lambda \xi(P), \tag{3.3}$$

and if $P_i \in \mathcal{P}(u_1, ..., u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, ..., u_N)$.

For the case where $u_1, ..., u_N$ are in general position, the following lemma was proved in [69].

Lemma 3.2. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, $P_i \in \mathcal{P}(u_1, ..., u_N)$ and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

Proof. Since $\xi(P) \in \text{Int }(P)$ by Lemma 3.1, we have

$$\liminf_{i\to\infty} \Phi_{P_i}(\xi(P_i)) \geq \liminf_{i\to\infty} \Phi_{P_i}(\xi(P)) = \Phi_P(\xi(P)).$$

Let z be any accumulation point of the sequence $\{\xi(P_i)\}$; namely, the limit of a subsequence $\{\xi(P_{i'})\}$. Since $\Phi_{P_i}(\xi(P_i))$ is bounded from below, and $h(P, u_k) - \xi(P_i) \cdot u_k$ is bounded from above for $k = 1, \ldots, N$, it follows that

$$\liminf_{i\to\infty} (h(P, u_k) - \xi(P_i) \cdot u_k) = \liminf_{i\to\infty} (h(P_i, u_k) - \xi(P_i) \cdot u_k) > 0$$

for $k = 1, \ldots, N$, and hence $z \in \text{Int }(P)$. We deduce that

$$\Phi_P(z) = \lim_{i'\to\infty} \Phi_P(\xi(P_{i'})) = \lim_{i'\to\infty} \Phi_{P_{i'}}(\xi(P_{i'})) \geq \liminf_{i\to\infty} \Phi_{P_i}(\xi(P_i)) \geq \Phi_P(\xi(P)).$$

Therefore Lemma 3.1 yields $z = \xi(P)$. □

The following lemma will be needed, as well.

Lemma 3.3. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, $P \in \mathcal{P}(u_1, ..., u_N)$, then*

$$\sum_{k=1}^N \gamma_k \frac{u_k}{h(P, u_k) - \xi(P) \cdot u_k} = 0.$$

Proof. We may assume that $\xi(P)$ is the origin because for $x, \xi \in \text{Int } P$, we have $\Phi_{P-x}(\xi - x) = \Phi_P(\xi)$. Since $\Phi_P(\xi)$ attains its maximum at the origin that is an interior point of P, differentiation gives the desired equation. □

Lemma 3.4. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, and there exists a $P \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P) = 0$, $V(P) = |\mu|$ such that*

$$\Phi_P(0) = \inf \left\{ \max_{\xi \in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = |\mu| \right\}.$$

Then,

$$V_P = \sum_{k=1}^N \gamma_k \delta_{u_k}.$$

Proof. According to Equation (3.3), it is sufficient to establish the lemma under the assumption that $|\mu| = 1$.

From the conditions, there exists a polytope $P \in \mathcal{P}_N(u_1, ..., u_N)$ with $\xi(P)$ is the origin and $V(P) = 1$ such that

$$\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\}.$$

For $\tau_1, ..., \tau_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope

$$P_t = \bigcap_{i=1}^{N} \{x : x \cdot u_i \leq h(P, u_i) + t\tau_i\} \in \mathcal{P}_N(u_1, ..., u_N).$$

In particular, $h(P_t, u_i) = h(P, u_i) + t\tau_i$ for $i = 1, \ldots, n$, and Lemma 7.5.3 in Schneider [61] yields that

$$\frac{\partial V(P_t)}{\partial t} = \sum_{i=1}^{N} \tau_i |F(P_t, u_i)|.$$

Let $\lambda(t) = V(P_t)^{-\frac{1}{n}}$. Then $\lambda(t)P_t \in \mathcal{P}_N(u_1, ..., u_N)$, $V(\lambda(t)P_t) = 1$, $\lambda(t)$ is C^1 and

$$\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \tau_i |F(P, u_i)|. \tag{3.5}$$

Define $\xi(t) := \xi(\lambda(t)P_t)$, and

$$\begin{aligned} \Phi(t) &:= \max_{\xi \in \lambda(t)P_t} \int_{S^{n-1}} \log\left(h(\lambda(t)P_t, u) - \xi \cdot u\right) d\mu(u) \\ &= \sum_{k=1}^{N} \gamma_k \log(\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k). \end{aligned} \tag{3.6}$$

It follows from Lemma 3.3, that

$$\sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k} = 0 \tag{3.7}$$

for $i = 1, ..., n$, where $u_k = (u_{k,1}, ..., u_{k,n})^T$. In addition, since $\xi(P)$ is the origin, we have

$$\sum_{k=1}^{N} \gamma_k \frac{u_k}{h(P, u_k)} = 0. \tag{3.8}$$

Let $F = (F_1, \ldots, F_n)$ be a function from a small neighbourhood of the origin in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ such that

$$F_i(t, \xi_1, ..., \xi_n) = \sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})}$$

for $i = 1, ..., n$. Then,

$$\begin{aligned} \left.\frac{\partial F_i}{\partial t}\right|_{(t,\xi_1,...,\xi_n)} &= \sum_{k=1}^{N} \gamma_k \frac{-u_{k,i}(\lambda'(t)h(P_t, u_k) + \lambda(t)\tau_k)}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})]^2} \\ \left.\frac{\partial F_i}{\partial \xi_j}\right|_{(t,\xi_1,...,\xi_n)} &= \sum_{k=1}^{N} \gamma_k \frac{u_{k,i} u_{k,j}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})]^2} \end{aligned}$$

are continuous on a small neighborhood of $(0,0,...,0)$ with

$$\left(\frac{\partial F}{\partial \xi}\bigg|_{(0,\ldots,0)}\right)_{n\times n} = \sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}u_k u_k^T,$$

where $u_k u_k^T$ is an $n\times n$ matrix. Since the unit vectors $u_1,...,u_N$ are not concentrated on a closed hemisphere, $\mathbb{R}^n = \mathrm{lin}\{u_1,...,u_N\}$. Thus, for any $x\in\mathbb{R}^n$ with $x\neq 0$, there exists a $u_{i_0}\in\{u_1,...,u_N\}$ such that $u_{i_0}\cdot x\neq 0$. Then,

$$\begin{aligned} x^T\left(\sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}u_k u_k^T\right)x &= \sum_{k=1}^{N}\frac{\gamma_k}{h(P,u_k)^2}(x\cdot u_k)^2 \\ &\geq \frac{\gamma_{i_0}}{h(P,u_{i_0})^2}(x\cdot u_{i_0})^2 > 0. \end{aligned}$$

Therefore, $(\frac{\partial F}{\partial \xi}\big|_{(0,\ldots,0)})$ is positive definite. By this, the fact that $F_i(0,...,0)=0$ for $i=1,...,n$, the fact that $\frac{\partial F_i}{\partial \xi_j}$ is continuous on a neighborhood of $(0,0,...,0)$ for all $1\leq i,j\leq n$ and the implicit function theorem, we have

$$\xi'(0) = (\xi_1'(0),...,\xi_n'(0))$$

exists.

From the fact that $\Phi(0)$ is a minimizer of $\Phi(t)$ (in Equation (3.6)), Equation (3.5), the fact that $\sum_{k=1}^N\gamma_k = 1$ and Equation (3.8), we have

$$\begin{aligned} 0 &= \Phi'(0) \\ &= \sum_{k=1}^{N}\gamma_k\frac{\lambda'(0)h(P,u_k)+\lambda(0)\frac{dh(P_t,u_k)}{dt}\big|_{t=0} - \xi'(0)\cdot u_k}{h(P,u_k)} \\ &= \sum_{k=1}^{N}\gamma_k\frac{-\frac{1}{n}(\sum_{i=1}^N\tau_i|F(P,u_i)|)h(P,u_k)+\tau_k - \xi'(0)\cdot u_k}{h(P,u_k)} \\ &= -\sum_{i=1}^{N}\frac{|F(P,u_i)|\tau_i}{n} + \sum_{k=1}^{N}\frac{\gamma_k\tau_k}{h(P,u_k)} - \xi'(0)\cdot\left[\sum_{k=1}^{N}\gamma_k\frac{u_k}{h(P,u_k)}\right] \\ &= \sum_{k=1}^{N}\left(\frac{\gamma_k}{h(P,u_k)} - \frac{|F(P,u_k)|}{n}\right)\tau_k. \end{aligned}$$

Since $\tau_1,...,\tau_N$ are arbitrary, we deduce that $\gamma_k = \frac{1}{n}h(P,u_k)|F(P,u_k)|$ for $k=1,...,N$. □

4. Existence of a solution of the extremal problem

In this section, we prove Lemma 4.7 about the existence of a solution of problem (3.2) for the case where the discrete measure is not concentrated on any closed hemisphere of S^{n-1} and satisfies the strict essential subspace concentration inequality. Having the results of the previous section, the essential new ingredient is the following statement (see Lemma 4.5).

If μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1} and satisfies the strict essential subspace concentration inequality, and $\{P_m\}$ is a sequence of polytopes of unit volume such that the set of outer unit normals of P_m is a subset of the support of μ, and $\lim_{m\to\infty} d(P_m)=\infty$ then

$$\lim_{m\to\infty}\Phi_{P_m}(\xi(P_m)) = \infty.$$

It is equivalent to prove that any subsequence of $\{P_m\}$ has some subsequence $\{P_{m'}\}$ such that $\lim_{m\to\infty}\Phi_{P_{m'}}(\xi(P_{m'}))=\infty$.

To indicate the idea, we sketch the argument for $n=2$. Let $\operatorname{supp}\mu=\{u_1,\dots,u_N\}$, and let $w_m=\min\{h_{P_m}(u)+h_{P_m}(-u):u\in S^1\}$ be the minimal width of P_m. Since $\lim_{m\to\infty}d(P_m)=\infty$ and $V(P_m)=1$, we have $\lim_{m\to\infty}w_m=0$. As P_m is a polygon, we may assume that $w_m=h_{P_m}(u_1)+h_{P_m}(-u_1)$ possibly after taking a subsequence and reindexing. If the angle of u_1 and u_i is $\alpha_i\in(0,\pi)$ then $V_1(F(P_m,u_i))\le w_m/\sin\alpha_i$, thus $\lim_{m\to\infty}d(P_m)=\infty$ implies that $-u_1\in\operatorname{supp}\mu$ for large m, say $u_2=-u_1$. Let $v\in S^1$ be orthogonal to u_1, and let $\gamma_i=\mu(\{u_i\})$ for $i=1,\dots,N$. We may translate P_m in a way such that $o\in\operatorname{Int}P_m$ in a way such that $h_{P_m}(u_1)=h_{P_m}(u_2)=w_m/2$, and $h_{P_m}(v)=h_{P_m}(-v)$ hold for large m. Thus $V(P_m)=1$ yields the existence of a constant $c_1>0$ such that $h_{P_m}(u_i)>c_1/w_m$ for $i=3,\dots,N$. Now $\operatorname{lin}u_1$ is an essential subspace with respect to μ, and hence $\gamma_1+\gamma_2<\gamma_3+\ldots+\gamma_N$ according to the strict essential subspace concentration inequality. Therefore writing $c_2=\min\{2,c_1\}$, we have

$$\begin{aligned}\liminf_{m\to\infty}\exp\left(\Phi_{P_m}(\xi(P_m))\right) &\ge \liminf_{m\to\infty}\exp\left(\Phi_{P_m}(o)\right)=\liminf_{m\to\infty}\prod_{i=1}^{N}h_{P_m}(u_i)^{\gamma_i}\\ &\ge \lim_{m\to\infty}\left(\frac{w_m}{2}\right)^{\gamma_1+\gamma_2}\left(\frac{c_1}{w_m}\right)^{\gamma_3+\ldots+\gamma_N}\ge\lim_{m\to\infty}\left(\frac{c_2}{w_m}\right)^{\gamma_3+\ldots+\gamma_N-\gamma_1-\gamma_2}=\infty.\end{aligned}$$

In the higher dimensional case, the idea is the very same. Only instead of one essential linear subspace like in the planar case, we will find essential subspaces $X_0\subset\ldots\subset X_{q-1}$ in a way such that for $j=0,\dots,q-1$, $P_m|_{X_j^\perp}$ is "much larger" than $P_m|_{X_j}$ for large m after taking suitable subsequence. This is achieved in the preparatory statements Lemmas 4.1 to 4.4.

Given N sequences, the first two observations will help to do book keeping of how the limits of the sequences compare.

Lemma 4.1. *Let $\{h_{1j}\}_{j=1}^{\infty},\dots,\{h_{Nj}\}_{j=1}^{\infty}$ be N $(N\ge 2)$ sequences of real numbers. Then, there exists a subsequence, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ and a rearrangement, $i_1,\dots,i_N$, of $1,\dots,N$ such that*

$$h_{i_1j_n}\le h_{i_2j_n}\le\ldots\le h_{i_Nj_n},$$

for all $n\in\mathbb{N}$.

Proof. We prove it by induction on N. We first prove the case for $N=2$. For $j\in\mathbb{N}$, consider the sequence

$$h_j=\max\{h_{1j},h_{2j}\}.$$

Since $\{h_j\}_{j=1}^{\infty}$ is an infinite sequence and h_j either equals to h_{1j} or equals to h_{2j} for all $j\in\mathbb{N}$, there exists an $i_2\in\{1,2\}$ and a subsequence, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ such that

$$h_{j_n}=h_{i_2j_n}$$

for all $n\in\mathbb{N}$. Let $i_1\in\{1,2\}$ with $i_1\ne i_2$. Then,

$$h_{i_1j_n}\le h_{i_2j_n},$$

for all $n\in\mathbb{N}$.

Suppose the lemma is true for $N=k$ (with $k\ge 2$), we next prove that the lemma is true for $N=k+1$. For $j\in\mathbb{N}$, consider the sequence

$$h_j=\max\{h_{1j},h_{2j},\dots,h_{k+1j}\}.$$

Since $\{h_j\}_{j=1}^{\infty}$ is an infinite sequence and h_j equals one of $h_{1j},h_{2j},\dots,h_{k+1j}$ for all $j\in\mathbb{N}$, there exists an $i_{k+1}\in\{1,2,\dots,k+1\}$ and a subsequence, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ such that

$$h_{j_n}=h_{i_{k+1}j_n}$$

for all $n \in \mathbb{N}$.

Consider the sequences $\{h_{ij_n}\}_{n=1}^{\infty}$ ($1 \le i \le k+1$ with $i \ne i_{k+1}$). By the inductive hypothesis, there exists a subsequence, j_{n_l}, of j_n and a rearrangement, $i_1, ..., i_k$, of $1, ..., \widehat{i_{k+1}}, ..., k+1$ such that

$$h_{i_1 j_{n_l}} \le h_{i_2 j_{n_l}} \le ... \le h_{i_k j_{n_l}}$$

for all $l \in \mathbb{N}$. By this and the fact that $h_{j_{n_l}} = h_{i_{k+1} j_{n_l}}$ for all $l \in \mathbb{N}$, we have

$$h_{i_1 j_{n_l}} \le h_{i_2 j_{n_l}} \le ... \le h_{i_k j_{n_l}} \le h_{i_{k+1} j_{n_l}}$$

for all $l \in \mathbb{N}$. □

Lemma 4.2. *Let $\{h_{1j}\}_{j=1}^{\infty}, ..., \{h_{Nj}\}_{j=1}^{\infty}$ be N ($N \ge 2$) sequences of real numbers with*

$$h_{1j} \le h_{2j} \le ... \le h_{Nj}$$

for all $j \in \mathbb{N}$, $\lim_{j\to\infty} h_{1j} = 0$ and $\lim_{j\to\infty} h_{Nj} = \infty$. Then, there exist $q \ge 1$,

$$1 = \alpha_0 < \alpha_1 < ... < \alpha_q \le N < N+1 = \alpha_{q+1}$$

and a subsequence, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ such that if $i = 1, ..., q$, then

$$\lim_{n\to\infty} \frac{h_{\alpha_i j_n}}{h_{\alpha_{i-1} j_n}} = \infty,$$

if $i = 0, ..., q$, and $\alpha_i \le k \le \alpha_{i+1} - 1$, then

$$\lim_{n\to\infty} \frac{h_{k j_n}}{h_{\alpha_i j_n}}$$

exists and equals to a positive number.

Proof. Let $\alpha_0 = 1$. By conditions,

$$\frac{h_{1j}}{h_{1j}} \le \frac{h_{2j}}{h_{1j}} \le ... \le \frac{h_{Nj}}{h_{1j}},$$

$\overline{\lim}_{j\to\infty} \frac{h_{ij}}{h_{1j}}$ either exists (equals to a positive number) or goes to ∞, and $\overline{\lim}_{j\to\infty} \frac{h_{Nj}}{h_{1j}} = \infty$. Thus, there exists an α_1 ($1 < \alpha_1 \le N$) such that for $1 \le i \le \alpha_1 - 1$,

$$\overline{\lim}_{j\to\infty} \frac{h_{ij}}{h_{1j}} < \infty$$

and

$$\overline{\lim}_{j\to\infty} \frac{h_{\alpha_1 j}}{h_{1j}} = \infty.$$

Hence, we can choose a subsequence, $\{j'_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{h_{\alpha_1 j'_n}}{h_{1 j'_n}} = \infty,$$

and for $1 \le i \le \alpha_1 - 1$,

$$\overline{\lim}_{n\to\infty} \frac{h_{i j'_n}}{h_{1 j'_n}} \le \overline{\lim}_{j\to\infty} \frac{h_{ij}}{h_{1j}} < \infty.$$

By choosing $\alpha_1 - 2$ times subsequences of j'_n, we can find a subsequence, $\{j''_n\}_{n=1}^{\infty}$, of $\{j'_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{h_{\alpha_1 j''_n}}{h_{1 j''_n}} = \infty,$$

and for $1 \le i \le \alpha_1 - 1$,

$$\lim_{n\to\infty} \frac{h_{i j''_n}}{h_{1 j''_n}}$$

exists and equals to a positive number.

By repeating (at most $N-\alpha_1$ times) similar arguments for the sequences $\{h_{ij''_n}\}_{n=1}^{\infty}$ ($\alpha_1 \leq i \leq N$), we can find $q \geq 1$,

$$1 = \alpha_0 < \alpha_1 < ... < \alpha_q \leq N < N+1 = \alpha_{q+1}$$

and a subset, $\{j_n\}_{n=1}^{\infty}$, of $\mathbb{N}$ that satisfy the conditions in the lemma. □

The following lemma compares positive hull and linear hull.

Lemma 4.3. *Suppose $u_1, ..., u_l \in S^{d-1}$ ($d \geq 2$), $\mathbb{R}^d = lin\{u_1, ..., u_l\}$, and $u_1, ..., u_l$ are not concentrated on a closed hemisphere of S^{d-1}, then*

$$\mathbb{R}^d = pos\{u_1, ..., u_l\}.$$

Moreover, there exists $\lambda > 0$ depending on $u_1, ..., u_l$ such that any $u \in S^{d-1}$ can be written in the form

$$u = a_{i_1} u_{i_1} + ... + a_{i_d} u_{i_d}$$

where $\{u_{i_1}, ..., u_{i_d}\} \subset \{u_1, ..., u_l\}$ and $0 \leq a_{i_1}, ..., a_{i_d} \leq \lambda$.

Proof. Let Q be the convex hull of $\{u_1, ..., u_l\}$, which is a polytope. Since $u_1, ..., u_l$ are not concentrated on a closed hemisphere of S^{d-1}, the origin is an interior point of Q. In particular, $rB^d \subset Q$ for some $r > 0$.

For $u \in S^{d-1}$, there exists some $t \geq r$ such that $tu \in \partial Q$. It follows that $tu \in F$ for some facet F of Q. We deduce from the Charateodory theorem that there exists vertices $u_{i_1}, ..., u_{i_d}$ of F that tu lies in their convex hull. In other words,

$$tu = \alpha_{i_1} u_{i_1} + ... + \alpha_{i_d} u_{i_d}$$

where $\alpha_{i_1}, \ldots, \alpha_{i_d} \geq 0$ and $\alpha_{i_1} + \ldots + \alpha_{i_d} = 1$. Therefore we choose $a_{i_j} = \alpha_{i_j}/t \leq 1/r$ for $j = 1, \ldots, d$, which in turn satisfy $u = a_{i_1} u_{i_1} + ... + a_{i_d} u_{i_d}$. In particular, we may take $\lambda = 1/r$. □

The following lemma will be the last preparatory statement.

Lemma 4.4. *Suppose μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1} with supp(μ)= $\{u_1, ..., u_N\}$ and $\mu(u_i) = \gamma_i$ for $i = 1, ..., N$. If P_m is a sequence of polytopes with $V(P_m) = 1$, $\xi(P_m)$ is the origin, the set of outer unit normals of P_m is a subset of $\{u_1, ..., u_N\}$, $\lim_{m\to\infty} d(P_m) = \infty$ and*

$$h(P_m, u_1) \leq h(P_m, u_2) \leq ... \leq h(P_m, u_N)$$

for all $m \in \mathbb{N}$. Then, there exist $q \geq 1$, and $1 = \alpha_0 < \alpha_1 < ... < \alpha_q \leq N < N+1 = \alpha_{q+1}$ such that if $j = 1, ..., q$, then

$$\lim_{m\to\infty} \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j-1}})} = \infty, \tag{4.0a}$$

and if $j = 0, ..., q$ and $\alpha_j \leq k \leq \alpha_{j+1} - 1$, then

$$\lim_{m\to\infty} \frac{h(P_m, u_k)}{h(P_m, u_{\alpha_j})} = t_{kj} < \infty. \tag{4.0b}$$

Moreover, $X_j = pos\{u_1, ..., u_{\alpha_{j+1}-1}\}$ are subspaces of $\mathbb{R}^n$ for all $0 \leq j \leq q$ and

$$1 \leq \dim(X_0) < \dim(X_1) < ... < \dim(X_q) = n.$$

Proof. By the conditions that $\lim_{m\to\infty} d(P_m) = \infty$, $V(K) = 1$ and $h(P_m, u_1) \le h(P_m, u_2) \le \dots \le h(P_m, u_N)$ for all $m \in \mathbb{N}$, we have,

$$\lim_{m\to\infty} h(P_m, u_1) = 0 \text{ and } \lim_{m\to\infty} h(P_m, u_N) = \infty.$$

From Lemma 4.2, we may assume that there exist $q \ge 1$, and

$$1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \le N < N + 1 = \alpha_{q+1}$$

that satisfy Equations (4.0a) and (4.0b).

For $j = 0, \dots, q - 1$, we consider the cone

$$\Sigma_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\},$$

and its negative polar

$$\Sigma_j^* = \{v \in \mathbb{R}^n : v \cdot u_i \le 0 \text{ for all } i = 1, \dots, \alpha_{j+1} - 1\}.$$

Let $0 \le j \le q - 1$, $1 \le p \le \alpha_{j+1} - 1$ and $v \in \Sigma_j^* \cap S^{n-1}$. From the condition that $\xi(P_m)$ is the origin and Lemma 3.3,

$$\sum_{i=1}^{N} \frac{\gamma_i (v \cdot u_i)}{h(P_m, u_i)} = 0.$$

By this and the fact that $v \in \Sigma_j^* \cap S^{n-1}$,

$$\begin{aligned} 0 \ge \gamma_p (v \cdot u_p) &= - \sum_{i \ne p} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i (v \cdot u_i) \\ &\ge - \sum_{i \ge \alpha_{j+1}} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i (v \cdot u_i) \\ &\ge - \sum_{i \ge \alpha_{j+1}} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i. \end{aligned}$$

By this, (4.0a) and (4.0b), we have, $\gamma_p(v \cdot u_p)$ is no bigger than 0, and no less than any negative number. Thus,

$$v \cdot u_p = 0$$

for all $p = 1, \dots, \alpha_{j+1} - 1$ and $v \in \Sigma_j^* \cap S^{n-1}$. Then, for any $u \in \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ and $v \in \Sigma_j^*$, $u \cdot v = 0$. Hence,

$$\Sigma_j^* \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \{0\}.$$

We claim that $\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$. Otherwise, there exists a vector $u_0 \in \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ such that $u_0 \ne 0$ and $u_0 \cdot u_p \le 0$ for all $p = 1, \dots, \alpha_{j+1} - 1$. This contradicts the fact that $\Sigma_j^* \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \{0\}$. Hence, $\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$. By Lemma 4.3,

$$\text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}.$$

Let $X_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$, $d_j = \dim X_j$ for $j = 0, \dots, q$, and $d_{-1} = 0$. Obviously, $d_0 \ge 1$ and $d_q = n$. We claim that $d_0 < d_1 < \dots < d_q$. Otherwise, there exist $0 \le k < l \le q$ such that $d_k = d_l$, and thus $X_k = X_l$. We write $\lambda > 0$ for the constant of Lemma 4.3 depending on $u_1, \dots, u_N$. By Lemma 4.3, there exist $u_{i_1}, \dots, u_{i_{d_k}} \in \{u_1, \dots, u_{\alpha_{k+1}-1}\}$ and $0 \le a_{i_1}, \dots, a_{i_{d_k}} \le \lambda$ such that

$$u_{\alpha_l} = a_{i_1} u_{i_1} + \dots + a_{i_{d_k}} u_{i_{d_k}}.$$

Hence,

$$\begin{aligned} h(P_m, u_{\alpha_l}) &= h(P_m, a_{i_1}u_{i_1} + ... + a_{i_{d_k}}u_{i_{d_k}}) \\ &\le a_{i_1}h(P_m, u_{i_1}) + ... + a_{i_{d_k}}h(P_m, u_{i_{d_k}}), \end{aligned}$$

for all $m \in \mathbb{N}$. But this contradicts (4.0a) and (4.0b). Therefore,

$$1 \le d_0 < d_1 < ... < d_q = n.$$

□

Lemma 4.5. *Suppose μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1}, and satisfies the strict essential subspace concentration inequality. If P_m is a sequence of polytopes with $V(P_m) = 1$, $\xi(P_m)$ is the origin, the set of outer unit normals of P_m is a subset of the support of μ and $\lim_{m\to\infty} d(P_m) = \infty$, then*

$$\int_{S^{n-1}} \log h(P_m, u) d\mu(u)$$

is not bounded from above.

Proof. Without loss of generality, we can suppose $|\mu| = 1$. Let $\mathrm{supp}(\mu) = \{u_1, ..., u_N\}$, and $\mu(\{u_i\}) = \gamma_i, i = 1, ..., N$. From Lemma 4.1, we may assume that

$$h(P_m, u_1) \le ... \le h(P_m, u_N), \tag{4.1}$$

for all $m \in \mathbb{N}$. Since $\lim_{m\to\infty} d(P_m) = \infty$ and $V(K) = 1$,

$$\lim_{m\to\infty} h(P_m, u_1) = 0 \text{ and } \lim_{m\to\infty} h(P_m, u_N) = \infty.$$

By Lemma 4.4, there exist $q \ge 1$, and

$$1 = \alpha_0 < \alpha_1 < ... < \alpha_q \le N < N + 1 = \alpha_{q+1}$$

such that if $j = 1, ..., q$, then

$$\lim_{m\to\infty} \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j-1}})} = \infty, \tag{4.2a}$$

and if $j = 0, ..., q$ and $\alpha_j \le k \le \alpha_{j+1} - 1$, then

$$\lim_{m\to\infty} \frac{h(P_m, u_k)}{h(P_m, u_{\alpha_j})} = t_{k,j} < \infty. \tag{4.2b}$$

Moreover, $X_j = \mathrm{pos}\{u_1, ..., u_{\alpha_{j+1}-1}\}$ are subspaces of $\mathbb{R}^n$ with respect to μ for all $0 \le j \le q$ with

$$1 \le d_0 < d_1 < ... < d_q = n,$$

where $d_j = \dim(X_j)$. In particular, $X_0, \ldots, X_{q-1}$ are essential subspaces.

Let $\widetilde{X}_0 = X_0$, and if $j = 1, ..., q$, then let

$$\widetilde{X}_j = X_{j-1}^{\perp} \cap X_j.$$

From the definition of X_j and $\widetilde{X}_j$, we have, $\tilde{X}_{j_1} \perp \tilde{X}_{j_2}$ for $j_1 \ne j_2$, $\dim \widetilde{X}_j = d_j - d_{j-1} > 0$ for $j = 0, ..., q$, and $\mathbb{R}^n$ is a direct sum of $\tilde{X}_0, ..., \tilde{X}_q$.

Let $\lambda > 0$ be the constant of Lemma 4.3 for $u_1, \ldots, u_N$. Suppose $0 \le j \le q$ and $u \in X_j \cap S^{n-1}$. By Lemma 4.3, there exists a subset, $\{u_{i_1}, ..., u_{i_{d_j}}\}$, of $\{u_1, ..., u_{\alpha_{j+1}-1}\}$ and $0 \le a_{i_1}, ..., a_{i_{d_j}} \le \lambda$ such that

$$u = a_{i_1}u_{i_1} + ... + a_{i_{d_j}}u_{i_{d_j}}.$$

Then,

$$\begin{aligned} h(P_m, u) &= h(P_m, a_{i_1} u_{i_1} + ... + a_{i_{d_j}} u_{i_{d_j}}) \\ &\leq a_{i_1} h(P_m, u_{i_1}) + ... + a_{i_{d_j}} h(P_m, u_{i_{d_j}}). \end{aligned}$$

By this, (4.2a) and (4.2b), if m is large, then

$$h(P_m, u) \leq t_j h(P_m, u_{\alpha_j}) \text{ for all } u \in X_j \cap S^{n-1}$$

where $t_j = d_j \lambda (t_{\alpha_{j+1}-1,j} + 1) > 0$. Hence, for $j = 0, ..., q$,

$$P_m|_{\tilde{X}_j} \subset t_j h(P_m, u_{\alpha_j}) \big(B^n \cap \tilde{X}_j \big).$$

By this and the fact that $\mathbb{R}^n$ is a direct sum of $\tilde{X}_0, ..., \tilde{X}_q$,

$$P_m \subset \sum_{j=0}^{q} t_j h(P_m, u_{\alpha_j}) \big(B^n \cap \tilde{X}_j \big),$$

where the summation is Minkowski sum. Let

$$\omega = \max_{0 \leq j \leq q} t_j \kappa_{d_j - d_{j-1}}^{\overline{\frac{1}{d_j - d_{j-1}}}},$$

where $\kappa_{d_j - d_{j-1}}$ is the volume of the $(d_j - d_{j-1})$-dimensional unit ball. Then, for $j = 0, ..., q$

$$V_{d_j - d_{j-1}} \Big(t_j h(P_m, u_{\alpha_j}) \big(B^n \cap \tilde{X}_j \big) \Big) \leq (\omega h(P_m, u_{\alpha_j}))^{d_j - d_{j-1}}.$$

From this, the fact that $\mathbb{R}^n$ is a direct sum of $\tilde{X}_0, ..., \tilde{X}_q$, and Fubini's formula, we have

$$\begin{aligned} 1 &= V(P_m) \\ &\leq V \Bigg(\sum_{j=0}^{q} t_j h(P_m, u_{\alpha_j}) \big(B^n \cap \tilde{X}_j \big) \Bigg) \\ &= \prod_{j=0}^{q} V_{d_j - d_{j-1}} \Big(t_j h(P_m, u_{\alpha_j}) \big(B^n \cap \tilde{X}_j \big) \Big) \\ &\leq \prod_{j=0}^{q} (\omega h(P_m, u_{\alpha_j}))^{d_j - d_{j-1}}. \end{aligned}$$

It follows from $0 = d_{-1} < d_0 < ... < d_q = n$ that if m is large, then

$$\sum_{j=0}^{q} \left(\frac{d_j}{n} - \frac{d_{j-1}}{n} \right) \log h(P_m, u_{\alpha_j}) \geq -\log \omega.$$

We rewrite the last inequality as

$$\log h(P_m, u_{\alpha_q}) \geq - \sum_{j=0}^{q-1} \frac{d_j}{n} \log \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j+1}})} - \log \omega. \tag{4.3}$$

For $j = 0, ..., q$, we set $\beta_j = \mu(X_j \cap S^{n-1}) = \sum_{i=1}^{\alpha_{j+1}-1} \gamma_i$, and $\beta_{-1} = 0$. We deduce from the facts that X_j is an essential subspace with $d_j = \dim(X_j)$, and from the condition that μ satisfies the strict essential subspace concentration condition that

$$\beta_j < \frac{d_j}{n} \quad \text{for } 0 \leq j \leq q-1. \tag{4.4}$$

By the fact that $h(P_m,u_1)\le h(P_m,u_2)\le ...\le h(P_m,u_N)$, the fact that $\beta_q=1$ and (4.3),

$$\begin{aligned}\sum_{i=1}^{N}\gamma_i\log h(P_m,u_i) &= \sum_{i=1}^{\alpha_1-1}\gamma_i\log h(P_m,u_i)+\sum_{i=\alpha_1}^{\alpha_2-1}\gamma_i\log h(P_m,u_i)+...+\sum_{i=\alpha_q}^{N}\gamma_i\log h(P_m,u_i)\\ &\ge \sum_{i=1}^{\alpha_1-1}\gamma_i\log h(P_m,u_{\alpha_0})+\sum_{i=\alpha_1}^{\alpha_2-1}\gamma_i\log h(P_m,u_{\alpha_1})+...+\sum_{i=\alpha_q}^{N}\gamma_i\log h(P_m,u_{\alpha_q})\\ &=\sum_{j=0}^{q}(\beta_j-\beta_{j-1})\log h(P_m,u_{\alpha_j})\\ &=\log h(P_m,u_{\alpha_q})+\sum_{j=0}^{q-1}\beta_j\log\frac{h(P_m,u_{\alpha_j})}{h(P_m,u_{\alpha_{j+1}})}\\ &\ge -\log\omega+\sum_{j=0}^{q-1}\left(\beta_j-\frac{d_j}{n}\right)\log\frac{h(P_m,u_{\alpha_j})}{h(P_m,u_{\alpha_{j+1}})}.\end{aligned}$$

It follows from (4.1), (4.2a), (4.4) that for $j=0,...,q-1$,

$$\lim_{m\to\infty}\left(\beta_j-\frac{d_j}{n}\right)\log\frac{h(P_m,u_{\alpha_j})}{h(P_m,u_{\alpha_{j+1}})}=\infty.$$

Therefore,

$$\lim_{m\to\infty}\sum_{i=1}^{N}\gamma_i\log h(P_m,u_i)=\infty.$$

$\square$

The following lemma will be needed (see, [71], Lemma 3.5).

Lemma 4.6. *If P is a polytope in $\mathbb{R}^n$ and $v_0\in S^{n-1}$ with $V_{n-1}(F(P,v_0))=0$, then there exists a $\delta_0>0$ such that for $0\le\delta<\delta_0$*

$$V(P\cap\{x:x\cdot v_0\ge h(P,v_0)-\delta\})=c_n\delta^n+...+c_2\delta^2,$$

where $c_n,...,c_2$ are constants that depend on P and v_0.

Now, we have prepared enough to prove the main result of this section.

Lemma 4.7. *Suppose the discrete measure $\mu=\sum_{k=1}^{N}\gamma_k\delta_{u_i}$ is not concentrated on a closed hemisphere. If μ satisfies the strict essential subspace concentration inequality, then there exists a $P\in\mathcal{P}_N(u_1,...,u_N)$ such that $\xi(P)=0$, $V(P)=|\mu|$ and*

$$\Phi_P(0)=\inf\left\{\max_{\xi\in\text{Int }(Q)}\Phi_Q(\xi):Q\in\mathcal{P}(u_1,...,u_N)\ \textit{and}\ V(Q)=|\mu|\right\},$$

where $\Phi_Q(\xi)=\int_{S^{n-1}}\log\left(h(Q,u)-\xi\cdot u\right)d\mu(u)$.

Proof. It is easily seen that it is sufficient to establish the lemma under the assumption that $|\mu|=1$.

Obviously, for $P,Q\in\mathcal{P}(u_1,...,u_N)$, if there exists an $x\in\mathbb{R}^n$ such that $P=Q+x$, then

$$\Phi_P(\xi(P))=\Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence $P_i\in\mathcal{P}(u_1,...,u_N)$ with $\xi(P_i)=0$ and $V(P_i)=1$ such that $\Phi_{P_i}(0)$ converges to

$$\inf\left\{\max_{\xi\in\text{Int }(Q)}\Phi_Q(\xi):Q\in\mathcal{P}(u_1,...,u_N)\text{ and }V(Q)=1\right\}.$$

Choose a fixed $P_0 \in \mathcal{P}(u_1, ..., u_N)$ with $V(P_0) = 1$, then

$$\inf \left\{ \max_{\xi \in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\} \leq \Phi_{P_0}(\xi(P_0)).$$

We claim that P_i is bounded. Otherwise, from Lemma 4.5, $\Phi_{P_i}(\xi(P_i))$ is not bounded from above. This contradicts the previous inequality. Therefore, P_i is bounded.

From Lemma 3.2 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in \mathcal{P}(u_1, ..., u_N)$, $V(P) = 1$, $\xi(P) = 0$ and

$$\Phi_P(0) = \inf \left\{ \max_{\xi \in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\}. \tag{4.5}$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, ..., N$. Otherwise, there exists an $i_0 \in \{1, ..., N\}$ such that

$$F(P, u_{i_0})$$

is not a facet of P.

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, ..., u_N),$$

and (by Lemma 4.6)

$$V(P_\delta) = 1 - (c_n\delta^n + ... + c_2\delta^2),$$

where $c_n, ..., c_2$ are constants that depend on P and direction u_{i_0}.

From Lemma 3.2, for any $\delta_i \to 0$ $\xi(P_{\delta_i}) \to 0$. We have,

$$\lim_{\delta \to 0} \xi(P_\delta) = 0.$$

Let δ be small enough so that $h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta$ for all $k \in \{1, ..., N\}$, and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = (1 - (c_n\delta^n + ... + c_2\delta^2))^{-\frac{1}{n}}.$$

From this and Equation (3.3), we have

$$\begin{aligned}
\prod_{k=1}^N (h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta) \cdot u_k)^{\gamma_k} &= \lambda \prod_{k=1}^N (h(P_\delta, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \\
&= \lambda \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \left[\frac{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0}} \right]^{\gamma_{i_0}} \\
&= \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \frac{(1 - \frac{\delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0}})^{\gamma_{i_0}}}{(1 - (c_n\delta^n + ... + c_2\delta^2))^{\frac{1}{n}}} \\
&\leq \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \frac{(1 - \frac{\delta}{d_0})^{\gamma_{i_0}}}{(1 - (c_n\delta^n + ... + c_2\delta^2))^{\frac{1}{n}}},
\end{aligned}$$

where $d_0 = d(P)$ is the diameter of P. Thus,

$$\Phi_{\lambda P_\delta}\left(\xi(\lambda P_\delta)\right) \leq \Phi_P\left(\xi(P_\delta)\right) + B(\delta), \tag{4.6}$$

where

$$B(\delta) = \gamma_{i_0} \log\left(1 - \frac{\delta}{d_0}\right) - \frac{1}{n} \log\left(1 - (c_n\delta^n + ... + c_2\delta^2)\right). \tag{4.7}$$

Obviously,

$$B'(\delta)=\gamma_{i_0}\frac{-1/d_0}{1-\delta/d_0}+\frac{1}{n}\frac{nc_n\delta^{n-1}+...+2c_2\delta}{1-(c_n\delta^n+...+c_2\delta^2)}<0, \tag{4.8}$$

when the positive δ is small enough. From this and the fact that $B_1(0)=0$,

$$B(\delta)<0$$

when the positive δ is small enough.

From this and Equations (4.6), (4.7), (4.8), there exists a $\delta_0>0$ such that $P_{\delta_0}\in\mathcal{P}(u_1,...,u_N)$ and

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0}))<\Phi_P(\xi(P_{\delta_0}))\leq\Phi_P(\xi(P))=\Phi_P(0),$$

where $\lambda_0=V(P_{\delta_0})^{-\frac{1}{n}}$. Let $P_0=\lambda_0 P_{\delta_0}-\xi(\lambda_0 P_{\delta_0})$, then $P_0\in\mathcal{P}(u_1,...,u_N)$, $V(P_0)=1$, $\xi(P_0)=0$ and

$$\Phi_{P_0}(0)<\Phi_P(0).$$

This contradicts Equation (4.5). Therefore, $P\in\mathcal{P}_N(u_1,...,u_N)$. □

5. Existence of the solution to the discrete logarithmic Minkowski problem

If μ is a Borel measure on S^{n-1} and ξ is a proper subspace of $\mathbb{R}^n$, it will be convenient to write μ_ξ for the restriction of μ to $S^{n-1}\cap\xi$. In this section, we prove the main result Theorem 1.5 of this paper based on the folowing idea. Let μ be discrete measure on S^{n-1}, $n\geq 2$, that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition. If μ satisfies the strict essential subspace concentration inequality, then Lemma 4.7 yields that μ is a cone volume measure. Otherwise there exist complementary proper subspaces ξ and ξ' such that $\operatorname{supp}\mu=S^{n-1}\cap(\xi\cup\xi')$, and μ_ξ and μ'_ξ are not concentrated on any closed hemisphere of $\xi\cap S^{n-1}$ and $\xi'\cap S^{n-1}$, respectively, and satisfy the essential subspace concentration condition. Therefore μ_ξ and μ'_ξ are cone volume measures on $\xi\cap S^{n-1}$ and $\xi'\cap S^{n-1}$, respectively, by induction on the dimension of the ambient space, which in turn imply that μ is a cone volume measure.

However, it is possible that $\dim\xi=1$. Therefore in order to execute the plan, we extend the notions occuring in Theorem 1.5 to $\mathbb{R}^1$. The role of a compact convex set containing the origin in its interior is played by some interval $K=[a,b]$ with $a<0$ and $b>0$, and closed hemispheres of $S^0=\{-1,1\}$ are $\{1\}$ and $\{-1\}$. The cone volume measure on S^0 associated to K satisfies $V_K(\{-1\})=|a|$ and $V_K(\{1\})=b$. In addition, we say that a non-trivial measure μ on S^0 satisfies the essential subspace concentration inequality if it is not concentrated on any closed hemisphere; namely, if $\mu(\{-1\})>0$ and $\mu(\{1\})>0$. These notions are in accordance with Definition 1.3 because if $n=1$, then there is no subspace ξ such that $0<\dim\xi<n$.

We note that the notion of strict essential subspace concentration inequality is defined and used only if the dimension $n\geq 2$.

The following lemma will be needed. The proof is the same that of Lemma 7.1 in [6].

Lemma 5.1. *Suppose $n\geq 2$, μ is a discrete measure on S^{n-1} that satisfies the essential subspace concentration condition. If ξ is an essential linear subspace with respect to μ for which*

$$\mu(\xi\cap S^{n-1})=\frac{1}{n}\mu(S^{n-1})\dim\xi,$$

then μ_ξ satisfies the essential subspace concentration condition.

For even measures, the following lemma was stated for even measures as Lemma 7.2 in [6]. However, the proof in [6] does not use the property that the measure is even.

Lemma 5.2. *Let ξ and ξ' be complementary subspaces in $\mathbb{R}^n$ with $0 < \dim \xi < n$. Suppose μ is a Borel measure on S^{n-1} that is concentrated on $S^{n-1} \cap (\xi \cup \xi')$, and so that*

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n}\mu(S^{n-1}) \dim \xi.$$

If μ_ξ and $\mu_{\xi'}$ are cone-volume measures of convex bodies in the subspaces ξ and ξ', then μ is the cone-volume measure of a convex body in $\mathbb{R}^n$.

In addition, we also need the following lemma.

Lemma 5.3. *Suppose μ is a Borel measure on S^{n-1}, $n \geq 2$, that is not concentrated on any closed hemisphere, and μ concentrated on two complementary subspaces ξ and ξ' of $\mathbb{R}^n$. Then, μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$ and $\mu_{\xi'}$ is not concentrated on any closed hemisphere of $\xi' \cap S^{n-1}$.*

Proof. We only need prove that μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Suppose μ_ξ is concentrated on a closed hemisphere, C, of $\xi \cap S^{n-1}$. Then, μ is concentrated on

$$S^{n-1} \cap \mathrm{pos}\{C \cup \xi'\}.$$

However, $S^{n-1} \cap \mathrm{pos}\{C \cup \xi'\}$ is a closed hemisphere of S^{n-1}. This contradicts the conditions of the lemma. Therefore, μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$. $\square$

Now, we have prepared enough to prove the main theorem of this paper.

Theorem 5.4. *If μ is a discrete measure on S^{n-1}, $n \geq 1$ that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition, then μ is the cone-volume measure of a polytope in $\mathbb{R}^n$.*

Proof. We prove Theorem 5.4 by induction on the dimension $n \geq 1$. If $n = 1$, then the theorem trivially holds, therefore let $n \geq 2$.

If μ satisfies the strict essential subspace concentration inequality, then μ is the cone-volume measure of a polytope in $\mathbb{R}^n$ according to Lemma 3.4 and Lemma 4.7.

Therefore we assume that there exists an essential subspace (with respect to μ), ξ, of $\mathbb{R}^n$, and a subspace, ξ', of $\mathbb{R}^n$ such that ξ, ξ' are complementary subspaces of $\mathbb{R}^n$, μ concentrated on $S^{n-1} \cap \{\xi \cup \xi'\}$ with

$$\mu(S^{n-1} \cap \xi) = \frac{\dim \xi}{n}\mu(S^{n-1}) \text{ and } \mu(S^{n-1} \cap \xi') = \frac{\dim \xi'}{n}\mu(S^{n-1}).$$

From the fact that μ is not concentrated on a closed hemisphere and Lemma 5.3, we have, μ_ξ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi$, and $\mu_{\xi'}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi'$. By Lemma 5.1, μ_ξ satisfies the essential subspace concentration condition on $\xi \cap S^{n-1}$, and $\mu_{\xi'}$ satisfies the essential subspace concentration condition on $\xi' \cap S^{n-1}$. From the induction hypothesis, μ_ξ is the cone-volume measure of a convex body in $\xi \cap \mathbb{R}^n$, and $\mu_{\xi'}$ is the cone-volume measure of a convex body in $\xi' \cap \mathbb{R}^n$. By Lemma 5.2, μ is the cone-volume measure of a convex body in $\mathbb{R}^n$. Since μ is discrete, μ is the cone-volume measure of a polytope in $\mathbb{R}^n$. $\square$

6. New inequalities for cone-volume measures

In this section, we establish some inequalities for cone-volume measures.

The following example shows that the cone-volume measure of a convex body does not need to satisfy the essential subspace concentration condition with respect to essential linear subspace.

Example 6.1. *Let $u_1,\dots,u_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $W=\{x\in u_1^\perp : |x\cdot u_i|\le 1,\ i=2,\dots,n\}$ be an $(n-1)$-dimensional cube. For $r>0$ and $i=1,\dots,n-1$, $\xi_i=\mathrm{lin}\{u_1,\dots,u_i\}$ is an essential subspace for the cone-volume measure of the truncated pyramid $P_r=[-ru_1+rW,u_1+W]$. If $r>0$ is small, then P_r approximates $[o,u_1+W]$, and thus*

$$V_{P_r}(\xi_i\cap S^{n-1})>V_{P_r}(\{u_1\})=V([o,u_1+W])>\tfrac{i}{n}V(P_r).$$

We next establish new inequalities for the cone-volume measures.

Lemma 6.2. *If K is a convex body in $\mathbb{R}^n$, $n\ge 3$, with $o\in$Int(K), then for $u\in S^{n-1}$*

$$V_K(\{u\})+V_K(\{-u\})+2(n-1)\sqrt{V_K(\{u\})V_K(\{-u\})}\le V(K), \tag{6.1}$$

with equality if and only if $F(K,-u)$ is a translate of $F(K,u)$, $K=[F(K,u),F(K,-u)]$, and $h(K,u)=h(K,-u)$.

In $\mathbb{R}^2$, we have

Lemma 6.3. *If K is a convex body containing the origin in its interior in $\mathbb{R}^2$, and $u\in S^1$, then*

$$\sqrt{V_K(\{u\})}+\sqrt{V_K(\{-u\})}\le\sqrt{V(K)}, \tag{6.2}$$

with equality if and only if K is a trapezoid with two sides parallel to $u^\perp$, and $u^\perp$ contains the intersection of the diagonals.

We obtain the following estimate from Lemma 6.2 and Lemma 6.3.

Corollary 6.4. *If K is a convex body in $\mathbb{R}^n$, $n\ge 2$ with $o\in$Int(K) and $u\in S^{n-1}$, then*

$$V_K(\{u\})\cdot V_K(\{-u\})\le\frac{1}{4n^2}\big(V(K)\big)^2,$$

with equality if and only if $F(K,-u)$ is a translate of $F(K,u)$, $K=[F(K,u),F(K,-u)]$, and $h(K,u)=h(K,-u)$.

We next prove Lemma 6.2 and Lemma 6.3 together.

Proof. For the case $|F(K,u)|\cdot|F(K,-u)|=0$, Lemma 6.2 and Lemma 6.3 are trivially true. Thus we prove Lemma 6.2 and Lemma 6.3 under the condition that $|F(K,u)|\cdot|F(K,-u)|>0$.

Let $V_K(\{u\})=\alpha>0$ and $V_K(\{-u\})=\beta>0$, let $h_K(u)=a$ and $h_K(-u)=b$, and for $0\le x\le a+b$ let

$$K_x=\big((a-x)u+u^\perp\big)\cap K.$$

Since K is a convex body,

$$\frac{x}{a+b}F(K,-u)+\frac{a+b-x}{a+b}F(K,u)\subset K_x.$$

From this and the Brunn-Minkowski inequality,

$$\begin{aligned}
|K_x| &\ge \left|\frac{x}{a+b}F(K,-u)+\frac{a+b-x}{a+b}F(K,u)\right|\\
&= \left|\left(\frac{x}{a+b}F(K,-u)+\frac{a+b-x}{a+b}F(K,u)\right)_{u^\perp}\right|\\
&= \left|\frac{x}{a+b}F(K,-u)|_{u^\perp}+\frac{a+b-x}{a+b}F(K,u)|_{u^\perp}\right| \qquad (6.3)\\
&\ge \left(\frac{x}{a+b}\big|F(K,-u)|_{u^\perp}\big|^{\frac{1}{n-1}}+\frac{a+b-x}{a+b}\big|F(K,u)|_{u^\perp}\big|^{\frac{1}{n-1}}\right)^{n-1}\\
&= \left(\frac{x}{a+b}\big|F(K,-u)\big|^{\frac{1}{n-1}}+\frac{a+b-x}{a+b}\big|F(K,u)\big|^{\frac{1}{n-1}}\right)^{n-1},
\end{aligned}$$

with equality if and only if $K_x = \frac{x}{a+b}F(u(K,-u) + \frac{a+b-x}{a+b}F(K,u)$, and $F(K,-u)|_{u^\perp}$ and $F(K,u)|_{u^\perp}$ are homothetic.

Let $t = \frac{a+b-x}{a+b}$. From (6.3) and Fubini's formula,

$$\begin{aligned} V(K) &= \int_0^{a+b} |K_x| dx \\ &\geq \int_0^{a+b} \left(\frac{x}{a+b} \big|F(K,-u)\big|^{\frac{1}{n-1}} + \frac{a+b-x}{a+b} \big|F(K,u)\big|^{\frac{1}{n-1}} \right)^{n-1} dx \\ &= (a+b) \int_0^1 \left(t|F(K,u)|^{\frac{1}{n-1}} + (1-t)|F(K,-u)|^{\frac{1}{n-1}} \right)^{n-1} dt \\ &= (a+b) \sum_{i=0}^{n-1} |F(K,u)|^{\frac{i}{n-1}} |F(K,-u)|^{\frac{n-1-i}{n-1}} \binom{n-1}{i} \int_0^1 t^i (1-t)^{n-1-i} dt \\ &= \frac{a+b}{n} \sum_{i=0}^{n-1} |F(K,u)|^{\frac{i}{n-1}} |F(K,-u)|^{\frac{n-1-i}{n-1}}. \end{aligned} \tag{6.4}$$

Let $S_1 = |F(K,u)|$ and $S_2 = |F(K,-u)|$. From (6.4) and the arithmetic-geometric inequality, we have

$$\begin{aligned} V(K) &= \frac{a+b}{n} \sum_{i=0}^{n-1} S_1^{\frac{i}{n-1}} S_2^{\frac{n-1-i}{n-1}} \\ &= \frac{a}{n} S_1 + \frac{b}{n} S_2 + \frac{1}{n} \sum_{i=1}^{n-1} \left(a S_1^{\frac{n-1-i}{n-1}} S_2^{\frac{i}{n-1}} + b S_2^{\frac{n-1-i}{n-1}} S_1^{\frac{i}{n-1}} \right) \\ &\geq \alpha + \beta + 2(n-1)\sqrt{\alpha\beta}. \end{aligned} \tag{6.5}$$

Thus, we get (6.1) and (6.2).

From the equality conditions for (6.3), (6.4) and the arithmetic-geometric inequality, we have, equality holds in (6.5) if and only if $F(K,u)|_{u^\perp}$ and $F(K,-u)|_{u^\perp}$ are homothetic, $K = [F(K,u), F(K,-u)]$, and

$$\frac{a}{b} = \left(\frac{S_1}{S_2} \right)^{\frac{2i-n+1}{n-1}}, \tag{6.6}$$

for all $1 \leq i \leq n-1$.

Therefore, equality holds in (6.2) ($n = 2$) if and only if K is a trapezoid with two sides parallel to $u^\perp$, and $u^\perp$ contains the intersection of the diagonals.

When $n \geq 3$, (6.6) hold for $i = 1, ..., n-1$. Thus, $\frac{a}{b} = \frac{S_1}{S_2} = 1$. Therefore, equality holds in (6.1) if and only if $F(K,-u)$ is a translation of $F(K,u)$, $K = [F(K,u), F(K,-u)]$, and $h_K(u) = h_K(-u)$. □

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