INVITATION TO INTERSECTION PROBLEMS FOR FINITE SETS

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ABSTRACT. Extremal set theory is dealing with families, $\mathcal{F}$ of subsets of an n-element set. The usual problem is to determine or estimate the maximum possible size of $\mathcal{F}$, supposing that $\mathcal{F}$ satisfies certain constraints. To limit the scope of this survey most of the constraints considered are of the following type: any r subsets in $\mathcal{F}$ have at least t elements in common, all the sizes of pairwise intersections belong to a fixed set, L of natural numbers, there are no s pairwise disjoint subsets. Although many of these problems have a long history, their complete solutions remain elusive and pose a challenge to the interested reader.

Most of the paper is devoted to sets, however certain extensions to other structures, in particular to vector spaces, integer sequences and permutations are mentioned as well. The last part of the paper gives a short glimpse of one of the very recent developments, the use of semidefinite programming to provide good upper bounds.

CONTENTS

1. INTRODUCTION

For a positive integer n let $[n]$ denote the set of the first n positive integers, $[n] = \{1, 2, \ldots, n\}$. Also let $2^{[n]}$ and $\binom{[n]}{k}$ denote the power set and the collection of all k-element subsets of $[n]$, respectively. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*, and elements of $\mathcal{F}$ are often called *members*. If $\mathcal{F} \subset \binom{[n]}{k}$, we call it k-*uniform*.

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Extremal set theory is a fast developing area within combinatorics which deals with determining or estimating the size $|\mathcal{F}|$ of a family satisfying certain restrictions.

The first result in extremal set theory was Sperner's Theorem, establishing the maximum size of an antichain, i.e., a family without a pair of members one containing the other.

Theorem 1.1 (Sperner [137]). *Suppose that $\mathcal{A} \subset 2^{[n]}$ satisfies $A \not\subset A'$ for all $A, A' \in \mathcal{A}$. Then it follows $|\mathcal{A}| \le \binom{n}{\lfloor n/2 \rfloor}$. Moreover, the only families achieving equality are $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$. (Note that for n even they coincide.)*

Sperner's Theorem dates back to 1928 but it remained an isolated result for decades. It was mostly due to the pioneering work of Paul Erdős that systematic research of similar problems started in the 1960's. Erdős' application of Sperner's theorem to Littlewood–Offord problem was also a very early result in Extremal Set Theory. By now extensions and analogues of Sperner's Theorem are very numerous and have been the subject of several survey articles and monographs, cf. e.g., [37]. For this reason we limit the scope of the present survey to another subfield *intersection theorems*. The first instance is the following.

Theorem 1.2 (Erdős–Ko–Rado [44]). *Let $n > k > t > 0$ be integers and let $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $|F \cap F'| \ge t$ for all $F, F' \in \mathcal{F}$. Then (i) and (ii) hold.*

(i) *If $t = 1$ and $n \ge 2k$ then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{1}$$

(ii) *If $t \ge 2$ and $n > n_0(k,t)$ then*

$$|\mathcal{F}| \le \binom{n-t}{k-t}.$$

Let us mention that Erdős, Ko, and Rado proved this result around 1938, when all three of them were in England. However, interest in combinatorics was very limited at that time. That is the reason that they postponed the publication of this fundamental result for more than 20 years. There are many proofs known for (i) and we are going to present one in the next section but let us show a simple proof of (ii) here.

Proof of (ii) in Theorem 1.2. For any set $T \in \binom{[n]}{t}$ the family $\{F \in \binom{n}{k} : T \subset F\}$ has size $\binom{n-t}{k-t}$ and satisfies the assumptions. Suppose now that no t-element set is contained in all members of $\mathcal{F}$.

We claim that one can find a set B, with $|B| < 3k$ such that

$$|F \cap B| \ge t + 1 \quad \text{for all } F \in \mathcal{F}. \tag{2}$$

To prove the claim we distinguish two cases. The first case is that $|F \cap F'| \ge t + 1$ for all $F, F' \in \mathcal{F}$. In this case $B = F$ will do for any $F \in \mathcal{F}$. The second case is that there exist $F_1, F_2 \in \mathcal{F}$ with $|F_1 \cap F_2| = t$. Now choose an arbitrary $F_3 \in \mathcal{F}$ with $F_1 \cap F_2 \not\subset F_3$. Choose $B = F_1 \cup F_2 \cup F_3$. Then $|F \cap F_i| \ge t$ for $i = 1, 2, 3$ forces $|F \cap B| \ge t + 1$, concluding the proof of the claim.

To finish the proof of (ii) just note that for every $F \in \binom{[n]}{k}$ and satisfying (2) one can find $B_0 \in \binom{B}{t+1}$ and $F_0 \in \binom{[n]}{k-t-1}$ in one or several ways such that $F = B_0 \cup F_0$. Consequently, $|\mathcal{F}| \le \binom{|B|}{t+1}\binom{n}{k-t+1} < \binom{3k}{t+1}\binom{n}{k-t-1}$. For $n > n_0(k,t)$ the RHS is much less than $\binom{n-t}{k-t}$. We mention that this type of argument is often referred to as a 'degrees of freedom argument.' □

To prove (i), Erdős, Ko, and Rado introduced an operation on families of sets, called *shifting*, which we are going to define in the next section. To avoid technicalities here we content ourselves with the following.

Definition 1.1. A family $\mathcal{F} \subset 2^{[n]}$ is called *shifted* if for all $F \in \mathcal{F}$, $j \in F$ and $i < j$, if $i \notin F$ then $(F \setminus \{j\}) \cup \{i\}$ is also in $\mathcal{F}$.

Let us introduce some convenient notation to give a reformulation of shiftedness. For $\mathcal{F} \subset 2^{[n]}$, $i, j \in [n]$ define

$$\mathcal{F}(i) = \{F \setminus \{i\} : i \in F \in \mathcal{F}\}, \quad \mathcal{F}(\bar{j}) = \{F : j \notin F \in \mathcal{F}\}, \quad \mathcal{F}(i, \bar{j}) = (\mathcal{F}(i))(\bar{j}).$$

Now we can easily see that $\mathcal{F}$ is shifted if and only if $\mathcal{F}(i,\bar{j}) \supset \mathcal{F}(j,\bar{i})$ for all $1 \le i < j \le n$. Note also the obvious equality: $|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$.

Proof of (i) in Theorem 1.2 for shifted families. Let us now prove (i) for shifted families by applying induction on n. In the case $n = 2k$ one has $\binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}$ and one can partition all k-element subsets of $[2k]$ into $\binom{2k-1}{k-1} = \binom{2k-1}{k}$ pairs (F, G) where in each pair $G = [2k] \setminus F$. The condition $F \cap F' \neq \emptyset$ implies that for each pair (F, G) at most one of the two is in $\mathcal{F}$. Consequently, $|\mathcal{F}| \le \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$, proving (1). Now suppose that $n > 2k$ and consider the two families $\mathcal{F}(n) \subset \binom{[n-1]}{k-1}$ and $\mathcal{F}(\bar{n}) \subset \binom{[n-1]}{k}$. By the induction hypothesis the latter satisfies $\mathcal{F}(\bar{n}) \le \binom{n-2}{k-1}$. The point is that if $\mathcal{F}$ is shifted then $G \cap G' \neq \emptyset$ for all $G, G' \in \mathcal{F}(n)$ as well. Indeed, supposing $G \cap G' = \emptyset$ we have $|[n-1] \setminus (G \cup G')| = (n-1) - 2(k-1)| = n - 2k + 1 > 0$. Take an element i from $[n-1] \setminus (G \cup G')$. By shiftedness $F' := G' \cup \{i\} \in \mathcal{F}$. However, $F := G \cup \{n\} \in \mathcal{F}$ by definition and $F \cap F' = \emptyset$, a contradiction. Therefore we can apply the induction hypothesis to $\mathcal{F}(n)$ as well to infer $|\mathcal{F}(n)| \le \binom{n-2}{k-2}$. Consequently, $|\mathcal{F}| = |\mathcal{F}(\bar{n})| + |\mathcal{F}(n)| \le \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$, proving (1). □

Let us give a proper name for the properties required in the Erdős–Ko–Rado Theorem.

Definition 1.2. Let $r \ge 2$ and $t \ge 1$ be integers. A family $\mathcal{F} \subset 2^{[n]}$ is called *r-wise t-intersecting* if for all $F_1, \dots, F_r \in \mathcal{F}$ one has $|F_1 \cap \cdots \cap F_r| \ge t$. For $r = 2$ we omit the word '2-wise,' and for $t = 1$ we just say 'intersecting' instead of '1-intersecting.'

What about t-intersecting families that are not necessarily k-uniform. In [44] it is shown that $|\mathcal{F}| \le 2^{n-1}$ for all intersecting $\mathcal{F} \subset 2^{[n]}$, and the upper bound follows from the simple observation that $F \in \mathcal{F}$ implies $[n] \setminus F \notin \mathcal{F}$. Moreover, they prove that every intersecting family $\mathcal{F} \subset 2^{[n]}$ can be extended to a family $\mathcal{G}$ of size 2^{n-1} with $\mathcal{F} \subset \mathcal{G} \subset 2^{[n]}$.

The case of t-intersecting families with $t \geq 2$ is more difficult. Answering a problem from the EKR paper [44] Katona proved the following.

Theorem 1.3 (Katona [99])**.** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, $n \geq t \geq 1$. Then either (i) or (ii) holds.*

(i) $n+t=2a$ *and*

$$|\mathcal{F}| \leq \sum_{k \geq a} \binom{n}{k}. \tag{3}$$

(ii) $n+t=2a+1$ *and*

$$|\mathcal{F}| \leq \binom{n-1}{a} + \sum_{k \geq a+1} \binom{n}{k}. \tag{4}$$

Moreover, if $t \geq 2$ then in the case of equality $\mathcal{F} = \{F \subset [n] : |F| \geq a\}$ holds for (i), and $\mathcal{F} = \binom{Y}{a} \cup \{F \subset [n] : |F| \geq a+1\}$ holds for (ii) with some $Y \in \binom{X}{n-1}$.

As we are going to show in the next section, upon proving the Katona Theorem one can assume that $\mathcal{F}$ is shifted. Using this assumption Wang [150] found an amazingly short proof.

Proof of (i) and (ii) in Theorem 1.3 for shifted families. The proof is based on the following.

Claim 1.1. *If $\mathcal{F} \subset 2^{[n]}$ is shifted and t-intersecting ($t \geq 2$), then $\mathcal{F}(1)$ is $(t-1)$-intersecting and $\mathcal{F}(\bar{1})$ is $(t+1)$-intersecting.*

Proof. Only the case of $\mathcal{F}(\bar{1})$ needs a proof. Let $F, F \in \mathcal{F}$ with $F, F' \subset [2,n] = \{2,3,\ldots,n\}$. Let $j \in F \cap F'$. By shiftedness, $F'' := (F' \setminus \{j\}) \cup \{1\}$ is also in $\mathcal{F}$. Using the t-intersecting property $|F \cap F'| = |F \cap F''| + 1 \geq t+1$ follows. □

Now one can prove (3) and (4) for all t by applying induction on n. Since for $t=1$ both formulae give 2^{n-1}, we assume $t \geq 2$. Let $n+t=2a$. In view of the claim we have $|\mathcal{F}(1)| \leq \sum_{k \geq a} \binom{n-1}{k-1}$ and $|\mathcal{F}(\bar{1})| \leq \sum_{k \geq a} \binom{n-1}{k}$. Now (3) follows from $|\mathcal{F}| = |\mathcal{F}(1)| + |\mathcal{F}(\bar{1})|$. The case $n+t=2a+1$ is done in the exactly the same way. □

Katona's original proof used the very important notion of shadows.

Definition 1.3. For a family $\mathcal{F} \subset 2^{[n]}$ and $0 \leq i \leq n$ we define the *i-shadow* $\Delta_i(\mathcal{F})$ of $\mathcal{F}$ by

$$\Delta_i(\mathcal{F}) = \{G \in \binom{[n]}{i} : G \subset F \text{ for some } F \in \mathcal{F}\}.$$

For given positive integers m, k, i with $k > i$, determine $\min |\Delta_i(\mathcal{F})|$ over all k-uniform families $\mathcal{F}$ consisting of m members. This problem was solved by Kruskal [111] and Katona [100]. This is a very important result which has applications beyond combinatorics. We shall discuss it in the next section. Here we state a slightly weaker version due to Lovász [118]. For a real number $x > k-1$ we define $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. Note that this is a monotone increasing function. Therefore for every positive m and fixed k there is a unique $x > k-1$ with $\binom{x}{k} = m$.

Theorem 1.4 (Lovász [118])**.** *Let* $1 \le i < k \le n$ *and* $\mathcal{F} \subset \binom{[n]}{k}$ *with* $|\mathcal{F}| = \binom{x}{k}$. *Then*

$$|\Delta_i(\mathcal{F})| \ge \binom{x}{i}. \tag{5}$$

The following short proof of item (i) of the Erdős–Ko–Rado Theorem is due to Daykin [25].

Proof of (i) in Theorem 1.2. Let $n \ge 2k > 0$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting. Consider the family of complements $\mathcal{F}^c := \{[n] \setminus F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k}$. Since $F \cap F' \neq \emptyset$ is equivalent to $F \not\subset [n] \setminus F'$ it follows $|\mathcal{F}| + |\Delta_k(\mathcal{F}^c)| \le \binom{n}{k}$. Should $|\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$ hold, by Theorem 1.4, one would deduce $|\Delta_k(\mathcal{F}^c)| > \binom{n-1}{k}$ implying $|\mathcal{F}| + |\Delta_k(\mathcal{F})| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$, a contradiction. □

Let us mention that Katona [99] proved a different shadow theorem which also implies (i) of Theorem 1.2.

Theorem 1.5 (Katona Intersection Shadow Theorem [99])**.** *Let* $1 \le t \le k \le n$ *and let* $\mathcal{F} \subset \binom{X}{k}$ *be* t*-intersecting. Then, for* $k - t \le l < k$*, we have*

$$|\Delta_l(\mathcal{F})|/|\mathcal{F}| \ge \binom{2k-t}{l} \Big/ \binom{2k-t}{k}. \tag{6}$$

Note that $k - t \le l < k$ ensures that the RHS is at least 1, i.e., $|\Delta_l(\mathcal{F})| \ge |\mathcal{F}|$ in this range. In the next section we shall discuss some extensions and analogues of Theorem 1.5.

Let us now present two conjectures dealing with r-wise t-intersecting families.

Definition 1.4 (Frankl Families)**.** Let $n \ge t$ be positive integers.

(i) (non-uniform case) Define

$$\mathcal{F}(n,r,t,i) := \{F \subset [n] : |F \cap [t+ri]| \ge t + (r-1)i\},$$

where $0 \le i < \frac{n-t}{r}$.

(ii) (uniform case) Let k be an integer with $t \le k \le \frac{r-1}{r}n$. Define

$$\mathcal{F}^{(k)}(n,r,t,i) := \mathcal{F}(n,r,t,i) \cap \tbinom{[n]}{k},$$

where $0 \le i \le \lfloor \frac{k-t}{r-1} \rfloor$.

Note that both type of families are r-wise t-intersecting.

Conjecture 1.1 (Frankl [47])**.** *If* $\mathcal{F} \subset 2^{[n]}$ *is* r*-wise* t*-intersecting, then*

$$|\mathcal{F}| \le \max_i |\mathcal{F}(n,r,t,i)|.$$

Moreover, if $\mathcal{F} \subset \binom{[n]}{k}$ *and* $k \le \frac{r-1}{r}n$ *then*

$$|\mathcal{F}| \le \max_i |\mathcal{F}^{(k)}(n,r,t,i)|. \tag{7}$$

For the case $r = 2$ the above conjectures are known to be true, and here is a brief history. For the non-uniform case Katona solved it (Theorem 1.3). For the uniform case a simple computation shows that if $n \geq (t+1)(k-t+1)$ then $\max_i |\mathcal{F}^{(k)}(n,2,t,i)| = |\mathcal{F}^{(k)}(n,2,t,0)| = \binom{n-t}{k-t}$. So Theorem 1.2 confirms (7) for $n > n_0(k,t)$. Also the case $n = 4m, k = 2m$ of the conjecture already appeared in the EKR paper [44], and was popularized by Erdős for a number of years (e.g. [41]). Frankl [50] proved (7) (for $t \geq 15$) for the exact range of n, that is $n \geq (t+1)(k-t+1)$, by using shifting and counting the lattice paths corresponding subsets in shifted family. Then Wilson [152] gave a completely different proof for the same result (for all t) by studying the spectra of a graph on $\binom{[n]}{k}$ reflecting the t-intersecting property. The case for small n was more difficult. Frankl and Füredi [67] proved it for the cases $n > (k-t+1)c\sqrt{t/\log t}$, where c is some absolute constant. Then it was Ahlswede and Khachatrian who finally established (7) in general for $r = 2$. They gave two proofs [1, 3], and both of them are purely combinatorial and based on shifting and clever exchange operations. For their methods we recommend an excellent survey by Bey and Engel [10].

We say that a t-intersecting family $\mathcal{F} \subset 2^{[n]}$ is *non-trivial* if $|\bigcap \mathcal{F}| < t$, where $\bigcap \mathcal{F} := \bigcap_{F \in \mathcal{F}} F$. So $\mathcal{F}$ is non-trivial intersecting family iff it is intersecting and $\bigcap \mathcal{F} = \emptyset$.

Theorem 1.6 (Hilton–Milner [95]). *Let $k \geq 3$ and $n \geq 2k$. If $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Moreover, if $n > 2k$ then equality holds if and only if $k \geq 3$ and*

$$\mathcal{F} \cong \{F \in \binom{[n]}{k} : 1 \in F,\ F \cap [2, k+1] \neq \emptyset\} \cup [2, k+1],$$

or $k = 3$ and $\mathcal{F} \cong \{F \in \binom{[n]}{3} : |F \cap [3]| \geq 2\}$.

For the corresponding result for non-trivial t-intersecting families, see [2, 51].

Finally we list some classic text on the subject.

- L. Babai, P. Frankl [8]: Linear Algebra Methods in Combinatorics, Preliminary Version 2. *Dept. of Comp. Sci., The univ. of Chicago*, 1992.
- B. Bollobás [12]: Combinatorics. Cambridge University Press.
- S. Jukna [97]: Extremal Combinatorics.
- C. Godsil, K. Meagher [88]: Erdős–Ko–Rado Theorems: Algebraic Approaches. Cambridge University Press, Cambridge, 2015.

We will not cover a topic on stability or supersaturation in this survey, see e.g., [24, 34].

2. Shadows and shifting

Recall that for $k > l \geq 0$ the l-shadow of a k-uniform family $\mathcal{F}$ is

$$\Delta_l(\mathcal{F}) = \{G : |G| = l,\ G \subset F \text{ holds for some } F \in \mathcal{F}\}.$$

Given positive integers m and k, the Kruskal–Katona Theorem determines the minimum of $|\Delta_l(\mathcal{F})|$ over all k-uniform families $\mathcal{F}$ with $|\mathcal{F}| = m$. The answer is 'take the first m sets of size k in colex order,'

where we define $A <_{\text{colex}} B$ iff $\max\{a \in A \setminus B\} < \max\{b \in B \setminus A\}$. For fixed k and each particular value of m one can use a simple algorithm to obtain the so-called cascade form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}$$

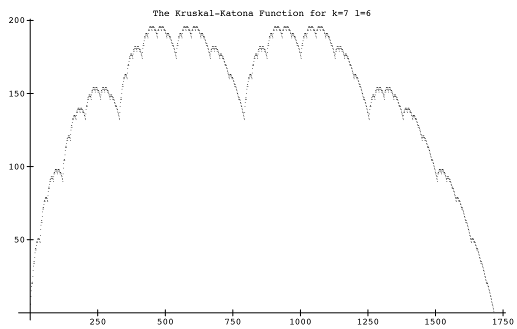
with

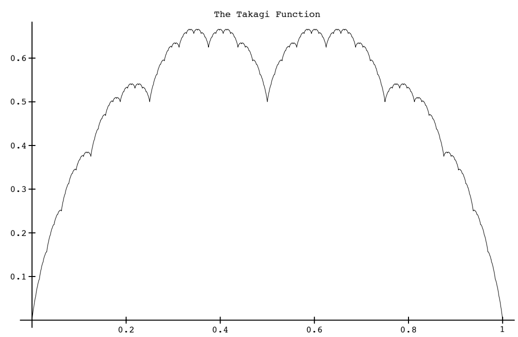
$$a_k > a_{k-1} > \cdots > a_t \geq t > 0.$$

Then the minimum of $|\Delta_l(\mathcal{F})|$ is

$$\binom{a_k}{l} + \binom{a_{k-1}}{l-1} + \cdots + \binom{a_t}{l-k+t},$$

where we define $\binom{a}{b} = 0$ for $a < b$. That is, the theorem does not provide an easily computable formula for the minimum size of the shadow. The actual function, to say the least, is not very smooth. As a matter of a fact it was shown in [71] that after some normalization the corresponding function (the left picture) converges uniformly to the Takagi function (the right picture), a continuous but nowhere differentiable function with self-similarity.





For many of the numerous application of the Kruskal–Katona Theorem we can bypass these tedious computations by using the version due to Lovász [118], Theorem 1.4. The functions $\binom{x}{k}$ and $\binom{x}{l}$ are differentiable and strictly monotone. Due to this second property it is sufficient to prove (5) for $l = k-1$ and apply it successively for $(k-1, k-2), \ldots, (l+1, l)$.

Let us present here a concise, clever proof of Theorem 1.4 due to Keevash [102] which was inspired by the proof of Lovász [118]. For this we recall some graph theoretic notions. It is sometimes convenient to view a family $\mathcal{F} \subset 2^{[n]}$ as a *hypergraph.* This case, an element of $[n]$ is called a *vertex*, and a member of the family is called a *hyperedge* (or simply an *edge*). For $\mathcal{F} \subset 2^{[n]}$ and a vertex $i \in [n]$ let $d_{\mathcal{F}}(i)$ denote the *degree* of i, that is, the number of edges (in $\mathcal{F}$) containing i. For $\mathcal{F} \subset \binom{[n]}{k}$ we say that $G \in \binom{[n]}{k+1}$ is a $(k+1)$-*clique* (or simply *clique*) of $\mathcal{F}$ if $\binom{G}{k} \subset \mathcal{F}$, and let $\mathcal{C}_{k+1}(\mathcal{F}) \subset \binom{[n]}{k+1}$ denote the set of cliques in $\mathcal{F}$. By definition it follows that

$$\mathcal{F} \subset \mathcal{C}_k(\Delta_{k-1}(\mathcal{F})). \tag{8}$$

For example, if $\mathcal{F} = \{\{1,2,4\},\{2,3,5\},\{1,3,6\}\}$, then $\mathcal{C}_3(\Delta_2(\mathcal{F})) = \mathcal{F} \cup \{\{1,2,3\}\}$.

Keevash observed the following, which implies Theorem 1.4 immediately.

Theorem 2.1 (Keevash [102]). *Let n, k be integers with $n > k \geq 1$, and let $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| = \binom{x}{k}$ for some real $x \geq k$. Then $|\mathcal{C}_{k+1}(\mathcal{F})| \leq \binom{x}{k+1}$.*

Proof of Theorem 1.4. It is sufficient to show the case $i = k-1$ (then applying this $k-i$ times and use the monotonicity of $\binom{x}{j}$). Let $|\Delta_{k-1}(\mathcal{F})| = \binom{y}{k-1}$. By (8) and Theorem 2.1 we have $\binom{x}{k} = |\mathcal{F}| \leq |\mathcal{C}_k(\Delta_{k-1}(\mathcal{F}))| \leq \binom{y}{k}$, and $x \leq y$. Thus $|\Delta_{k-1}(\mathcal{F})| = \binom{y}{k-1} \geq \binom{x}{k-1}$. □

For $\mathcal{F} \subset \binom{[n]}{k}$ and $i \in [n]$ let

$$\mathcal{C}_k(\mathcal{F}, i) := \{G \setminus \{i\} : i \in G \in \mathcal{C}_{k+1}(\mathcal{F})\}.$$

Observe that $F \in \mathcal{C}_k(\mathcal{F}, i)$ if and only if $F \in \mathcal{F}$ and $F \cup \{i\} \in \mathcal{C}_{k+1}(\mathcal{F})$. So if $i \in F' \in \mathcal{F}$ then $F' \notin \mathcal{C}_k(\mathcal{F}, i)$. Thus we have

$$|\mathcal{C}_k(\mathcal{F}, i)| \leq |\mathcal{F}| - d_{\mathcal{F}}(i), \tag{9}$$

$$\mathcal{C}_k(\mathcal{F}, i) \subset \mathcal{C}_k(\mathcal{F}(i)), \tag{10}$$

where $\mathcal{F}(i) := \{F \setminus \{i\} : i \in F \in \mathcal{F}\}$. (Using this notation we can write $\mathcal{C}_k(\mathcal{F}, i) = (\mathcal{C}_{k+1}(\mathcal{F}))(i)$.)

Proof of Theorem 2.1. We apply induction on k. The case $k = 1$ is clear, so let $k \geq 2$. We claim that

$$|\mathcal{C}_k(\mathcal{F}, i)| \leq \left(\frac{x}{k} - 1\right) d_{\mathcal{F}}(i) \text{ for all } i \in [n]. \tag{11}$$

Fix $i \in [n]$. First suppose that $d_{\mathcal{F}}(i) \geq \binom{x-1}{k-1}$. Then

$$|\mathcal{F}| = \binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1} \leq \frac{x}{k} d_{\mathcal{F}}(i),$$

and (11) follows from (9). Next suppose that $d_{\mathcal{F}}(i) = \binom{y-1}{k-1} \leq \binom{x-1}{k-1}$ for some $y \leq x$. Then, by induction hypothesis, we have $|\mathcal{C}_k(\mathcal{F}(i))| \leq \binom{y-1}{k}$. This together with (10) gives us that

$$|\mathcal{C}_k(\mathcal{F}, i)| \leq \binom{y-1}{k} = \left(\frac{y}{k} - 1\right)\binom{y-1}{k-1} \leq \left(\frac{x}{k} - 1\right)\binom{y-1}{k-1} = \left(\frac{x}{k} - 1\right) d_{\mathcal{F}}(i),$$

proving (11). Finally we have

$$|\mathcal{C}_{k+1}(\mathcal{F})| = \frac{1}{k+1}\sum_{i \in [n]} |\mathcal{C}_k(\mathcal{F}, i)| \leq \frac{1}{k+1}\sum_{i \in [n]} \left(\frac{x}{k} - 1\right) d_{\mathcal{F}}(i).$$

Noting that $\sum_{i \in [n]} d_{\mathcal{F}}(i) = k\binom{x}{k}$, we get $|\mathcal{C}_{k+1}(\mathcal{F})| \leq \frac{1}{k+1}\left(\frac{x}{k} - 1\right) k\binom{x}{k} = \binom{x}{k+1}$. □

One of the advantages of this 'shifting-free' proof is that it also works in vector spaces. In fact Chowdhury and Patkós [23] obtained a vector space version of Theorems 1.4 and 2.1 along this line. We mention that shifting is a very strong proof technique to deal with families of subsets, but a vector space version of shifting has not been found yet.

As n tends to infinity the ration $\binom{x}{l}/\binom{x}{k}$ tends to zero. However, if $\mathcal{F}$ is t-intersecting then

$$|\Delta_l(\mathcal{F})| \geq |\mathcal{F}| \text{ for } k-t \leq l < k \tag{12}$$

holds. This was proved by Katona [99] in a stronger form, Theorem 1.5. Looking at the t-intersecting family $\binom{[2k-t]}{k}$ shows that the inequality (6) is best possible.

Let us use (12) to give a short proof of the Erdős–Ko–Rado Theorem *without computation.* Recall the notation

$$\mathcal{F}(1) = \{F \setminus \{1\} : 1 \in F \in \mathcal{F}\},$$
$$\mathcal{F}(\bar{1}) = \{F \in \mathcal{F} : 1 \notin F\},$$

and the identity $|\mathcal{F}| = |\mathcal{F}(1)| + |\mathcal{F}(\bar{1})|$. Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family, $n \geq 2k$. We need to prove

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{13}$$

Proof. This proof is taken from [68]. Without computation means that we are not even proving (13) as a formula but in the conceptually simpler form:

$$\text{number of members of } |\mathcal{F}| \leq \text{ number of sets in } \binom{[2,n]}{k-1}. \tag{14}$$

First take away from both sides the members corresponding to $\mathcal{F}(1)$. On the LHS remain $F \in \mathcal{F}$ with $1 \notin F$, i.e., $\mathcal{F}(\bar{1})$. Set $\mathcal{G} = \{[2,n] \setminus F : F \in \mathcal{F}(\bar{1})\}$.

Claim 2.1. (i) $\mathcal{G} \subset \binom{[2,n]}{n-1-k}$ *is* $(n-2k)$*-intersecting, and* $|\Delta_{k-1}(\mathcal{G})| \geq |\mathcal{F}(\bar{1})|$.
(ii) $\Delta_{k-1}(\mathcal{G}) \cap \mathcal{F}(1) = \emptyset$.

Before proving the claim let us note that by the claim the number of the sets remaining on the RHS is at least as much as those on the LHS, therefore concluding the proof of the RHS of (14). Now we prove the claim.

(i) Let $F, F' \in \mathcal{F}(\bar{1})$, and let $G = [2,n] \setminus F$ and $G' = [2,n] \setminus F'$. Since $|F \cap F'| \geq 1$ it follows that $|G \cup G'| \leq n-2$ and

$$|G \cap G'| = |G| + |G'| - |G \cup G'| \geq 2(n-1-k) - (n-2) = n-2k.$$

Applying (6) to $\mathcal{G}$ yields (i).

(ii) It is a restatement of $F \cap F' \neq \emptyset$ for $1 \notin F \in \mathcal{F}$ and $1 \in F' \in \mathcal{F}$. This completes the proof of (13). □

Katona proved (6) using the shifting technique. Let $1 \leq i < j \leq n$. We define the *shifting* operator s_{ij} on $[n]$ and also on $2^{[n]}$ as follows. For $F \subset [n]$ let

$$s_{ij}(F) := \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } F \cap \{i,j\} = \{j\} \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Then, for $\mathcal{F} \subset 2^{[n]}$, let

$$s_{ij}(\mathcal{F}) := \{s_{ij}(F) : F \in \mathcal{F}\}.$$

A family $\mathcal{F}$ is called *shifted* if $s_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$.

This is a formal definition, but now we redefine $s_{ij}(\mathcal{F})$ in an intuitive way. Every family in $2^{[n]}$ is uniquely determined by the four families

$$\mathcal{F}(i,j), \mathcal{F}(i,\bar{j}), \mathcal{F}(\bar{i},j), \mathcal{F}(\bar{i},\bar{j}),$$

where, e.g., $\mathcal{F}(i,\bar{j}) = \{F \setminus \{i,j\} : F \cap \{i,j\} = \{i\}\}$. Now $s_{ij}(\mathcal{F})$ is the unique family $\mathcal{G}$ satisfying

$$\begin{aligned}
\mathcal{G}(i,j) &= \mathcal{F}(i,j),\\
\mathcal{G}(\bar{i},\bar{j}) &= \mathcal{F}(\bar{i},\bar{j}),\\
\mathcal{G}(i,\bar{j}) &= \mathcal{F}(i,\bar{j}) \cup \mathcal{F}(\bar{i},j),\\
\mathcal{G}(\bar{i},j) &= \mathcal{F}(i,\bar{j}) \cap \mathcal{F}(\bar{i},j).
\end{aligned}$$

With this in mind it is easy to prove the following.

Lemma 2.1. (i) *If $\mathcal{F}$ is r-wise t-intersecting then $s_{ij}(\mathcal{F})$ is r-wise t-intersecting.*
(ii) $\Delta_l(s_{ij}(\mathcal{F})) \subset s_{ij}(\Delta_l(\mathcal{F}))$.

In view of (ii), in proving lower bounds for $\Delta_l(\mathcal{F})$ we can assume that $\mathcal{F}$ is shifted.

Lemma 2.2 (Frankl [52])**.** *If $\mathcal{F}$ is shifted k-uniform with $|\mathcal{F}| = \binom{x}{k}$, $x \geq k$, then*

$$|\mathcal{F}(\bar{1})| \leq \binom{x-1}{k}. \tag{15}$$

Proof. We prove the statement simultaneously with Lovász version of the Kruskal–Katona Theorem, i.e.,

$$|\Delta_{k-1}(\mathcal{F})| \geq \binom{x}{k-1}. \tag{16}$$

Without loss of generality $\mathcal{F}$ is shifted. Apply induction on $\lfloor x \rfloor$. Note that $|\mathcal{F}(\bar{1})| > 0$ implies $\{2, 3, \ldots, k+1\} \in \mathcal{F}$. Whence by shiftedness $\binom{[k+1]}{k} \subset \mathcal{F}$, forcing $x \geq k+1$. The inequality (15) is true for $\lfloor x \rfloor = k$, which is our base case. (16) is checked in the same way.

Now the induction step. Suppose for contradiction that $|\mathcal{F}(\bar{1})| = \binom{y}{k}$, $y > x-1$. As $\lfloor x \rfloor \geq k+1$, $\lfloor y \rfloor \geq k$ and using the induction hypothesis

$$|\Delta_{k-1}(\mathcal{F}(\bar{1}))| \geq \binom{y}{k-1} > \binom{x-1}{k-1}.$$

However, by shiftedness $\Delta_{k-1}(\mathcal{F}(\bar{1})) \subset \mathcal{F}(1)$ yielding $|\mathcal{F}(1)| > \binom{x-1}{k-1}$. Consequently

$$\binom{x}{k} = |\mathcal{F}| = |\mathcal{F}(\bar{1})| + |\mathcal{F}(1)| > \binom{x-1}{k} + \binom{x-1}{k-1} = \binom{x}{k},$$

a contradiction. Now $|\mathcal{F}(1)| = |\mathcal{F}| - |\mathcal{F}(\bar{1})| \geq \binom{x}{k} - \binom{x-1}{k} = \binom{x-1}{k-1}$ follows from (15). Then (16) is a consequence of $|\Delta_{k-1}(\mathcal{F})| \geq |\mathcal{F}(1)| + |\Delta_{k-2}(\mathcal{F}(1))|$ and the induction hypothesis. □

It is worth noting that the main reason that the whole proof works and that the full Kruskal–Katona Theorem can be proved by this approach is the following obvious fact.

Proposition 2.1 (Frankl [57]). *If $\mathcal{F} \subset \binom{n}{k}$ is shifted then $\Delta_{k-1}(\mathcal{F}) = \Delta_{k-1}(\mathcal{F} \setminus \mathcal{F}(\bar{1}))$.*

In [57] the same approach is used to prove strong bounds on the size of complexes. Here we recall the definition. We say that a family $\mathcal{C} \subset 2^{[n]}$ (resp. $\mathcal{F} \subset 2^{[n]}$) is called a complex (resp. filter) if it satisfies $C' \subset C \in \mathcal{C}$ implies $C' \in \mathcal{C}$ (resp. $\mathcal{F} \ni F \subset F'$ implies $F' \in \mathcal{F}$). For a complex $\mathcal{C}$ define

$$\mathcal{A}(\mathcal{C}) = \{A \in \mathcal{C} : \nexists B \in \mathcal{C},\ A \subset B, A \neq B\}.$$

Clearly, $\mathcal{A}(\mathcal{C})$ is an antichain and it determines $\mathcal{C}$:

$$\mathcal{C} = \{C \subset [n] : \exists A \in \mathcal{A}(\mathcal{C}),\ C \subset A\}.$$

Set $I(\mathcal{C}) = \mathcal{C} \setminus \mathcal{A}(\mathcal{C})$. We call it the *interior* of $\mathcal{C}$.

Theorem 2.2 (Frankl [57]). *Suppose that $\mathcal{C} \subset 2^{[n]}$ is a complex of size at least $\binom{n}{0} + \cdots + \binom{n}{k-1} + \binom{x}{k}$ for some $1 \leq k \leq x \leq n$, then $|I(\mathcal{C})| \geq \binom{n}{0} + \cdots + \binom{n}{k-2} + \binom{x}{k-1}$.*

For an filter $\mathcal{F} \subset 2^{[n]}$ define its *exterior* $\mathcal{E}(\mathcal{F})$ by

$$\mathcal{E}(\mathcal{F}) = \{E \subset [n] : \exists F \in \mathcal{F},\ E \subset F,\ |F \setminus E| = 1\}.$$

Theorem 2.3 (Frankl [57]). *Suppose that $\mathcal{F} \subset 2^{[n]}$ is an filter and that $|\mathcal{F}| = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{x}{k}$ with $1 \leq k \leq x \leq n$, then $|\mathcal{E}(\mathcal{F})| \geq \binom{n}{n-1} + \cdots + \binom{n}{k} + \binom{x}{k-1}$.*

The above two theorems imply important, classical results of Kleitman [108] and Harper [94], which incidentally both appeared in the first volume of Journal of Combinatorial Theory, 50 years ago!

Theorem 2.4 (Kleitman). *Let $\mathcal{C} \subset 2^{[n]}$ be a complex with $|\mathcal{A}(\mathcal{C})| \geq \binom{n}{k}$ for some integer $1 \leq k \leq n/2$, then $|\mathcal{C}| \geq \binom{n}{0} + \cdots + \binom{n}{k}$.*

Note that the original proof was incomplete. In [90] a full proof due to A. M. Odlyzko is reproduced.

For a family $\mathcal{F} \subset 2^{[n]}$ define its *full boundary* $\sigma(\mathcal{F})$ by $\sigma(\mathcal{F}) = \{G \subset [n] : \exists F \in \mathcal{F},\ |F \triangle G| \leq 1\}$.

Theorem 2.5 (Harper (handy version) [94]). *If $|\mathcal{F}| = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{x}{k}$ where $n \geq x \geq k \geq 1$, then $|\sigma(\mathcal{F})| \geq \binom{n}{n} + \cdots + \binom{n}{k} + \binom{x}{k-1}$.*

Note that in Harper's theorem $\mathcal{F}$ is not required to be a complex or filter, but that can be taken care of via 'down shifting,' which we will not cover in this survey.

The following result gives a lower bound for the size of shadows in a family with independence number restrictions.

Theorem 2.6 (Frankl [59]). *If $\mathcal{F} \subset \binom{[n]}{k}$ contains no $s+1$ pairwise disjoint sets, then $|\Delta_{k-1}(\mathcal{F})| \geq |\mathcal{F}|/s$.*

We mention that there are some similar results in [59].

We say that two families $\mathcal{A}$ and $\mathcal{B}$ are cross intersecting if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We can use the following inequality concerning cross intersecting families to prove the Hilton–Milner Theorem.

Lemma 2.3 ([77])**.** *If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are non-empty cross intersecting families with $n \geq a+b$, $a \leq b$, then it follows that $|\mathcal{A}|+|\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1$.*

More generally, Wang and Zhang obtained the following inequality.

Theorem 2.7 (Wang–Zhang [151])**.** *Let n, a, b, t be positive integers with $n \geq 4$, $a, b \geq 2$, $t < \min\{a, b\}$, $a+b < n+t$, $(n,t) \neq (a+b, 1)$, and $\binom{n}{a} \leq \binom{n}{b}$. If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross t-intersecting, that is, $|A \cap B| \geq t$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|\mathcal{A}|+|\mathcal{B}| \leq \binom{n}{b} - \sum_{i=0}^{t-1}\binom{a}{i}\binom{n-a}{b-i} + 1$.*

Proof of the inequality in Theorem 1.6. Suppose that $|\mathcal{F}|$ is maximal with respect to the conditions. The covering number $\tau(\mathcal{F})$ is defined by the minimal integer t such that there exists a t-element set T satisfying $T \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. First we deal with an important special case $\tau(\mathcal{F}) \leq 2$, that is, there is an $A := \{a, b\} \in \binom{[n]}{2}$ such that $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. It follows from the maximality of $|\mathcal{F}|$ that $\{G \in \binom{[n]}{k} : A \subset G\} \subset \mathcal{F}$. Define

$$\mathcal{A} := \{F \setminus \{a\} : F \in \mathcal{F},\ F \cap A = \{a\}\},$$
$$\mathcal{B} := \{F \setminus \{b\} : F \in \mathcal{F},\ F \cap A = \{b\}\}.$$

Then $\mathcal{A}, \mathcal{B}$ are cross-intersecting families on $[n] \setminus A$, and Lemma 2.3 yields $|\mathcal{A}|+|\mathcal{B}| \leq \binom{n-2}{k-1} - \binom{(n-2)-(k-1)}{k-1} + 1$. Thus, $|\mathcal{F}| \leq \binom{n-2}{k-1} - \binom{n-k-1}{k-1} + 1 + \binom{n-2}{k-2} = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, as desired.

Next consider the case when $\mathcal{F}$ is shifted and $\tau(\mathcal{F}) \geq 3$. Let $Y := [n] \setminus \{1, 2\}$ and define

$$\mathcal{A} := \{F \setminus \{1\} : F \in \mathcal{F},\ F \cap \{1,2\} = \{1\}\} \subset \tbinom{Y}{k-1},$$
$$\mathcal{B} := \{F \setminus \{2\} : F \in \mathcal{F},\ F \cap \{1,2\} = \{2\}\} \subset \tbinom{Y}{k-1},$$
$$\mathcal{C} := \{F \setminus \{1,2\} : F \in \mathcal{F},\ \{1,2\} \subset F\} \subset \tbinom{Y}{k-2},$$
$$\mathcal{D} := \{F \in \mathcal{F} : F \cap \{1,2\} = \emptyset\} \subset \tbinom{Y}{k}.$$

Since $\tau(\mathcal{F}) \geq 3$ and $\mathcal{F}$ is shifted we have $\{2, 3, \ldots, k+1\} \in \mathcal{F}$ and $\{3, 4, \ldots, k+1\} \in \mathcal{A} \cap \mathcal{B}$. Thus $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross intersecting families, and we can apply Lemma 2.3 to get $|\mathcal{A}|+|\mathcal{B}| \leq \binom{n-2}{k-1} - \binom{n-k-1}{k-1} + 1$.

On the other hand, $\mathcal{C}, \mathcal{D}$ are cross-intersecting and $\mathcal{D}$ is 2-intersecting. (To see the latter, suppose to the contrary that there are $D_1, D_2 \in \mathcal{D}$ such that $D_1 \cap D_2 = \{x\}$. Then, by the shiftedness, it follows that both $D_1' := (D_1 \setminus \{x\}) \cup \{1\}$, and $D_2' := (D_2 \setminus \{x\}) \cup \{2\}$ are in $\mathcal{F}$, but $D_1' \cap D_2' = \emptyset$, a contradiction.) Let $\mathcal{D}^c := \{[n] \setminus D : D \in \mathcal{D}\} \subset \binom{Y}{(n-2)-k}$ and $\mathcal{S} := \Delta_{k-2}(\mathcal{D}^c) \subset \binom{Y}{k-2}$. Then by the cross intersecting property of $\mathcal{C}$ and $\mathcal{D}$ we have $\mathcal{C} \cap \mathcal{S} = \emptyset$. Since $\mathcal{D}$ is 2-intersecting, $\mathcal{D}^c$ is $(n-2)-(2k-2) = (n-2k)$-intersecting. Thus Theorem 1.5 on shadows in intersecting families implies that $|\mathcal{S}| \geq |\mathcal{D}^c| = |\mathcal{D}|$. Therefore, $|\mathcal{C}|+|\mathcal{D}| \leq |\mathcal{C}|+|\mathcal{S}| \leq |\binom{Y}{k-2}| = \binom{n-2}{k-2}$. Again, we obtain $|\mathcal{F}| = |\mathcal{A}|+|\mathcal{B}|+|\mathcal{C}|+|\mathcal{D}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

Now to the general case. Apply shifting operations s_{ij} repeatedly to $\mathcal{F}$. Either we obtain a shifted non-trivial intersecting family of the same size (and we are done by the first or second case according to the covering number) or at some point the family stops to be non-trivial. That is for some non-trivial intersecting $\mathcal{G} \subset \binom{[n]}{k}$ with $|\mathcal{F}| = |\mathcal{G}|$ we have that $\bigcap_{H \in s_{ij}(\mathcal{G})} H \neq \emptyset$. In this case clearly $\{i\} = \bigcap_{H \in s_{ij}(\mathcal{G})} H$ and consequently $\{i, j\} \cap G \neq \emptyset$ for all $G \in \mathcal{G}$. Thus we can apply the first special case to $\mathcal{G}$ and we are done. □

3. Independence number and the Erdős matching conjecture

For a family $\mathcal{F} \subset 2^{[n]}$ with $\emptyset \notin \mathcal{F}$ let $\alpha(\mathcal{F})$ denote its *independence number*, i.e., the maximum number s such that there exist pairwise disjoint sets $F_1, \ldots, F_s \in \mathcal{F}$. One of the classical results of extremal set theory is the following.

Theorem 3.1 (Kleitman [109]). *Let $\mathcal{F} \subset 2^{[n]}$ satisfy $\alpha(\mathcal{F}) \leq s - 1$.*

(i) *If $n = sq - 1$ for some positive integer q then*

$$|\mathcal{F}| \leq \sum_{q \leq i \leq n} \binom{n}{i}. \tag{17}$$

(ii) *If $n = sq$ ($q \geq 1$ integer) then*

$$|\mathcal{F}| \leq 2 \sum_{q \leq i \leq n-1} \binom{n-1}{i} = \binom{n-1}{q} + \sum_{q < i \leq n} \binom{n}{i}. \tag{18}$$

Noting that from a family $\mathcal{G} \subset 2^{[n-1]}$ with $\alpha(\mathcal{G}) = s - 1$ one can construct $\mathcal{F} := \{F \subset [n] : F \cap [n-1] \in \mathcal{G}\}$ satisfying $\alpha(\mathcal{F}) = s - 1$ and $|\mathcal{F}| = 2|\mathcal{G}|$, one can see that (18) implies (17). For the cases $n \not\equiv 0$ or $-1 \pmod{s}$ the methods of [109] do not provide the exact answer. As a matter of fact the exact answer is unknown except for $s = 3$ (cf. Quinn [129]).

Definition 3.1. Let us call $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ *cross-dependent* if there are no $F_i \in \mathcal{F}_i$, $i = 1, \ldots, s$ which are pairwise disjoint.

Example 3.1. Let $n = qs + p$ with $q \geq 1$, $0 \leq p < s$. Define $\mathcal{H}_i = \{F \subset [n] : |F| \geq q + 1\}$ for $1 \leq i \leq p + 1$ and $\mathcal{H}_j = \{F \subset [n] : |F| \geq q\}$ for $p + 1 < j \leq s$. It is easy to see that these families are cross-dependent.

Theorem 3.2 (Frankl–Kupavskii [62]). *Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ are cross-dependent, $n = qs + p$ with $q \geq 1$, $0 \leq p < s$ then*

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leq \sum_{i=1}^{s} |\mathcal{H}_i| = (s - p - 1)\binom{n}{q} + s \sum_{l > q} \binom{n}{l}. \tag{19}$$

Note that by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_s = \mathcal{F}$ (19) implies (17) and (18).

There is an attractive open problem related to the independence number.

Conjecture 3.1 (Erdős–Kleitman [43]). *Suppose that $\mathcal{F} \subset 2^{[n]}$ satisfies $\alpha(\mathcal{F}) = s$ but for all $E \in 2^{[n]} \setminus \mathcal{F}$, $\alpha(\mathcal{F} \cup \{E\}) \geq s + 1$, then*

$$|\mathcal{F}| \geq 2^n - 2^{n-s} = 2^n(1 - 2^{-s}).$$

The case $s = 1$ was already proved in [44]. However, for $s \geq 2$ no non-trivial lower bound is known. Even to prove $|\mathcal{F}| \geq (\frac{1}{2} + \epsilon)2^n$ is a challenging open problem.

Let us consider now the uniform case.

Example 3.2. For $n \geq k(s+1)$ and $1 \leq i \leq k$ define

$$\mathcal{A}_i(n,k) = \{A \in \binom{[n]}{k} : |A \cap [i(s+1)-1]| \geq i\}.$$

Then $\alpha(\mathcal{A}_i(n,k)) = s$ is easy to verify. With rather tedious computation the first author verified that for every n, k with $n \geq sk$, the maximum of $|\mathcal{A}_i(n,k)|$ is either $|\mathcal{A}_1(n,k)|$ or $|\mathcal{A}_k(n,k)|$. However, no proof of it was ever published.

Conjecture 3.2 (Erdős matching conjecture [38]). *Suppose that $\mathcal{A} \subset \binom{[n]}{k}$ satisfies $\alpha(\mathcal{A}) = s$ where $s \geq 1$, $n \geq k(s+1)$. Then*

$$|\mathcal{A}| \leq \max\{|\mathcal{A}_1(n,k)|, |\mathcal{A}_k(n,k)|\}. \tag{20}$$

For $k = 2$, (20) is an old result of Erdős and Gallai [42]. Erdős [38] proved (20) for $n > n_0(k,s)$. The bounds for $n_0(k)$ were subsequently improved by Bollobás, Daykin and Erdős [13] to $2k^3 s$, by Huang, Loh, and Sudakov [96] to $3k^2 s$, by Frankl, Łuczak, Mieczkowska [69] to $3k^2 s/(2\log k)$. Let us note that for $s = 1$ the condition reduces to $\mathcal{A}$ being intersecting.

As mentioned in section 2, shifting maintains the property $\alpha(\mathcal{A}) \leq s$. Therefore it is sufficient to deal with shifted families. One can use this to prove the following general bound.

Proposition 3.1 (Frankl [56]). *Suppose that $\mathcal{A} \subset \binom{[n]}{k}$ satisfies $\alpha(\mathcal{A}) = s$, $n \geq k(s+1)$, then*

$$|\mathcal{A}| \leq s\binom{n-1}{k-1}. \tag{21}$$

Note that for $n > n_0(k,s)$ one has $|\mathcal{A}_1(k,s)| = (s - o(1))\binom{n-1}{k-1}$, i.e., (21) is *only* slightly worse than (20).

Proof. Let $s \geq 1$ be fixed. Apply induction on n (simultaneously for all k). The case $k = 1$ is trivial and the case $n = k(s+1)$ follows from (18). (To see the latter, let $\mathcal{G} := \mathcal{A} \cup \bigcup_{i>k} \binom{[n]}{i}$. Then, $\tau(\mathcal{G}) = s$, and $|\mathcal{G}| = |\mathcal{A}| + \sum_{k<i\leq n} \binom{n}{i}$. So, by (18), we have that $|\mathcal{A}| \leq \binom{n-1}{k} = \binom{ks+k-1}{k} = \frac{ks}{k}\binom{ks+k-1}{k-1} = s\binom{n-1}{k-1}$.) Without loss of generality let $\mathcal{A}$ be shifted. Then both $\mathcal{A}(\bar{n})$ and $\mathcal{A}(n)$ satisfy $\alpha(\mathcal{A}(n)) = s$, $\alpha(\mathcal{A}(\bar{n})) = s$. By the induction hypothesis $|\mathcal{A}(\bar{n})| \leq s\binom{n-1}{k-1}$, $|\mathcal{A}(n)| \leq s\binom{n-1}{k-2}$. Now (21) follows from $|\mathcal{A}| = |\mathcal{A}(n)| + |\mathcal{A}(\bar{n})|$. □

The current record on $n_0(k,s)$ is due to the first author, and is slightly less than $2ks$. In particular, it is less than double $k(s+1)$, the first case for which the question arises. However, in all these cases $\mathcal{A}_1(n,k)$ is the optimal family. On the other hand for $n = k(s+1)$ it follows from Kleitman's Theorem case (ii) that (20) is true with $\mathcal{A}_k(n,k)$ providing the maximum. Very recently the first author showed (20) for a narrow range.

Theorem 3.3 (Frankl [61]). *Let $\mathcal{F} \subset \binom{[n]}{k}$, $\alpha(\mathcal{F}) = s$. For every k there exists a positive $\epsilon = \epsilon(k)$ such that $|\mathcal{F}| \leq |\mathcal{A}_k(n,k)| = \binom{k(s+1)-1}{k}$ holds for $k(s+1) \leq n < k(s+1) + \epsilon s$.*

Łuczak, Mieczkowska [119] proved (20) in the case $k = 3$ and s very large. In [60] (20) is proved for $k = 3$ and all s.

4. Conditions on the size of the union

Definition 4.1. For positive integers r, t, $r \geq 2$, a family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-union if $|F_1 \cup \cdots \cup F_r| \leq n - t$ for all $\mathcal{F}_1, \ldots, \mathcal{F}_r \in \mathcal{F}$.

Note that $\mathcal{F}$ is r-wise t-union if and only if the family of complements $\mathcal{F}^c = \{[n] \setminus F : F \in \mathcal{F}\}$ is r-wise t-intersecting. There are two reasons that we use the union terminology in this section. The first is that for most of the examples it is more direct to verify the union condition. The second is that there are some beautiful theorems and conjectures that were made and are more natural for the union terminology. Let us start with such a result due to Brace and Daykin.

Definition 4.2. For $n - t \geq ri$ let us define the families

$$\mathcal{D}_i(n, r, t) = \{D \subset [n] : |D \cap [t + ri]| \leq i\}.$$

In memory of David E. Daykin we call them the Daykin families. Since the union of any r members restricted to $[t + ri]$ has size at most ri, $\mathcal{D}_i(n, r, t)$ has the r-wise t-union property.

Theorem 4.1 (Brace–Daykin [18])**.** *Suppose that $\mathcal{F} \subset 2^{[n]}$ has the r-wise 1-union property. If $\bigcup_{F \in \mathcal{F}} = [n]$ then*

$$|\mathcal{F}| \leq |\mathcal{D}_1(n, r, 1)|. \tag{22}$$

Note that for $r = 2$, (22) gives $|\mathcal{F}| \leq 2^{n-1}$, which was already pointed out by Erdős, Ko and Rado. However, for $r \geq 3$ one has $|\mathcal{D}_1(n, r, 1)| = \frac{r+2}{2^{r+1}} 2^n$ and already for $r = 3$, $\frac{5}{16}$ is much smaller than $\frac{1}{2}$. That is, the Brace–Daykin Theorem is a strong stability result showing that if we exclude the trivial construction then the maximum size of an r-wise 1-union family drops considerably.

Note that if $\mathcal{F}$ is r-wise t-union then the complex generated by $\mathcal{F}$, i.e., $\{G : \exists F \in \mathcal{F}, G \subset F\}$ has the same property. For this reason from now on we always assume that $\mathcal{F}$ is a complex. Then $\bigcup_{F \in \mathcal{F}} F = [n]$ is equivalent to saying that $\binom{[n]}{1} \subset \mathcal{F}$.

Conjecture 4.1 (Frankl [58])**.** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is r-wise t-union, $r \geq 3$ and $\binom{[n]}{1} \subset \mathcal{F}$. Then*

$$|\mathcal{F}| \leq |\mathcal{D}_1(n, r, t)| \text{ for } t < 2^r - r - 1. \tag{23}$$

Note that for $t \geq 2^r - r - 1$ one has $|\mathcal{D}_1(n, r, t)| \geq |\mathcal{D}_0(n, r, t)| = 2^{n-t}$, i.e., (23) ceases to be a stability result. Moreover, for $t \geq 2^{r+1} - 3r$ one has already $|\mathcal{D}_1(n, r, t)| < |\mathcal{D}_2(n, r, t)|$, i.e., (23) is no longer true.

Since the property $\binom{[n]}{l} \subset \mathcal{F}$ is invariant under shifting for all l, upon proving (22) or (23) one can always assume that $\mathcal{F}$ is shifted. Throughout this section, unless otherwise stated we shall always assume that $\mathcal{F}$ is shifted.

Let $\delta(\mathcal{F}) = \min_{i \in [n]} |\mathcal{F}(i)|$ be the *minimum degree* of the family $\mathcal{F}$. Note that $\delta(\mathcal{D}_1(n, r, 1)) = 2^{n-r-1}$. There is a beautiful conjecture due to Daykin.

Conjecture 4.2 (Daykin [26]). *Suppose that* $\mathcal{F} \subset 2^{[n]}$, $\delta(\mathcal{F}) > 2^{n-r-1}$, $r \geq 3$. *Then there exist* $F_1, \ldots, F_r \in \mathcal{F}$ *such that* $F_1 \cup \cdots \cup F_r = [n]$.

The Daykin conjecture was proved in [27] for $r \geq 25$. In [58] it is established for $r \geq 5$. However the cases $r = 3$ and 4 remain wide open. Especially in the case $r = 3$ some new ideas seem to be needed. Let us mention also that in [58] (23) is proved for all but six values of (r, t), namely, $r = 3$, $2 \leq t \leq 4$, and $r = 4$, $t = 8, 9, 10$.

Let us explain the reason that such problems are easier to tackle in the cases of relatively large r. For a fixed $l \geq 1$ let us say that $\mathcal{F}$ is l-complete if $\binom{[n]}{l} \subset \mathcal{F}$ holds.

Observation 4.1. *If* $\mathcal{F}$ *is* r*-wise* t*-union and* l*-complete, then it is* $(r-1)$*-wise* $(t+l)$*-union as well.*

Using this observation s times for some $1 \leq s \leq r-2$ one concludes that $\mathcal{F}$ is $(r-s)$-wise $(t+sl)$-union. This property can be used to obtain relatively strong upper bounds on $|\mathcal{F}|$ even though the *exact* value of

$$m(n, r, t) := \max\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-union}\}$$

is unknown for fixed $r \geq 3$ and e.g., $t > 2^{2r}$. The reason is that there are some relatively good upper bounds for the general case.

Let α_r denote the unique positive root of the polynomial $\frac{x^r - 2x + 1}{x - 1}$, $\alpha_2 = 1$, $\alpha_3 = \frac{\sqrt{5}-1}{2}$.

$$\begin{aligned} m(n, r, t) &\leq 2^n \alpha_r^t, \\ m(n+s, r, t+s) &\leq m(n, r, t)(2\alpha_r)^s. \end{aligned} \tag{24}$$

Since $m(n, r, t) \leq m(n+s, r, t+s)$ is obvious, and $\alpha_r \to \frac{1}{2}$ as $r \to \infty$, the bound (24) is quite accurate for large values of r.

These bounds are obtained based on the *Frankl random walk method*. We briefly explain the main idea. The *walk* associated to a set $F \subset [n]$ is an n-step walk on the integer grid $\mathbb{Z}^2$ starting at the origin $(0,0)$ whose i-th step is *up* (going from (x, y) to $(x, y+1)$) if $i \in F$, and is *right* (going from (x, y) to $(x+1, y)$) if $i \notin F$. Suppose that $\mathcal{G} \subset 2^{[n]}$ is a shifted r-wise t-intersecting family. Then one can show that for each $G \in \mathcal{G}$ the walk corresponding to G hits a line $y = (r-1)x + t$. This enables us to bound the size of $\mathcal{G}$ by counting the number of all n-step lattice walks that hit the line, or equivalently, by the probability that a random walk starting from the origin, with one step up or to the right, hits the line. See [56, 70, 147] for more details.

Definition 4.3. The families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$ are called r-cross t-union if $|F_1 \cup \cdots \cup F_r| \leq n - t$ holds for all choices of $F_i \in \mathcal{F}_i$, $i = 1, \ldots, r$.

Let us mention the following result.

Theorem 4.2 (Frankl [58]). *Suppose that* $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$ *are* r*-cross* t*-union,* $n \geq t$*, then the following hold.*

(i) $|\mathcal{F}_1| \cdots |\mathcal{F}_r| \leq (2^{n-t})^r$ *for* $t \leq 2^r - r - 2$ *with equality if and only if* $\mathcal{F}_1 = \cdots = \mathcal{F}_r = 2^{[t+1,n]}$.

(ii) $|\mathcal{F}_1| \cdots |\mathcal{F}_r| \leq (2^{n-2^r-r-2}\alpha^{t-2^r-r-2})^r$ *for* $t > 2^r - r - 2$.

Let us turn to uniform families, $\mathcal{F} \subset \binom{[n]}{k}$. If $rk \le n - t$ then $\mathcal{F}$ is automatically r-wise t-union. Therefore we assume that $rk > n - t$. The first non-trivial result is the following generalization of the Erdős–Ko–Rado Theorem.

Theorem 4.3 (Frankl [49]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise 1-union, $n \le kr$, then*

$$|\mathcal{F}| \le \binom{n-1}{k}. \tag{25}$$

Proof. Let us use Katona's cyclic permutation method (cf. [101]). Let $(x_1, \ldots, x_n)$ be a random cyclic ordering of $1, \ldots, n$. By cyclic we mean that x_1 is considered to be the next element after x_n. Note that there are altogether $(n-1)!$ cyclic orderings. $A_i := \{x_i, x_{i+1}, \ldots, x_{i+k-1}\}$ is called an interval. Here again $x_{i+j} = x_{i+j-n}$ for $i + j > n$. There are n intervals of length k.

Claim 4.1. *Out of the n intervals of length k at most $n - k$ are members of $\mathcal{F}$.*

Note that the claim implies (25). Indeed, the probability that $A_i \in \mathcal{F}$ is $|\mathcal{F}|/\binom{n}{k}$. Therefore the expected number of intervals A_i that are in $\mathcal{F}$ is $n|\mathcal{F}|/\binom{n}{k}$. By the claim $n|\mathcal{F}|/\binom{n}{k} \le n - k$, or equivalently,

$$|\mathcal{F}| \le \binom{n}{k}\frac{n-k}{n} = \binom{n-1}{k}.$$

Now let us turn to the proof of the claim. Let $s = \lceil n/k \rceil$ be the minimum integer such that $ks \ge n$ holds. Obviously, $2 \le s \le r$ holds. Let us first consider the case $n = sk$. One can divide the n intervals into k groups of s each:

$$\begin{gathered} A_1, A_{k+1}, \ldots, A_{(s-1)k+1} \\ A_2, A_{k+2}, \ldots, A_{(s-1)k+2} \\ \cdots \\ A_k, A_{2k}, \ldots, A_{sk}. \end{gathered}$$

Since in each group the s intervals form a partition of $[n]$, at least one of them is missing from $\mathcal{F}$. These amount to at least k missing sets, as desired.

Now let $ks = n + t$ for some $1 \le t < k$. Without loss of generality $A_n \in \mathcal{F}$. Let us define $A_{n+i} = A_n$ for $i = 1, \ldots, t$ and consider the above k groups of s intervals each. It is easy to verify that the union of the s sets in each group is still $[n]$. Therefore at least one interval from each group is missing from $\mathcal{F}$. Since $A_n \in \mathcal{F}$, there is no overlapping and the proof of the claim is complete. □

Let us mention that unless $r = 2$ and $n = 2k$, $\binom{[n-1]}{k}$ is the unique optimal family, cf. Theorem 11.1 in [56].

One can further extend Theorem 4.3 as follows.

Theorem 4.4 ([78]). *Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ are r-cross 1-union, $n \le kr$, then $\prod_{i=1}^{r} |\mathcal{F}_i| \le \binom{n-1}{k}^r$.*

This is an easy consequence of the next result, which is a variant of the Kruskal–Katona theorem. To state the result we need a definition. For $\mathcal{F} \subset \binom{[n]}{k}$ choose a unique real $x \geq k$ so that $|\mathcal{F}| = \binom{x}{k}$, and let $\|\mathcal{F}\|_k := x$.

Theorem 4.5 ([78]). *Let $n \leq rk$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ be r-cross 1-union families. Then $\sum_{i=1}^r \|\mathcal{F}_i\|_k \leq r(n-1)$.*

We propose the following conjectures.

Conjecture 4.3. *Let $1/r \leq k_i/n \leq 1$ and $\mathcal{F}_i \subset \binom{[n]}{k_i}$ for $1 \leq i \leq r$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross 1-union, then $\prod_{i=1}^r |\mathcal{F}_i|/\binom{n}{k_i} \leq \prod_{i=1}^r (1 - k_i/n)$.*

The corresponding product measure version should be the following. For $0 < p \leq 1$ and $\mathcal{F} \subset 2^{[n]}$ let $\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} p^{|F|}(1-p)^{n-|F|}$.

Conjecture 4.4. *Let $1/r \leq p_i \leq 1$ and $\mathcal{F}_i \subset 2^{[n]}$ for $1 \leq i \leq r$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross 1-union, then $\prod_{i=1}^r \mu_{p_i}(\mathcal{F}_i) \leq \prod_{i=1}^r (1 - p_i)$.*

If both conjectures are true, then it would be more interesting to find a general result which contains them as special cases.

5. Excluding simplices

One of the first problems concerning multiple intersections, as so many other problems, is due to Erdős.

Definition 5.1. Three sets F_0, F_1, F_2 are forming a triangle if $F_i \cap F_j \neq \emptyset$ for $0 \leq i < j \leq 2$ but $F_0 \cap F_1 \cap F_2 = \emptyset$.

Erdős [39] posed the following question. Let $k \geq 3$. Is it true that if $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a triangle and $3k \leq 2n$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$? This is now known to be true in a stronger sense as we will see below, see Theorem 5.4. Chvátal introduced the more general notion of a *simplex*.

Definition 5.2. We say that $\{F_0, \ldots, F_d\} \subset 2^{[n]}$ is a *d-simplex* if $F_0 \cap \cdots \cap F_d = \emptyset$ but $\bigcap_{i \in I} F_i \neq \emptyset$ for all $I \subset \{0, 1, \ldots, d\}$ with $|I| = d$.

Let us note that if $|F_0| = \cdots = |F_d| = d$ then the only d-simplex is $\binom{[d+1]}{d}$, the complete d-graph on $d+1$ vertices. To determine the maximum of $|\mathcal{F}|$ for $\mathcal{F} \subset \binom{[n]}{d}$ not containing a d-simplex is Turán's problem (cf. e.g., [40, 149, 103]) and seems to be beyond reach for $d \geq 3$.

We say that $\mathcal{F} \subset \binom{[n]}{k}$ is a *star* if $\mathcal{F} = \{F \in \binom{[n]}{k} : i \in F\}$ for some fixed $i \in [n]$.

Conjecture 5.1 (Chvátal [22]). *Suppose that $k \geq d+1 \geq 2$, $n \geq k(d+1)/d$, and $\mathcal{F} \subset \binom{[n]}{k}$ contains no d-simplex. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$, moreover, equality holds if and only if $\mathcal{F}$ is a star.*

Chvátal proved this conjecture for the case $k = d+1$. Frankl [53] proved Erdős's conjecture for $k \geq 5$ and $n > n_0(k)$. Let us also mention that the case $\frac{d-1}{d}n < k \leq \frac{d}{d+1}n$ follows from (23).

Let $s(n, k, d)$ denote the maximum of $|\mathcal{F}|$, $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ contains no d-simplex.

Theorem 5.1 (Frankl–Füredi [66])**.**

$$s(n,k,d) \le \binom{n}{k-1}. \tag{26}$$

Let us mention that the proof of (26) is a simple linear independence argument. Also, for $2k > n$ the bound is trivial because of $\binom{n}{k-1} \ge \binom{n}{k}$. For $n \ge n_0(k)$ Frankl and Füredi proved Chvátal's conjecture.

Theorem 5.2 (Frankl–Füredi [66])**.** *For $n > n_0(k,d)$, $s(n,k,d) = \binom{n-1}{k-1}$ holds. Moreover, the only family achieving equality is a star.*

Definition 5.3. We say that $\{H_0, \ldots, H_d\} \subset \binom{[n]}{k}$ is a *special d-simplex* if for some $(d+1)$-element set $C = \{x_0, \ldots, x_d\}$ one has $H_i \cap C = C \setminus \{x_i\}$, moreover, the sets $H_i \setminus C$ are pairwise disjoint for $0 \le i \le d$. Note that $|\bigcup_{i=0}^{d} H_i| = (d+1)(k-d+1)$.

Theorem 5.3 (Frankl–Füredi [66])**.** *Suppose that $k \ge d+3$, $n > n_0(k)$, and $\mathcal{F} \subset \binom{[n]}{k}$ contains no special d-simplex. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, moreover equality holds if and only if $\mathcal{F}$ is a star.*

They conjectured that the same is true for $k = d+1$ and $k = d+2$ as well. In the case $d = 2$ they did actually prove it.

A nontrivial intersecting family of size $d+1$ is a family of $d+1$ distinct sets $F_0, \ldots, F_d$ that have pairwise nonempty intersection, but $\bigcap_{i=0}^{d} F_i = \emptyset$.

Theorem 5.4 (Mubayi–Verstraëte [123])**.** *Suppose that $k \ge d+1 \ge 3$, $n \ge (d+1)k/d$, and $\mathcal{F} \subset \binom{[n]}{k}$ contains no nontrivial intersecting family of size $d+1$. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, moreover equality holds if and only if $\mathcal{F}$ is a star.*

Definition 5.4. We say that $\{F_0, \ldots, F_d\} \subset \binom{[n]}{k}$ is a *d-cluster* if $\bigcap_{i=0}^{d} F_i = \emptyset$ and $|\bigcup_{i=0}^{d} F_i| \le 2k$. If, moreover, it is also d-simplex, then we call it a *d-cluster-simplex.*

Mubayi posed the following conjecture, which is a generalization of a conjecture due to Frankl and Füredi in [63].

Conjecture 5.2 (Mubayi [124])**.** *Suppose that $k \ge d+1 \ge 2$, $n \ge k(d+1)/d$, and $\mathcal{F} \subset \binom{[n]}{k}$ contains no d-cluster. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, moreover, equality holds if and only if $\mathcal{F}$ is a star.*

The above conjecture holds for $d = 2$; this was first verified by Frankl and Füredi for $n > k^2 + 3k$ in [63], and then completed (for $n \ge 3k/2$) by Mubayi [124]. Chen, Liu and Wang [21] observed that the case $k = d+1$ of Conjecture 5.2 is reduced to Conjecture 5.1, which is true by a result of Chvátal.

Conjecture 5.3 (Keevash–Mubayi [106])**.** *Suppose that $k \ge d+1 \ge 2$, $n > k(d+1)/d$, and $\mathcal{F} \subset \binom{[n]}{k}$ contains no d-cluster-simplex. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, moreover, equality holds if and only if $\mathcal{F}$ is a star.*

6. Intersections and unions

We have seen so far problems related to intersections and also to unions. As noted earlier an intersection problem is basically the same as the corresponding union problem considered on the family of the complements. However, if we impose *both* intersection and union conditions to the same family then we get different problems. Let us present the first non-trivial case which was solved independently by many authors, including Marica and Schonheim [120], Daykin and Lovász [28], Seymour [134], Anderson [7], and Kleitman [107].

Theorem 6.1 (IU-Theorem)**.** *Suppose that* $\mathcal{F} \subset 2^n$ *satisfies*

$$F \cap F' \neq \emptyset \text{ for all } F, F' \in \mathcal{F}, \tag{27}$$

and

$$F \cup F' \neq [n] \text{ for all } F, F' \in \mathcal{F}. \tag{28}$$

Then

$$|\mathcal{F}| \leq 2^{n-2}. \tag{29}$$

It is easy to see that (29) is best possible. Let $Y \cup Z$ be an arbitrary partition of $[n]$ and let $\mathcal{G} \subset 2^Y$, $\mathcal{H} \subset 2^Z$ be families satisfying $|\mathcal{G}| = 2^{|Y|-1}$, $G \cap G' \neq \emptyset$ for all $G, G' \in \mathcal{G}$; $|\mathcal{H}| = 2^{|Z|-1}$, $H \cup H' \neq Z$ for all $H, H' \in \mathcal{G}$. Define $\mathcal{F} = \{G \cup H : G \in \mathcal{G},\, H \in \mathcal{H}\}$ then $|\mathcal{F}| = |\mathcal{G}| \cdot |\mathcal{H}| = 2^{n-2}$ and clearly $\mathcal{F}$ satisfies both (27) and (28).

From the actual proof of the following result it follows that all optimal families $\mathcal{F}$ come from the above construction.

Lemma 6.1 (Kleitman's Lemma [107])**.** *If* $\mathcal{A} \subset 2^{[n]}$ *is a complex and* $\mathcal{B} \subset 2^{[n]}$ *is a filter then*

$$|\mathcal{A} \cap \mathcal{B}|/2^n \leq (|\mathcal{A}|/2^n)(|\mathcal{B}|/2^n). \tag{30}$$

Of course (30) is equivalent to $|\mathcal{A} \cap \mathcal{B}|2^n \leq |\mathcal{A}||\mathcal{B}|$. We write it in this fraction form because it is more about negative correlation. Namely, if we consider the uniform distribution on $2^{[n]}$ where each set $S \subset [n]$ has probability $1/2^n$ then $|\mathcal{A}|/2^n$, $|\mathcal{B}|/2^n$, and $|\mathcal{A} \cap \mathcal{B}|/2^n$ are the probabilities that a randomly chosen set $S \subset [n]$ is in $\mathcal{A}, \mathcal{B}$ or both, respectively. I.e., (30) expresses that the probability of both events happening is not larger than the product of the individual probabilities. That is negative correlation. Extensions of Kleitman's Lemma were discovered and applies in theoretical physics. See also [135] for some other extensions.

Proof. The case $n = 1$ is very easy to check. To prove the general case let us apply induction. Consider the four families $\mathcal{A}(1), \mathcal{A}(\bar{1}), \mathcal{B}(1), \mathcal{B}(\bar{1})$ on $[n-1]$, and note that $\mathcal{A}(1)$ and $\mathcal{A}(\bar{1})$ are complexes with $\mathcal{A}(1) \subset \mathcal{A}(\bar{1})$, while $\mathcal{B}(1)$ and $\mathcal{B}(\bar{1})$ are filters with $\mathcal{B}(1) \supset \mathcal{B}(\bar{1})$. This implies that we may apply the induction hypothesis to both pairs $(\mathcal{A}(1), \mathcal{B}(1))$ and $(\mathcal{A}(\bar{1}), \mathcal{B}(\bar{1}))$, and also see inequality

$$(|\mathcal{A}(1)| - |\mathcal{A}(\bar{1})|)(|\mathcal{B}(1)| - |\mathcal{B}(\bar{1})|) \leq 0. \tag{31}$$

From the induction hypothesis we obtain

$$\frac{|\mathcal{A}\cap\mathcal{B}|}{2^n} = \frac{|\mathcal{A}(1)\cap\mathcal{B}(1)|}{2^n} + \frac{|\mathcal{A}(\bar{1})\cap\mathcal{B}(\bar{1})|}{2^n}$$
$$\leq \frac{1}{2}\left(\frac{|\mathcal{A}(1)|}{2^{n-1}}\cdot\frac{|\mathcal{B}(1)|}{2^{n-1}} + \frac{|\mathcal{A}(\bar{1})|}{2^{n-1}}\cdot\frac{|\mathcal{B}(\bar{1})|}{2^{n-1}}.\right)$$

In order to prove (30) it is sufficient to show that the RHS is not more than

$$\frac{|\mathcal{A}(1)|+|\mathcal{A}(\bar{1})|}{2^n}\cdot\frac{|\mathcal{B}(1)|+|\mathcal{B}(\bar{1})|}{2^n}.$$

Multiplying by 2^{2n} it is equivalent to

$$2(|\mathcal{A}(1)||\mathcal{B}(1)|+|\mathcal{A}(\bar{1})||\mathcal{B}(\bar{1})|) \leq (|\mathcal{A}(1)|+|\mathcal{A}(\bar{1})|)(|\mathcal{B}(1)|+|\mathcal{B}(\bar{1})|).$$

This inequality is equivalent to (31). □

Let us mention that Kleitman discovered this inequality in order to prove the following result which was conjectured by Erdős.

Theorem 6.2 (Kleitman [107]). *Suppose that $\mathcal{F}_1,\dots,\mathcal{F}_s$ are intersecting families on $[n]$. Then*

$$|\mathcal{F}_1\cup\cdots\cup\mathcal{F}_s| \leq 2^n - 2^{n-s}$$

for all $2\leq s\leq n$.

Let $g(n,t)$ denote the maximum of $|\mathcal{G}|$ over $\mathcal{G}\subset 2^{[n]}$, $\mathcal{G}$ is t-intersecting. Note that $g(n,t)$ is determined by the Katona Theorem (Theorem 1.3). Katona [101] conjectured the validity of the following.

Theorem 6.3 (Frankl [48]). *Suppose that $\mathcal{F}\subset 2^{[n]}$ is t-intersecting and at the same time $F\cup F'\neq[n]$ for all $F,F'\in\mathcal{F}$. Then*

$$|\mathcal{F}| \leq g(n-1,t). \tag{32}$$

By considering a t-intersecting family $\mathcal{F}\subset 2^{[n-1]}$ of maximal size shows that, if true, the bound (32) is optimal.

Proof. Without loss of generality $\mathcal{F}$ is shifted, i.e., $1\leq i<j\leq n$ and $F\cap\{i,j\}=\{j\}$ imply that $(F\setminus\{j\})\cup\{i\}$ is also in $\mathcal{F}$.

Claim 6.1. *For all $F,F'\in\mathcal{F}$ one has*

$$|F\cap F'\cap[n-1]| \geq t. \tag{33}$$

Let us first show that (33) implies (32). Consider

$$\mathcal{F}^* = \{F^*\subset[n] : \exists F\in\mathcal{F},\ F\subset F^*\},$$
$$\mathcal{F}_* = \{F_*\subset[n] : \exists F\in\mathcal{F},\ F_*\subset F\}.$$

Then (33) holds for all $F,F'\in\mathcal{F}^*$ as well while $F\cup F'\neq[n]$ follows for $F,F'\in\mathcal{F}_*$. By this second property

$$|\mathcal{F}_*| \leq 2^{n-1}.$$

From (33) it follows that both $\mathcal{F}^*(n)$ and $\mathcal{F}^*(\bar{n})$ are t-intersecting, yielding

$$|\mathcal{F}^*| = |\mathcal{F}^*(n)|+|\mathcal{F}^*(\bar{n})| \leq 2g(n-1,t).$$

So (32) follows from Kleitman's Lemma.

Now we prove the claim. Suppose the contrary. Since $\mathcal{F}$ is t-intersecting we must have

$$|F \cap F'| = t \text{ and } n \in F \cap F'.$$

By the union condition $F \cup F' \neq [n]$, i.e., we can find $i \in [n-1]$, $i \notin F \cup F'$. However $\mathcal{F}$ is shifted, implying $F'' := (F' \setminus \{n\}) \cup \{i\} \in \mathcal{F}$. As $|F \cap F''| = |F \cap F'| - 1 = t - 1$, we obtained a contradiction concluding the proof of both the claim and (32). $\square$

Let us present a version for two families. Let $g_2(n, t) = \max |\mathcal{F}||\mathcal{G}|$ where the maximum is over all cross t-intersecting $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$. It was shown by Matsumoto and Tokushige in [121] that $g_2(n, t)$ is $g(n, t)^2$ if $n-t$ is even, and $\max\{g(n, t)^2, g(n, t-1)g(n, t+1)\}$ if $n - t$ is odd.

Theorem 6.4 (Frankl [48]). *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross t-intersecting and cross 1-union. Then $|\mathcal{F}||\mathcal{G}| \leq g_2(n-1, t)$ holds.*

The proof is almost the same except that one proves the two families version of the claim. The interesting thing is that it does not rely on any knowledge of $g_2(n, t)$ or $g(n, t)$.

Well, the situation in general, namely, if $\mathcal{F}$ is required to be s-union for some $s \geq 2$, is much more difficult.

Definition 6.1. Let $h(n, t, s)$ denote the maximum of $|\mathcal{F}|$ over all $\mathcal{F} \subset 2^{[n]}$ that are both t-intersecting and s-union.

Conjecture 6.1 (Frankl [48]). *Suppose that $n \geq t + s$. Then*

$$h(n, t, s) = \max_q g(q, t) g(n - q, s).$$

If one defines $h_2(n, t, s)$ analogously as $\max |\mathcal{F}||\mathcal{G}|$ for $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$, where $\mathcal{F}, \mathcal{G}$ are both t-intersecting and s-union, then one can make the following conjecture.

Conjecture 6.2. *$h_2(n, t, s) = \max_q g_2(q, t) g_2(n - q, s)$.*

7. Intersecting families with fixed covering number

For a family $\mathcal{F} \subset 2^{[n]}$ its covering number $\tau(\mathcal{F})$ is the minimal integer t such that there exists a t-element set T satisfying $T \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The covering number is a very important notion both in graph theory and extremal set theory. There is a vast, excellent literature on problems related to the covering numbers. Even though it is almost thirty years old, we recommend the excellent survey of Füredi [86]. Here we only deal with some natural questions related to intersecting families.

Note that if $\mathcal{F}$ is intersecting then every member F of $\mathcal{F}$ is a *cover*, i.e., $F \cap F' \neq \emptyset$ for all $F' \in \mathcal{F}$. In particular, if $\mathcal{F}$ is k-uniform then $\tau(\mathcal{F}) \leq k$ holds.

Theorem 7.1 (Erdős–Lovász [45]). *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $\tau(\mathcal{F}) = k$ then*

$$|\mathcal{F}| \leq k^k. \tag{34}$$

The above result shows that

$$\max\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform, intersecting and } \tau(\mathcal{F}) = k\}$$

exists. Let us denote this quantity by $r(k)$. For a family $\mathcal{G} \subset 2^{[n]}$ let us define

$$\mathcal{C}_t(\mathcal{G}) = \{C \in \binom{[n]}{t} : C \cap G \neq \emptyset \text{ for all } G \in \mathcal{G}\}.$$

Also set $c_t(\mathcal{G}) = |\mathcal{C}_t(\mathcal{G})|$. Obviously, $c_t(\mathcal{G}) = 0$ for $t < \tau(\mathcal{G})$.

Theorem 7.2 (Gyárfás [92]). *Let $\mathcal{G} \subset \binom{[n]}{k}$ and $t = \tau(\mathcal{G})$. Then*

$$c_t(\mathcal{G}) \leq k^t. \tag{35}$$

Let us reproduce the proof.

Proof. First we consider the case $t = 1$. Let $G \in \mathcal{G}$ be an arbitrary edge. If $\{x\}$ is a cover then $x \in G$. Thus $c_1(\mathcal{G}) \leq |G| = k$. Now we apply induction. Fix again $G \in \mathcal{G}$ and for every $x \in G$ consider $\mathcal{C}(x) = \{C \setminus \{x\} : x \in C \in \mathcal{C}_t(\mathcal{G})\}$ together with $\mathcal{G}(\bar{x}) = \{G' \in \mathcal{G} : x \notin G'\}$. Since $C \cap G \neq \emptyset$ for all $C \in \mathcal{C}_t(\mathcal{G})$,

$$\sum_{x \in G} |\mathcal{C}(x)| \geq |\mathcal{C}_t(\mathcal{G})|. \tag{36}$$

On the other hand provided $\mathcal{C}(x) \neq \emptyset$ we have $\tau(\mathcal{G}(\bar{x})) = t-1$ and $\mathcal{C}(x) \in \mathcal{C}_{t-1}(\mathcal{G}(\bar{x}))$. By the induction hypothesis $|\mathcal{C}(x)| \leq k^{t-1}$ follows. Using (36) we obtain the validity of (35). □

If $\mathcal{G}$ consists of t pairwise disjoint k-sets then equality holds in (35). Also, if $\mathcal{F}$ is k-uniform, intersecting and $\tau(\mathcal{F}) = k$ then $\mathcal{C}_k(\mathcal{F}) \supset \mathcal{F}$. Therefore (35) implies (34).

Erdős and Lovász [45] showed the following recursive lower bound for $r(k)$.

$$r(k+1) \geq (k+1)r(k) + 1. \tag{37}$$

Proof. If $\mathcal{F}$ realizes the bound k then let F_0 be a $(k+1)$-set disjoint to all $F \in \mathcal{F}$ and define $\mathcal{F}_0 = \{F_0\} \cup \{F \cup \{x\} : F \in \mathcal{F}, x \in F_0\}$. It is easy to check that $\mathcal{F}_0$ is $(k+1)$-uniform, intersecting and $\tau(\mathcal{F}_0) = k+1$. Since $|\mathcal{F}_0| = 1 + (k+1)|\mathcal{F}|$, we are done. □

Starting with $r(1) = 1$, using (37) one obtains $r(2) \geq 3$, $r(3) \geq 10$, $r(4) \geq 41$ etc. In general, $r(k) \geq \lfloor k!(e-1) \rfloor$ follows. Lovász [116] conjectured that one has equality here (and (37)). However, this was disproved in [73] by an example showing $r(4) \geq 42$.

Then Majumder and Mukherjee [126] showed that there are at least two non-isomorphic 4-uniform families of size 42 with covering number 4, and $r(5) \geq 234$. For general k, in [73, 126], the following lower bound is given.

Theorem 7.3. $r(k) > (\lfloor k/2 \rfloor + 1)^{k-1}$.

Let us give a construction for $k = 2d$, $d \geq 2$. Let $A_1, \ldots, A_{2d-1}$ be pairwise disjoint sets of size $d+1$ and let y be an extra vertex. Define two families $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \{A : |A| = 2d,\ y \in A,\ |A \cap A_j| = 1 \text{ for } i = 1, \ldots, 2d-1\},$$

and $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{2d-1}$ where

$$\mathcal{B}_i = \{B : |B| = 2d,\ A_i \subset B,\ |B \cap A_j| = 1 \text{ for } j = i+1, \ldots, i+d-1\},$$

where addition is modulo $2d - 1$. Note that $|\mathcal{A}| = (d+1)^{2d-1}$, $\mathcal{A} \cup \mathcal{B}$ is k-uniform, intersecting and $\tau(\mathcal{A} \cup \mathcal{B}) = k$ is not too hard to verify. The bounds for $r(k)$ are still quite far apart. We believe that $r(k) = O((\mu k)^k)$ holds with some $\mu < 1$ (cf. [73]).

The covering number $\tau(\mathcal{F})$ can be considered as a measure of nontriviality for the intersecting family $\mathcal{F}$. By the Erdős–Ko–Rado Theorem, $|\mathcal{F}| \leq \binom{n-1}{k-1}$ for all intersecting families $\mathcal{F} \subset \binom{[n]}{k}$. The family giving equality is all k-sets through a fixed vertex, it has covering number 1. The Hilton–Milner Theorem (Theorem 1.6) determines the maximum size of $\mathcal{F}$ for intersecting families with $\tau(\mathcal{F}) \geq 2$.

Definition 7.1. Fix $k > t > 0$ and define

$$c(n,k,t) = \max\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k},\ \mathcal{F} \text{ is intersecting},\ \tau(\mathcal{F}) \geq t\}.$$

To determine $c(n,k,t)$ seems to be very difficult. Even in the case $n > n_0(k,t)$ only partial results are known.

Theorem 7.4 (Frankl [52])**.** *For $k \geq 4$*

$$c(n,k,3) = (k^2 - k + 1 + o(1))\binom{n-3}{k-3}.$$

The following result was proved for $k \geq 9$ in [72], then completed by Furuya and Takatou [83, 84].

Theorem 7.5. *For $k \geq 5$, $c(n,k,4) = (k^3 - 3k^2 + 6k - 4 + o(1))\binom{n-4}{k-4}$.*

Actually in both of the above results the exact value and the essentially unique optimal families are determined for $n > n_0(k,t)$.

Let us close this section by the following conjecture.

Conjecture 7.1 ([73])**.** *For $k > k(t)$, $c(n,k,t) = (k^{t-1} - \binom{t-1}{2}k^{t-2} + p(k,t) + o(1))\binom{n-t}{k-t}$ holds where $p(k,t)$ is a polynomial of k and t with the degree of k being at most $t-3$.*

One can also consider

$$n(k) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform, intersecting and } \tau(\mathcal{F}) = k\}.$$

Erdős and Lovász [45] proved $n(k) \geq 8k/3 - 3$ for all $k \geq 2$. The general belief was that $n(k)/k$ tends to infinity. Therefore it came as a big surprise when Kahn [98] proved

$$n(k) = O(k).$$

To determine the exact value of $n(k)$ appears to be hopelessly difficult.

We mention one more related problem. Let $m(k)$ denote the minimum size of k-uniform maximal intersecting families. Clearly we have $m(k) \geq n(k)$, and so $m(k) \geq 8k/3 - 3$. Dow et al. [32] improved the bound by showing $m(k) \geq 3k$ for $k \geq 4$.

Conjecture 7.2 (Kahn [98])**.** *$m(k) = O(k)$.*

8. Intersections in vector spaces, permutations, and graphs

Fix a finite field $\mathbb{F}$ and let V_n denote an n-dimensional vector space over $\mathbb{F}$. Then many intersection problems on families of subsets of $[n]$ can be translated to the corresponding problems on families of subspaces of V_n. Let $\begin{bmatrix} V_n \\ k \end{bmatrix}$ denote the set of k-dimensional subspaces of V_n, and let $\mathcal{L}_n := \bigcup_{i=0}^{n} \begin{bmatrix} V_n \\ i \end{bmatrix}$. Let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the cardinality of $\begin{bmatrix} V_n \\ k \end{bmatrix}$, that is, $\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1}$. We say that r families of subspaces $\mathcal{F}_1, \dots, \mathcal{F}_r \subset \mathcal{L}_n$ are *r-cross t-intersecting* if

$$\dim(F_1 \cap F_2 \cap \cdots \cap F_r) \geq t$$

for all $F_i \in \mathcal{F}_i$, $1 \leq i \leq r$. As usual when we say r-cross t-intersecting we omit r (resp. t) if $r = 2$ (resp. $t = 1$). If $\mathcal{F}, \dots, \mathcal{F}$ (r times) are r-cross t-intersecting, then $\mathcal{F}$ is called r-wise t-intersecting. Chowdhury and Patkós established a vector space version of Theorem 4.3.

Theorem 8.1 ([23]). *Let $(r-1)/r \geq k/n$. If $\mathcal{F} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ is r-wise intersecting, then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Moreover, equality holds if and only if $\mathcal{F} = \{F \in \begin{bmatrix} V_n \\ k \end{bmatrix} : L \subset F\}$ for some $L \in \begin{bmatrix} V_n \\ 1 \end{bmatrix}$ unless $r = 2$ and $n = 2k$.*

The proof of the above result in [23] is a combinatorial one, which is based on the following vector space version of Theorem 1.4.

Theorem 8.2 (Chowdhury–Patkós [23]). *Let $\mathcal{F} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ and let $y \in \mathbb{R}$ be such that $|\mathcal{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$. Then $|\{G \in \begin{bmatrix} V_n \\ k-1 \end{bmatrix} : G \subset F \text{ for some } F \in \mathcal{F}\}| \geq \begin{bmatrix} y \\ k-1 \end{bmatrix}$.*

We propose the following conjecture.

Conjecture 8.1. *Let $(r-1)/r \geq \max\{k_1/n, \dots, k_r/n\}$. If $\mathcal{F}_1 \subset \begin{bmatrix} V_n \\ k_1 \end{bmatrix}, \dots, \mathcal{F}_r \subset \begin{bmatrix} V_n \\ k_r \end{bmatrix}$ are r-cross intersecting, then $\prod_{i=1}^{r} |\mathcal{F}_i| \leq \prod_{i=1}^{r} \begin{bmatrix} n-1 \\ k_i-1 \end{bmatrix}$.*

This conjecture is true if $\mathcal{F}_1 = \cdots = \mathcal{F}_r$ by Theorem 8.1 and if $r = 2$ by Theorem 10.3 in section 10.

Theorem 8.3. *Let $k \geq t \geq 1$. Suppose that two families $\mathcal{A}, \mathcal{B} \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ are cross t-intersecting. Then we have*

$$|\mathcal{A}||\mathcal{B}| \leq \begin{cases} \begin{bmatrix} n-t \\ k-t \end{bmatrix}^2 & \text{if } n \geq 2k, \\ \begin{bmatrix} 2k-t \\ k \end{bmatrix}^2 & \text{if } 2k-t < n \leq 2k. \end{cases}$$

Extremal configurations are following:

(i) *If $n > 2k$ and $|\mathcal{A}||\mathcal{B}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}^2$, then $\mathcal{A} = \mathcal{B} = \{F \in \begin{bmatrix} V_n \\ k \end{bmatrix} : T \subset F\}$ for some $T \in \begin{bmatrix} V_n \\ t \end{bmatrix}$.*

(ii) *If $2k-t < n < 2k$ and $|\mathcal{A}||\mathcal{B}| = \begin{bmatrix} 2k-t \\ k \end{bmatrix}^2$, then $\mathcal{A} = \mathcal{B} = \begin{bmatrix} Y \\ k \end{bmatrix}$ for some $Y \in \begin{bmatrix} V_n \\ 2k-t \end{bmatrix}$.*

(iii) *If $n = 2k$ and $|\mathcal{A}||\mathcal{B}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}^2 = \begin{bmatrix} 2k-t \\ k \end{bmatrix}^2$, then $\mathcal{A} = \mathcal{B} = \{F \in \begin{bmatrix} V_n \\ k \end{bmatrix} : T \subset F\}$ for some $T \in \begin{bmatrix} V_n \\ t \end{bmatrix}$ or $\mathcal{A} = \mathcal{B} = \begin{bmatrix} Y \\ k \end{bmatrix}$ for some $Y \in \begin{bmatrix} V_n \\ 2k-t \end{bmatrix}$.*

This result (except item (iii)) for the case $\mathcal{A} = \mathcal{B}$ was first obtained by Frankl and Wilson [81] using linear algebra. Tanaka [141] gave another algebraic proof for the case $\mathcal{A} = \mathcal{B}$ (including item (iii)). Then using an idea in [5] (see also [36]) Theorem 8.3 can be easily deduced from the Frankl–Wilson's proof with Tanaka's result for (iii).

Blokhuis et al. obtained a vector space version of the Hilton–Milner theorem, see [11].

Leffman obtained a vector space version of Theorem 1.3. Define

$$\mathcal{K}[n,t] = \begin{cases} \bigcup_{k=d}^{n} \begin{bmatrix} V_n \\ k \end{bmatrix} & \text{if } n+t = 2d, \\ (\bigcup_{k=d+1}^{n} \begin{bmatrix} V_n \\ k \end{bmatrix}) \cup \begin{bmatrix} V_{n-1} \\ d \end{bmatrix} & \text{if } n+t = 2d+1, \end{cases}$$

where V_{n-1} is an $(n-1)$-dimensional subspace of V_n.

Theorem 8.4 (Leffman [114]). *Let $1 \le t \le n$ and let $\mathcal{F} \subset \mathcal{L}_n$ be t-intersecting. Then $|\mathcal{F}| \le |\mathcal{K}[n,t]|$. Moreover if $t > 1$ then equality holds if and only if $\mathcal{F} \cong \mathcal{K}[n,t]$.*

Leffman also obtained the upper bound for the size of a family $\mathcal{F} \subset \mathcal{L}_n$ satisfying $\dim(F \cap F') \notin \{s, s+1, \dots, t\}$ for all $F, F' \in \mathcal{F}$, see [115].

Let S_n denote the symmetric group, the group of all permutations of $[n]$. Two permutations $\sigma, \tau \in S_n$ are said to t-intersect if there is some $T \in \binom{[n]}{t}$ such that $\sigma(i) = \tau(i)$ for all $i \in T$. We say that a family of permutations $I \subset S_n$ is t-intersecting if any two permutations in I t-intersect.

Deza and Frankl [31] observed that if $I \subset S_n$ is 1-intersecting, then

$$|I| \le (n-1)!. \tag{38}$$

Then, Cameron and Ku [20], and independently, Larose and Malvenuto [113] proved that if equality holds in (38), then I is a 1-coset. Deza and Frankl conjectured that the similar statement holds for t-intersecting families of permutations provided n is large enough. They verified the conjecture for $t = 2, 3$ with infinitely many values of n.

Theorem 8.5 (Ellis–Friedgut–Pilpel [36]). *If $n > n_0(t)$ and $I \subset S_n$ is t-intersecting, then $|I| \le (n-t)!$. Equality holds if and only if I is a t-coset of S_n.*

Conjecture 8.2 (Ellis [33]). *If $n > n_0(t)$ and $I \subset S_n$ with no two permutations in I agreeing on exactly $t-1$ points, then $I \le (n-t)!$. Equality holds if and only if I is a t-coset of S_n.*

The above conjecture is known to be true if $t = 1$ by (38), and $t = 2$ by Ellis [33], see also [105].

Let $\mathcal{F}$ be a family of graphs on the same vertex set. Then $\mathcal{F}$ is called triangle-intersecting if for every $G, H \in \mathcal{F}$, $G \cap H$ contains a triangle. If we fix a triangle and take all subgraphs of K_n containing this triangle, then we get a triangle-intersecting family of size $2^{\binom{n}{2}-3}$. Sós conjectured that example gives the maximum and the only family (up to isomorphism) which has the maximum size, see [136]. Ellis, Filmus, and Friedgut verified this conjecture in the following stronger sense.

A family $\mathcal{F}$ of subgraphs of K_n is called odd-cycle-intersecting if for every $G, H \in \mathcal{F}$, $G \cap H$ contains an odd cycle. For a real $p \in (0,1)$ and a graph $G \in \mathcal{F}$ let $\mu_p(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}$, and let $\mu_p(\mathcal{F}) = \sum_{G\in\mathcal{F}} \mu_p(G)$.

Theorem 8.6 ([35])**.** *Let $p \le 1/2$ and let $\mathcal{F}$ be an odd-cycle-intersecting family of subgraphs of K_n. Then $\mu_p(\mathcal{F}) \le p^3$ with equality iff all graphs in $\mathcal{F}$ contains a fixed triangle.*

For some related and other results, see a nice survey by Borg [14].

9. L-systems

Let n, k be positive integers with $n \ge k$, and let $L \subset \{0, 1, \ldots, k-1\}$. We say that a family of k-element subsets $\mathcal{F} \subset \binom{[n]}{k}$ is an (n, k, L)*-system* if

$$|F \cap F'| \in L$$

holds for all *distinct* $F, F' \in \mathcal{F}$. We also call it a (k, L)-system or just an L-system for short. Let $m(n, k, L)$ denote the maximum size of (n, k, L)-systems. If there exist positive reals α, c, c' depending only on k and L such that

$$cn^\alpha < m(n, k, L) < c'n^\alpha,$$

then we define $\alpha(k, L) = \alpha$, and we say that (k, L)-systems have *exponent* α.

Conjecture 9.1. *For every k and L, the exponent $\alpha(k, L)$ exists.*

In this section we only consider pairs k, L such that the corresponding exponents exist (and if the conjecture is true, then this is a void restriction). No irrational exponent is known so far.

Theorem 9.1 (Frankl [55])**.** *For every rational number $q \ge 1$ there are infinitely many choices of k and L such that $\alpha(k, L) = q$.*

As an example, let us construct a family showing $\alpha(k, L) \ge 2.5$. To this end let $k = 10$ and $L = \{0, 1, 3, 6\}$. We need an (n, k, L)-system $\mathcal{F}$ with $|\mathcal{F}| = \Theta(n^{2.5})$. Let p be a positive integer, $V := \binom{[p]}{2}$, and let $\mathcal{F} := \{\binom{A}{2} : A \in \binom{[p]}{5}\}$. If $F, F' \in \mathcal{F}$ ($F \ne F'$) with $F = \binom{A}{2}$, $F' = \binom{A'}{2}$, then $|F \cap F| = \binom{|A\cap A'|}{2}$, which is one of $0, 1, 3, 6$. Thus $\mathcal{F}$ is a (k, L)-system on V, where $n := |V| = \binom{p}{2}$ and $|\mathcal{F}| = \binom{p}{5}$, as required. On the other hand one can also show that $\alpha(10, L) \le 2.5$ by using Theorem 9.3.

Deza, Erdős, and Frankl obtained the following general upper bound for $m(n, k, L)$.

Theorem 9.2 ([30])**.** *Let $n \ge 2^k k^3$, and let $\mathcal{F}$ be an (n, k, L)-system, where $L = \{l_1, l_2, \ldots, l_s\}$ with $0 \le l_1 < l_2 < \cdots < l_s < k$. Then we have the following.*

(i) $|\mathcal{F}| \le \displaystyle\prod_{l\in L} \frac{n-l}{k-l}$.

(ii) *If $|\mathcal{F}| \ge 2^{s-1}k^2n^{s-1}$, then $|\bigcap\mathcal{F}| \ge l_1$, where $\bigcap\mathcal{F} = \bigcap_{F\in\mathcal{F}} F$.*

(iii) *If $s \ge 2$ and $|\mathcal{F}| \ge 2^k k^2 n^{s-1}$, then*

$$(l_2 - l_1)|(l_3 - l_2)| \cdots |(l_s - l_{s-1})|(k - l_s),$$

where $a|b$ means a divides b.

On the other hand no general lower bound for $m(n,k,L)$ or even $\alpha(k,L)$ is known. However, Füredi proposed a conjecture which (if true) would give a strong lower bound for $\alpha(k,L)$ in terms of a combinatorial invariant. Before stating the conjecture we must first introduce some notion and auxiliary results. Let $k \in \mathbb{N}$ and $L \subset \{0,1,\dots,k-1\}$ be given. A family $\mathcal{I} \subset 2^{[k]}$ is called a closed L-system if $\mathcal{I}$ is an L-system and $I \cap I' \in \mathcal{I}$ for all (not necessarily distinct) $I, I' \in \mathcal{I}$. The *rank* of $\mathcal{I}$ is defined by

$$\operatorname{rank}(\mathcal{I}) := \min\{t \in \mathbb{N} : \Delta_t(\mathcal{I}) \neq \tbinom{[k]}{t}\},$$

where Δ_t denotes the t-shadow, and then the *rank* of (k,L)-system is defined by

$$\operatorname{rank}(k,L) := \max\{\operatorname{rank}(\mathcal{I}) : \mathcal{I} \subset 2^{[k]} \text{ is a closed } L\text{-system}\}.$$

We say that $\mathcal{I} \subset 2^{[k]}$ is an *intersection structure* of a (k,L)-system if $\mathcal{I}$ is a closed L-system whose rank is $\operatorname{rank}(k,L)$. A *generator set* $\mathcal{I}^*$ of $\mathcal{I}$ is the collection of all maximal elements of $\mathcal{I}$, that is

$$\mathcal{I}^* := \{I \in \mathcal{I} : \nexists I' \in \mathcal{I} \text{ such that } I \subset I',\ I \neq I'\}.$$

We can retrieve $\mathcal{I}$ from $\mathcal{I}^*$ by taking all possible intersections.

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and an edge $F \in \mathcal{F}$ define the *restriction* of $\mathcal{F}$ on F by

$$\mathcal{F}_{|F} := \{F \cap F' : F' \in \mathcal{F} \setminus \{F\}\} \subset 2^F.$$

Moreover, if $\mathcal{F}$ is k-partite with k-partition $[n] = X_1 \sqcup \cdots \sqcup X_k$, namely, if $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$ and $1 \le i \le k$, then we define the *projection* $\pi : \{G \subset [n] : |G \cap X_i| \le 1 \text{ for all } i\} \to [k]$ by

$$\pi(G) := \{i : |G \cap X_i| = 1\},$$

and write $\pi(\mathcal{F}_{|F})$ for $\{\pi(G) : G \in \mathcal{F}_{|F}\}$. Füredi proved the following fundamental result, which was conjectured by Frankl.

Theorem 9.3 ([85, 87])**.** *Given $k \ge 2$ and $L \subset \{0,1,\dots,k-1\}$ there exists a positive constant $c = c(k,L)$ such that every (k,L)-system $\mathcal{F} \subset \binom{[n]}{k}$ contains a k-partite subfamily $\mathcal{F}^* \subset \mathcal{F}$ with k-partition $[n] = X_1 \cup \cdots \cup X_k$ satisfying (i)–(iii).*

(i) $|\mathcal{F}^*| > c|\mathcal{F}|$.
(ii) *If $F_1, F_2 \in \mathcal{F}^*$, then $\pi(\mathcal{F}^*{}_{|F_1}) = \pi(\mathcal{F}^*{}_{|F_2})$. We write* $\mathrm{IS}(\mathcal{F}^*)$ *for this common family in $2^{[k]}$.*
(iii) $\mathrm{IS}(\mathcal{F}^*)$ *is a closed L-system.*

In the above situation, we say that $\mathrm{IS}(\mathcal{F}^*)$ is the *intersection structure* of $\mathcal{F}^*$. We also say that $\mathcal{F}^*$ is a *canonical (k,L)-system* or a *canonical family.* It is an immediate consequence of Theorem 9.3 that for fixed (k,L)-system the rank is an upper bound for the exponent.

Theorem 9.4. *Let k, L be given. If the exponent $\alpha = \alpha(k,L)$ exists then we have*

$$\alpha(k,L) \le \operatorname{rank}(k,L).$$

Proof. Let $\mathcal{F}$ be an (n,k,L)-system with $|\mathcal{F}| = \Theta(n^\alpha)$. Choose a canonical family $\mathcal{F}^*$ from Theorem 9.3. Let $\mathcal{I} = \mathrm{IS}(\mathcal{F}^*) \subset 2^{[k]}$ be the intersection structure and let $t = \operatorname{rank}(\mathcal{I})$. By the definition of rank there is an $I \in \binom{[k]}{t}$ such that $I \notin \Delta_t(\mathcal{I})$.

Then for every G with $\pi(G) = I$ (so $|G \cap X_i| = 1$ for $i \in I$) there is at most one $F \in \mathcal{F}^*$ such that $G \subset F$. Thus the size $|\mathcal{F}^*|$ is at most the number of choices for G, and

$$|\mathcal{F}^*| \leq \prod_{i \in I} |V_i| = O(n^t).$$

Then (i) of Theorem 9.3 yields $|\mathcal{F}| = O(n^t)$ as needed. □

On the other hand, Füredi conjectures the following:

Conjecture 9.2 ([87]). $\alpha(k, L) > \operatorname{rank}(k, L) - 1$.

This conjecture is true if $\operatorname{rank}(k, L) = 2$. In fact if $\mathcal{I} \subset 2^{[k]}$ is a closed L-system with rank at least 2, then there is an (n, k, L)-system $\mathcal{F}$ with $|\mathcal{F}| = \Omega(n^{k/(k-1)})$, see [87]. It is also true if $k \leq 12$ for all L, see [74, 144], where all corresponding exponents are determined. We say that $\mathcal{B} \subset \binom{[k]}{b}$ is a *Steiner system* $S(t, b, k)$ if every $T \subset \binom{[k]}{t}$ there is a unique $B \in \mathcal{B}$ such that $T \subset B$. Thus an $S(t, b, k)$ is a $(k, b, [0, t-1])$-system of size $\binom{k}{t}/\binom{b}{t}$. If there exists a Steiner system $S(t, b, k)$ then we have $\operatorname{rank}(k, L) = t + 1$ for $L = [0, t-1] \cup \{b\}$. Rödl and Tengan found a construction which verifies the conjecture in this situation.

Theorem 9.5 ([132]). *Suppose that a Steiner system $S(t, b, k)$ exists. Then there is $\epsilon > 0$ and a sequence $\mathcal{F}_n$ of k-partite (kn, k, L)-system with $L = [0, t-1] \cup \{b\}$ and $|\mathcal{F}_n| = \Omega(n^{t+\epsilon})$.*

An obvious necessary condition for the existence of $S(t, b, k)$ is that $\binom{b-i}{t-i}$ divides $\binom{k-i}{t-i}$ for all $0 \leq i < t$. Very recently Keevash [104] published a deep result that if this necessary condition is satisfied and $k > k_0(b, t)$ then an $S(t, b, k)$ exists.

Now let $t < k$ and consider an (n, k, L)-system $\mathcal{F}$ with $L = [0, t-1]$. Then for every $T \in \binom{[n]}{t}$ there is at most one $F \in \mathcal{F}$ such that $T \subset F$. This gives $\binom{n}{t} \geq |\mathcal{F}|\binom{k}{t}$ and

$$m(n, k, [0, t-1]) \leq \binom{n}{t}/\binom{k}{t}. \tag{39}$$

Erdős and Hanani conjectured that the bound in (39) is always almost tight provided n is large enough for fixed t and k. Then Rödl proved this conjecture using probabilistic method, which is one of the basic tools used in [104]. Rödl's proof technique was further extended by Frankl and Rödl to obtain the following result stating that almost regular hypergraphs have almost perfect matchings. Here we include a stronger version given by Pippenger.

Theorem 9.6 ([75, 128]). *Let $\mathcal{H} \subset \binom{X}{h}$ satisfy the following.*

1. *There is D such that $\#\{H \in \mathcal{H} : x \in H\} = D$ for all $x \in X$.*
2. *For all $\{x, y\} \in \binom{X}{2}$, $\#\{H \in \mathcal{H} : \{x, y\} \subset H\} = o(D)$ as $D \to \infty$.*

Then there exist pairwise disjoint $H_1, \ldots, H_m \in \mathcal{H}$ with $m \sim |X|/h$ (as $D \to \infty$ and hence $|X| \to \infty$).

See [6] for a proof of even more general cases. Let us present how Theorem 9.6 implies the following.

Theorem 9.7 (Rödl [131])**.** *It follows* $m(n,k,[0,t-1]) = (1-o(1))\binom{n}{t}/\binom{k}{t}$.

Proof. Since we have a trivial inequality (39), it suffices to show that for every $\epsilon > 0$ there is n_0 such that if $n > n_0$ then we can find $m(n,k,[0,t-1])$-system $\mathcal{F}$ with $|\mathcal{F}| > (1-\epsilon)\binom{n}{t}/\binom{k}{t}$.

Let $X = \binom{[n]}{t}$ and $h = \binom{k}{t}$. Define $\mathcal{H} := \{\binom{F}{t} : F \in \binom{[n]}{k}\} \subset \binom{X}{h}$. Then $\mathcal{H}$ is D-regular, where $D = \binom{n-t}{k-t}$. Moreover, for a pair $\{x,y\} \subset X$, we have

$$\#\{H \in \mathcal{H} : \{x,y\} \subset F\} \leq \binom{n-t-1}{k-t-1} = \frac{k-t}{n-t} D = o(D).$$

Thus, by Theorem 9.6, we have a matching $H_1, \ldots, H_m \in \mathcal{H}$ with $m \sim \binom{n}{t}/\binom{k}{t}$. For $1 \leq i \leq m$ we can write $H_i = \binom{F_i}{t}$ for some $F_i \in \binom{[n]}{k}$. Then $|F_i \cap F_j| < t$ for $i \neq j$, and $\mathcal{F} := \{F_1, \ldots, F_m\}$ is a desired $m(n,k,[0,t-1])$-system. □

We mention that Theorem 9.5 is also an application of Theorem 9.6. For some other related results for special L, where L is a union of intervals, see [64, 65, 76, 125].

Linear algebra method is also one the useful tools for studying L-systems. The typical one is the following result due to Ray-Chaudhuri and Wilson, see also [4] for a proof using space of multilinear polynomials.

Theorem 9.8 ([130])**.** *Let* $\mathcal{F}$ *be an* (n,k,L)*-system with* $|L| = s$. *Then* $|\mathcal{F}| \leq \binom{n}{s}$.

Frankl and Wilson obtained a modular version of Theorem 9.8 which has many applications. We write $a \in L \pmod{p}$ if $a \equiv l \pmod{p}$ for some $l \in L$.

Theorem 9.9 (Frankl–Wilson [80])**.** *Let* $n > k \geq s$ *be positive integers, and let* p *be a prime. Let* $L \subset [0, p-1]$ *be a set of* s *integers. Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *satisfies the following:*

(i) $k \notin L \pmod{p}$.
(ii) *If* $F, F' \in \mathcal{F}$ *with* $F \neq F'$, *then* $|F \cap F'| \in L \pmod{p}$.

Then $|\mathcal{F}| \leq \binom{n}{s}$ *follows.*

The condition that p is a prime cannot be dropped in general.

Example 9.1 (Frankl [54])**.** Let $\mathcal{G} := \{G \in \binom{[m]}{11} : \{1,2,3\} \subset G\}$. Then we have $|G \cap G'| \in [3,10]$ for distinct $G, G' \in \mathcal{G}$. Let $p = 6$, $L = \{0,3,4\}$, and let $\mathcal{F} := \{\binom{G}{2} : G \in \mathcal{G}\}$ be a k-uniform family on $n := \binom{m}{2}$ vertices, where $k = \binom{11}{2} = 55 \equiv 1 \pmod{p}$. Then it follows that $|F \cap F'| \in \{\binom{i}{2} : 3 \leq i \leq 10\} \equiv L \pmod{p}$ for distinct $F, F' \in \mathcal{F}$. So $\mathcal{F}$ satisfies (i) and (ii) in Theorem 9.9, but $|\mathcal{F}| = |\mathcal{G}| = \binom{m-3}{8} = \Theta(n^4) \gg \binom{n}{3}$.

Grolmusz obtained a much stronger superpolynomial lower bound. He used a low-degree polynomial representing the Boolean OR function mod m due to Barrington, Eigel and Rudich along with the construction of [54], see also [112].

Theorem 9.10 ([91]). *Let m be a positive integer, and suppose that m has r different prime divisors. Then there exists $c = c(m) > 0$ such that for every integer $h > 0$ there exists a uniform family $\mathcal{H}$ on n vertices such that for $H, H' \in \mathcal{H}$*

$$|H \cap H'| \begin{cases} \equiv 0 \pmod{m} & \text{if } H = H', \\ \not\equiv 0 \pmod{m} & \text{if } H \neq H', \end{cases}$$

and

$$|\mathcal{H}| \geq \exp\left(\frac{c(\log n)^r}{(\log\log n)^{r-1}}\right).$$

In other words, $|\mathcal{H}|$ grows faster than any polynomial of n.

In [91] Grolmusz posed the following question.

Problem 9.1. *Let $\mathcal{F} \subset 2^{[n]}$. Suppose that for $F, F' \in \mathcal{F}$ it follows that*

$$|F \cap F'| \begin{cases} \equiv & 0 \pmod{6} \text{ if } F = F', \\ \not\equiv & 0 \pmod{6} \text{ if } F \neq F'. \end{cases}$$

Then is it true that $|\mathcal{F}| = 2^{o(n)}$?

On the other hand, Babai et al. [9] showed that under the condition of Theorem 9.9 modulo a prime power, it follows that $|\mathcal{F}| \leq \sum_{k=0}^{f(s)} \binom{n}{k}$, where $f(s) \leq 2^{s-1}$.

Finally we list some randomly chosen problems concerning L-systems.

Conjecture 9.3 (Frankl–Füredi[65]). *If $l \geq l'$, then $m(n,k,L) = (1+o(1))\binom{n}{l}\binom{k+l'}{l'}\binom{k+l'}{l}^{-1}$, where $L = [0, l-1] \cup [k-l'+1, k-1]$.*

This conjecture is true if $k - l$ has a prime power divisor q with $q > l'$, see [65].

Conjecture 9.4 (Snevily [140]). *Let p be a prime, and let K and L be disjoint subsets of $[0, p-1]$. Let $\{F_1, F_2, \ldots, F_m\} \subset 2^{[n]}$ be a family such that $|F_i \cap F_j| \pmod{p}$ is in K if $i = j$, and in L if $i \neq j$. Then $m \leq \binom{n}{|L|}$.*

For some recent related results, see [93, 19].

Conjecture 9.5. *Let n, k, p, r be positive integers with $0 \leq r < p$, $p|k$, and let $\mathcal{F} \subset \binom{[n+r]}{k+r}$. Suppose that $|F \cap F'| \equiv r \pmod{p}$ for all distinct $F, F' \in \mathcal{F}$. If $n > n_0(k)$, then $|\mathcal{F}| \leq \binom{\lfloor n/p \rfloor}{k/p}$, where $n_0(k)$ is a polynomial in k.*

This conjecture is true if we drop the condition that $n_0(k)$ is a polynomial, see [79]. When $p = 2$ this result has an application in classification of antipodal sets in oriented real Grassmann manifolds, see [142, 143].

Conjecture 9.6. *$\alpha(24, \{0,1,2,3,4,8\}) = 6$.*

Theorem 9.5 with the existence of the Witt design $S(5,8,24)$ implies that the above exponent is more than 5. On the other hand, Theorem 9.2 yields that the exponent is at most 6. Using the structure of $S(5,6,12)$ it is shown that $\alpha(12, \{0,1,2,3,4,6\}) = 6$ in [144].

10. Application of semidefinite programming

The aim of this section is to present a semidefinite programming (SDP) approach to some intersection problems for readers not familiar with SDP. We consider simple concrete problems, then encode them into SDP problems and solve them. These problems in the next three subsections can be solved as linear programming (LP) problems, or by using completely different method, say, shifting technique. Moreover rewriting the problems into SDP setting looks complicated at the first sight. So one may wonder if SDP approach is a right way. Where is the merit? Well, we will generalize problems step by step, and the corresponding SDP problems can be obtained in almost the same way, only with slight changes. So we can start with an easy problem and reach a rather difficult one without much effort. In the last subsection we obtain Theorem 10.2 which is just a 'correctly' rewritten and extended version of easy Proposition 10.3. This theorem provides some nontrivial results, and the SDP approach is the only known way to prove some of them so far, e.g., Theorem 10.3 and Theorem 10.4. Extending the celebrated LP bound due to Delsarte [29], Schrijver established the SDP bound in [133] and obtained better upper bounds for the size of codes in many cases by solving the corresponding SDP problems directly (using computer). Along this line, de Klerk and Pasechnik obtained the exact value of the independence number of an orthogonality graph, see [110]. In this section, we follow Schrijver's idea, but we will not solve the original problem (called primal form) directly, instead we will find a solution to its 'dual' problem and use the 'weak duality' property, which then will give a sharp bound for the original problem. The authors learned most of the material concerning SDP in this section from Hajime Tanaka.

10.1. **A quick introduction to SDP and its weak duality.** Since SDP is an extension of linear programming (LP) we briefly recall LP and one of its basic properties called weak duality.

The LP problem in *primal form* is

$$\text{(P):}\quad \begin{array}{ll} \text{minimize} & \boldsymbol{c}^\mathsf{T}\boldsymbol{x}, \\ \text{subject to} & A\boldsymbol{x}=\boldsymbol{b}, \\ & \boldsymbol{x}\geq 0, \end{array}$$

where $A\in\mathbb{R}^{m\times n}$ (the set of all $m\times n$ real matrices), $\boldsymbol{b}\in\mathbb{R}^m$, $\boldsymbol{c}\in\mathbb{R}^n$ are given, $\boldsymbol{x}\in\mathbb{R}^n$ is the variable. By $\boldsymbol{x}\geq 0$ we mean that $\boldsymbol{x}$ is nonnegative, that is, every entry of $\boldsymbol{x}$ is nonnegative. We say that $\boldsymbol{x}$ is feasible in (P) if $\boldsymbol{x}$ satisfies the constraints $A\boldsymbol{x}=\boldsymbol{b}$ and $\boldsymbol{x}\geq 0$.

The corresponding *dual form* is

$$\text{(D):}\quad \begin{array}{ll} \text{maximize} & \boldsymbol{b}^\mathsf{T}\boldsymbol{y}, \\ \text{subject to} & \boldsymbol{y}^\mathsf{T}A\leq \boldsymbol{c}^\mathsf{T}, \end{array}$$

where $\boldsymbol{y}\in\mathbb{R}^n$ is the variable. Then we have the following easy but useful fact.

Proposition 10.1 (Weak duality for LP)**.** *If $\boldsymbol{x}$ is feasible in* (P) *and $\boldsymbol{y}$ is feasible in* (D)*, then $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}\geq\boldsymbol{b}^\mathsf{T}\boldsymbol{y}$.*

Proof. Indeed, $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}\geq(\boldsymbol{y}^\mathsf{T}A)\boldsymbol{x}=\boldsymbol{y}^\mathsf{T}(A\boldsymbol{x})=\boldsymbol{y}^\mathsf{T}\boldsymbol{b}=\boldsymbol{b}^\mathsf{T}\boldsymbol{y}$. □

We can use this fact in the following way. Suppose that if $\boldsymbol{x}$ is feasible in (P) and $\boldsymbol{y}$ is feasible in (D). Suppose, moreover, that $\boldsymbol{x}$ and $\boldsymbol{y}$ happen to satisfy $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}$. Then, by the weak duality, we see that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal. In application it often happens that one easily finds a feasible solution $\boldsymbol{x}$ for (P) which is a candidate for an optimal solution. To prove that $\boldsymbol{x}$ is in fact optimal, it suffices to find a feasible solution $\boldsymbol{y}$ for (D) satisfying $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}$. (Of course finding such $\boldsymbol{y}$ is usually more difficult than finding $\boldsymbol{x}$.)

Now we proceed to SDP. In this case variables are taken from the set of real symmetric matrices of a fixed order, say n, denoted by $\mathrm{S}\mathbb{R}^{n\times n}$. For two matrices $A, B \in \mathrm{S}\mathbb{R}^{n\times n}$ we define the inner product by $A \bullet B := \mathrm{tr}(A^{\mathsf{T}}B)$. We say that $A \in \mathrm{S}\mathbb{R}^{n\times n}$ is *positive semidefinite* if $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ (here we assume that $\boldsymbol{x}$ is a column vector), and write $A \succeq 0$. We write $A \geq 0$ if every entry of A is nonnegative. We recall some basic facts from linear algebra.

Fact 10.1. *Let $A, B, X \in \mathrm{S}\mathbb{R}^{n\times n}$.*

(i) *All eigenvalues of A are nonnegative iff $A \succeq 0$.*
(ii) *If $A \succeq 0$ then* $\mathrm{tr}(A) \geq 0$.
(iii) *If $A \succeq 0$ and $B \succeq 0$ then $A \bullet B \geq 0$.*
(iv) *Let $\boldsymbol{a} \in \mathbb{R}^n$ and $A := \boldsymbol{a}(\boldsymbol{a}^{\mathsf{T}}) \in \mathrm{S}\mathbb{R}^{n\times n}$. If $X \in \mathrm{S}\mathbb{R}^{n\times n}$ then $A \bullet X = \boldsymbol{a}^{\mathsf{T}}X\boldsymbol{a}$.*

Proof. First suppose that A has nonnegative eigenvalues $\alpha_1, \ldots, \alpha_n$. Then there exists a nonsingular $P \in \mathrm{S}\mathbb{R}^{n\times n}$ such that $P^{-1}AP$ is a diagonal matrix D with diagonals $\alpha_1, \ldots, \alpha_n$. In this case let $\sqrt{D}$ be a diagonal matrix with diagonals $\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}$, and let $\sqrt{A} := P^{-1}\sqrt{D}P$. Then $\sqrt{A} \in \mathrm{S}\mathbb{R}^{n\times n}$ and $\sqrt{A}\sqrt{A} = A$. So, for any $\boldsymbol{x} \in \mathbb{R}^n$, we have $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}\sqrt{A}\sqrt{A}\boldsymbol{x} = (\sqrt{A}\boldsymbol{x})^{\mathsf{T}}\sqrt{A}\boldsymbol{x} = |\sqrt{A}\boldsymbol{x}|^2 \geq 0$, which means that $\mathcal{A} \succeq 0$. Next suppose that $A \succeq 0$. If α is an eigenvalue of A with eigenvector $\boldsymbol{x} \neq \boldsymbol{0}$, then we have $A\boldsymbol{x} = \alpha\boldsymbol{x}$. It follows that $0 \leq \boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}(\alpha\boldsymbol{x}) = \alpha|\boldsymbol{x}|^2$, which yields $\alpha \geq 0$. This gives us (i), and since $\mathrm{tr}(A)$ is the sum of all eigenvalues, (ii) follows. By (i) we see that A has a square root $\sqrt{A} \succeq 0$. Then it follows that $A \bullet B = \mathrm{tr}(A^{\mathsf{T}}B) = \mathrm{tr}(\sqrt{A}\sqrt{A}B) = \mathrm{tr}(\sqrt{A}B\sqrt{A})$. If $B \succeq 0$ then for every $\boldsymbol{x} \in \mathbb{R}^n$ we have $\boldsymbol{x}^{\mathsf{T}}(\sqrt{A}B\sqrt{A})\boldsymbol{x} = (\sqrt{A}\boldsymbol{x})^{\mathsf{T}}B(\sqrt{A}\boldsymbol{x}) \geq 0$, which means that $\sqrt{A}B\sqrt{A} \succeq 0$. So (ii) implies $\mathrm{tr}(\sqrt{A}B\sqrt{A}) \geq 0$, and noting that the LHS equals to $A \bullet B$ we get (iii). Finally, just noting that $\boldsymbol{a}$ is a column vector, (iv) follows from the definition and simple computation. □

The SDP problem in primal form is

$$\begin{aligned}(\mathrm{P}):\quad &\text{minimize} && C \bullet X,\\ &\text{subject to} && A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m,\\ & && X \succeq 0,\end{aligned}$$

where $A_i \in \mathrm{S}\mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $C \in \mathrm{S}\mathbb{R}^{n\times n}$ are given, and $X \in \mathrm{S}\mathbb{R}^{n\times n}$ is the variable. The corresponding dual form is

$$\begin{aligned}(\mathrm{D}):\quad &\text{maximize} && \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y},\\ &\text{subject to} && \textstyle\sum_{i=1}^{m} y_iA_i + S = C, \qquad S \succeq 0,\end{aligned}$$

where $\boldsymbol{y} \in \mathbb{R}^m$, $S \in \mathrm{S}\mathbb{R}^{n\times n}$ are the variables. As in LP we have the following weak duality in SDP as well.

Proposition 10.2 (Weak duality for SDP)**.** *If X is feasible in* (P) *and* (y, S) *is feasible in* (D)*, then* $C \bullet X \geq \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y}$.

Proof. By the constraints in (P) and (D) we have

$$C \bullet X - \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y} = \left(\sum_i y_i A_i + S \right) \bullet X - \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y} = \sum_i y_i b_i + S \bullet X - \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y} = S \bullet X.$$

Since $X, S \succeq 0$ it follows from Fact 10.1 (iii) that $X \bullet S \geq 0$. □

See, e.g., [148] for more about semidefinite programming in general.

10.2. **Bounding the independence number of a graph.** Let G be a graph on the vertex set Ω with $|\Omega| = n$. Let $U \subset \Omega$ be an independent set, that is, there are no edges between any two vertices in U, and let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ be the characteristic (column) vector of U. Finally let

$$X := \frac{1}{|U|} \boldsymbol{x} \, \boldsymbol{x}^{\mathsf{T}} \in \mathbb{SR}^{n \times n}.$$

Then $X \succeq 0$, in fact, for any $\boldsymbol{y} \in \mathbb{R}^n$ it follows that $\boldsymbol{y}^{\mathsf{T}} X \boldsymbol{y} = \frac{1}{|U|} \boldsymbol{y}^{\mathsf{T}} \boldsymbol{x} \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y} = \frac{1}{|U|} |\boldsymbol{x}^{\mathsf{T}} \boldsymbol{y}|^2 \geq 0$. Also $X \geq 0$ (all entries are nonnegative). Moreover simple computation shows that

$$\begin{aligned} I \bullet X &= \boldsymbol{x}^{\mathsf{T}} I \boldsymbol{x} = \frac{1}{|U|} (x_1^2 + \cdots + x_n^2) = 1, \\ J \bullet X &= \boldsymbol{x}^{\mathsf{T}} J \boldsymbol{x} = \frac{1}{|U|} (x_1 + \cdots + x_n)^2 = |U|, \end{aligned}$$

where I is the identity matrix and J is the all ones matrix. If $A = (a_{ij})$ is the adjacency matrix of G, then it follows that

$$A \bullet X = \frac{1}{|U|} \sum_{i,j} a_{ij} x_i x_j = 0.$$

Indeed if $a_{ij} \neq 0$ then the vertices i and j are adjacent, and so $x_i x_j = 0$ because $\boldsymbol{x}$ is a characteristic vector of an independent set. In other words, $a_{ij} x_i x_j$ is always 0. Consequently X is a feasible solution to the following SDP problem in primal form:

$$\begin{aligned} \text{(P):} \quad & \text{maximize} && J \bullet X, \\ & \text{subject to} && I \bullet X = 1, \\ &&& A \bullet X = 0, \\ &&& X \succeq 0, \quad X \geq 0, \end{aligned}$$

where $A \in \mathbb{SR}^{n \times n}$ is given, and $X \in \mathbb{SR}^{n \times n}$ is the variable. The corresponding dual form is

$$\begin{aligned} \text{(D):} \quad & \text{minimize} && \alpha, \\ & \text{subject to} && \alpha I - J = S + Z + \gamma A, \\ &&& S \succeq 0, \quad Z \geq 0, \end{aligned}$$

where $\alpha, \gamma \in \mathbb{R}$, and $S, Z \in \mathbb{SR}^{n \times n}$ are the variables.

Proposition 10.3 (Weak duality). *If X is feasible in* (P) *and* (α, γ, S, Z) *is feasible in* (D)*, then* $J \bullet X \leq \alpha$.

Proof. We take the bullet product with X on both sides of $\alpha I - J = S + Z + \gamma A$. From the LHS we get $\alpha - J \bullet X$. From the RHS we get $S \bullet X + Z \bullet X \geq 0$, where we use $S \bullet X \geq 0$ (because $S \succeq 0$ and $X \succeq 0$), $Z \bullet X \geq 0$ (because $Z \geq 0$ and $X \geq 0$), and $A \bullet X = 0$. □

So any feasible solution α to (D) gives an upper bound for the optimal solution to (P), which provides an upper bound for the size of an independent set U in G. In summary, the independence number of G is at most α.

Let G be a d-regular graph with n vertices. Let A be the adjacency matrix with eigenvalues $d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$. Since $0 = \operatorname{tr}(A) = \sum_i \lambda_i$ and $\lambda_1 = d > 0$ it follows that $\lambda_n < 0$.

Corollary 10.1 (Hoffman's ratio bound). *The independence number of G is at most*

$$\alpha := \frac{-\lambda_n}{\lambda_1 - \lambda_n} n.$$

Proof. For $1 \leq i \leq n$ let $\boldsymbol{x}_i$ be the eigenvector corresponding to λ_i, where we take $\boldsymbol{x}_1 = \mathbf{1}$ (the all ones vector). We notice that for $2 \leq i \leq n$, $\boldsymbol{x}_i$ is perpendicular to $\boldsymbol{x}_1$ with respect to the standard inner product. We will check that α (defined above), $\gamma := \alpha/\lambda_n$, and $Z := 0$ give a feasible solution to (D). To this end it suffices to show that $S := \alpha I - J - Z - \gamma A = \alpha I - J - (\alpha/\lambda_n) A$ is positive semidefinite, or equivalently, all eigenvalues of S are nonnegative. In fact it follows

$$(\alpha I - J - (\alpha/\lambda_n)A)\, \boldsymbol{x}_1 = (\alpha - n - (\alpha/\lambda_n)\lambda_1)\boldsymbol{x}_1 = 0\boldsymbol{x}_1,$$

and for $2 \leq i \leq n$

$$(\alpha I - J - (\alpha/\lambda_n)A)\, \boldsymbol{x}_i = (\alpha - (\alpha/\lambda_n)\lambda_i)\boldsymbol{x}_i = \alpha(1 - \lambda_i/\lambda_n)\boldsymbol{x}_i,$$

where $\alpha(1 - \lambda_i/\lambda_n) \geq 0$, as desired. □

Let us define the *Kneser graph* $G(n, k, t) = (\Omega, E)$ with $\Omega := \binom{[n]}{k}$ and $u, v \in \Omega$ are adjacent if and only if $|u \cap v| < t$. (Note that u and v are k-element subsets of $[n]$.) Recall that $U \subset \Omega$ is a t-intersecting family if $|u \cap v| \geq t$ for all $u, v \in U$. This is equivalent to the statement that U is an independent set in $G(n, k, t)$.

Let $t = 1$, $n \geq 2k$ and let A be the adjacency matrix of $G(n, k, 1)$. With some efforts one can show that the set of eigenvalues of A is

$$\left\{ (-1)^i \binom{n-k-i}{k-i} : i = 0, 1, \ldots, k \right\}$$

with corresponding multiplicities $\binom{n}{i} - \binom{n}{i-1}$, see, e.g., [89]. If we rearrange the eigenvalues of A as $\lambda_1 \geq \cdots \geq \lambda_N$ where $N = \binom{n}{k}$, then $\lambda_1 = \binom{n-k}{k}$ and $\lambda_N = -\binom{n-k-1}{k-1}$. Thus it follows from Corollary 10.1 that the independence number of $G(n, k, 1)$ is at most

$$\frac{-\lambda_N}{\lambda_1 - \lambda_N} N = \binom{n-1}{k-1}.$$

So this gives us an alternative proof of the Erdős–Ko–Rado theorem for the 1-intersecting case, see Lovász [117].

It is much more difficult to deal with the t-intersecting case for $t \geq 2$ along this line, but Wilson found a way to do this. Instead of the adjacency matrix he used a *pseudo* adjacency matrix $A = (a_{ij})$ of a regular graph G, that is, A is indexed by $\Omega = V(G)$, and

- if $i, j \in \Omega$ and $ij \notin E(G)$, then $a_{ij} = 0$, and
- the all ones vector $\mathbf{1}$ is one of the eigenvectors of A.

If $ij \in E(G)$, then a_{ij} can take any real number. Wilson [152] succeeded to construct a pseudo adjacency matrix of the Kneser graph $G(n, k, t)$ with largest eigenvalue $\binom{n}{k}\binom{n-t}{k-t}^{-1} - 1$ and least eigenvalue -1, where $n \geq (t+1)(k-t+1)$. Thus it follows from Corollary 10.1 that the independence number of $G(n, k, t)$ is at most $\binom{n-t}{k-t}$. In other words, the maximum size of k-uniform t-intersecting family on n vertices is $\binom{n-t}{k-t}$.

10.3. **The measure version.** In the previous subsection we considered the maximum size of an independent set of a graph, and the maximum size of an intersecting families. It is sometimes useful to consider the corresponding measure version described in detail shortly. The SDP approach also works in the measure setting, in fact, it is a natural generalization of the SDP problem we discussed in the previous subsection.

Let $G = (\Omega, E)$ be a regular graph, and let

$$\mu : \Omega \to [0, 1]$$

be a probability measure, that is, $\sum_{x \in \Omega} \mu(x) = 1$. Now we are interested in the maximum of $\mu(U)$, where $U \subset \Omega$ runs over all independent sets in G. If we take a uniform measure $\mu(x) = 1/|\Omega|$ for all $x \in \Omega$, then we get the original problem in the previous subsection. But there is another important measure called *product measure.* To define this let $\Omega = 2^{[n]}$ and let $p \in (0, 1)$ be a fixed real. Then the measure is defined by $\mu(x) = p^{|x|}(1-p)^{n-|x|}$ for $x \in \Omega$, where $|x|$ denotes the cardinality of x as a subset of $[n]$.

Let $U \subset \Omega := \{v_1, v_2, \ldots, v_n\}$ be an independent set, and let $\boldsymbol{x}$ be the characteristic vector of U. Let

$$X := \frac{1}{\mu(U)} \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}} \in \mathbb{SR}^{n \times n}.$$

Then $X \succeq 0$ and $X \geq 0$. Let $\Delta \in \mathbb{SR}^{n \times n}$ be a diagonal matrix with diagonals $\mu(\{v_1\}), \mu(\{v_2\}), \ldots, \mu(\{v_n\})$. Then it follows that

$$\Delta \bullet X = 1,$$
$$\Delta J \Delta \bullet X = \mu(U).$$

Let $E_{ij} \in \mathbb{SR}^{n \times n}$ denote the matrix (indexed by Ω) with a 1 in the (v_i, v_j)-entry and 0 elsewhere. If $v_i \sim v_j$, namely, v_i and v_j are adjacent, then

$$E_{ij} \bullet X = 0.$$

Therefore X is a feasible solution to the following SDP problem in primal form:

$$\begin{array}{lll}\text{(P):} & \text{maximize} & \Delta J\Delta \bullet X,\\ & \text{subject to} & \Delta \bullet X = 1,\\ & & E_{ij} \bullet X = 0 \text{ for } v_i \sim v_j,\\ & & X \succeq 0,\ \ X \geq 0,\end{array}$$

where $X \in \mathbb{SR}^{n\times n}$ is the variable. The corresponding dual form is

$$\begin{array}{lll}\text{(D):} & \text{minimize} & \alpha,\\ & \text{subject to} & \alpha\Delta - \Delta J\Delta = S + Z + \sum_{v_i\sim v_j} \gamma_{ij}E_{ij},\\ & & S \succeq 0, \quad Z \geq 0,\end{array}$$

where $\alpha, \gamma_{ij} \in \mathbb{R}$, and $S, Z \in \mathbb{SR}^{n\times n}$ are the variables. Here we remark that if $\gamma_{ij} \equiv 1$ then $\sum_{v_i\sim v_j} \gamma_{ij}E_{ij}$ is the adjacency matrix of G. One can verify the weak duality as in the proof of Proposition 10.3.

Proposition 10.4 (Weak duality). *If X is feasible in* (P) *and $(\alpha, \gamma_{ij}, S, Z)$ is feasible in* (D)*, then $\Delta J\Delta \bullet X \leq \alpha$.*

As a consequence any feasible solution α to (D) provides an upper bound for the maximum measure $\mu(U)$ where U runs over all independent sets in G.

Example 10.1. Let $G = (\Omega, E)$ be a graph with $\Omega = 2^{[n]}$ and $u \sim v$ iff $u \cap v = \emptyset$, namely two vertices u, v in G are adjacent if and only if they are disjoint as subsets of $[n]$. Fix $p, q \in (0, 1)$ with $p + q = 1$, and let $\mu : \Omega \to [0, 1]$ be the *product measure*, that is, $\mu(v) := p^{|v|}q^{n-|v|}$. If $p \leq 1/2$ and $U \subset \Omega$ is an independent set (in other words, $U \subset 2^{[n]}$ is intersecting), then

$$\mu(U) \leq p.$$

Proof. It suffices to find a feasible solution to (D) with optimal value $\alpha = p$. If $n = 1$ then $U = \{\{1\}\}$ is the only (nonempty) independent set in G, and $\mu(U) = p$ follows. But we construct an optimal solution to (D) carefully, because it can be expanded to the general case quite easily. The trick due to Friedgut [82] is to use tensor product.

So let $n = 1$. Let $c = p/q$ and define

$$A := \begin{bmatrix} 1-c & c \\ 1 & 0 \end{bmatrix}, \quad D := \begin{bmatrix} 1 & 0 \\ 0 & -c \end{bmatrix}, \quad V := \begin{bmatrix} 1 & \sqrt{c} \\ 1 & -\sqrt{1/c} \end{bmatrix}, \quad \Delta := \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix},$$

where the rows and columns are indexed in the order $\emptyset$, $\{1\}$. Then it follows that

$$AV = VD, \quad V^{\mathsf{T}}\Delta V = I, \quad (\Delta A)^{\mathsf{T}} = \Delta A, \tag{40}$$

from which it follows that

$$V^{\mathsf{T}}(\Delta A)V = D. \tag{41}$$

We also have that

$$V^{\mathsf{T}}(\Delta J\Delta)V = E_{11}, \tag{42}$$

that is, $(1, 1)$-entry is a 1 and 0 elsewhere. Then we construct an optimal solution by letting $\alpha = p$, $Z = 0$, and $\sum \gamma_{ij}E_{ij} = q\Delta A$ (that is, $\gamma_{11} = q(q-p)$, $\gamma_{12} = \gamma_{12} = -pq$). In this case $S = p\Delta - \Delta J\Delta + q\Delta A$ is the zero matrix and $S \succeq 0$, as needed.

Next we move to the general case $n \geq 2$. Let $A_n := A \otimes \cdots \otimes A$ be an $2^n \times 2^n$ matrix obtained by taking n-folded tensor of the 2×2 matrix A. We naturally identify $2^{\{1\}} \times \cdots \times 2^{\{n\}}$ with $\Omega = 2^{[n]}$, and we understand that A is indexed by Ω. Then it follows that

$$\text{if } u \not\sim v \text{ in } G \text{ then } (u,v)\text{-entry of } A_n \text{ is } 0. \tag{43}$$

We define D_n, V_n, Δ_n in the same manner. These new matrices again satisfy (40), (41) and (42). Now an optimal solution to (D) will be given by $\alpha = p$, $Z = 0$, and $\sum \gamma_{ij} E_{ij} = q\Delta_n A_n$, where γ_{ij}'s are well-defined by (43). We have to show that $S := p\Delta_n - \Delta_n J \Delta_n + q\Delta_n A_n$ is positive semidefinite. By (40), (41) and (42) we have

$$V_n^{\mathsf{T}} S V_n = pI - E_{11} + qD_n \cong \bigoplus_{z \in 2^{[n]}} S^{(z)},$$

where

$$S^{(\emptyset)} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S^{(z)} := \left(p + q(-c)^{|z|}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ (z \neq \emptyset).$$

Since V_n is nonsingular it follows that $S \succeq 0$ if and only if $V_n^{\mathsf{T}} S V_n \succeq 0$. So we need to check that $S^{(z)} \succeq 0$ for all $z \subset [n]$. This follows from $p + q(-c)^j = p + q(-p/q)^j \geq 0$ for all $1 \leq j \leq n$. Indeed, this is clear if j is even, and if j is odd then this is equivalent to $p/q \geq (p/q)^j$, which follows from $0 < p/q < 1$ (because $p < 1/2 < q$). □

The matrices used in the above proof were introduced by Friedgut in [82]. Actually he found matrices corresponding to t-intersecting families to show the following result.

Theorem 10.1. *Let $0 < p < \frac{1}{t+1}$ and let μ be the product measure. If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then $\mu(\mathcal{F}) \leq p^t$.*

His proof can be viewed as a measure version of Wilson's proof of the Erdős–Ko-Rado Theorem. Both proofs are within scope of the Delsarte LP bound [29]. In the next subsection we present examples of intersection problems that cannot be reduced to LP problems and require the full strength of the SDP approach.

10.4. **The bipartite graph version.** In this subsection, we consider a problem of bounding the measures of cross independent sets in a bipartite graph as an SDP problem.

Let Ω_1, Ω_2 be finite sets, and let $\Omega := \Omega_1 \sqcup \Omega_2$. Let G be a bipartite graph with bipartition $V(G) = \Omega = \Omega_1 \sqcup \Omega_2$. We say that $U_1 \subset \Omega_1$ and $U_2 \subset \Omega_2$ are *cross independent* if there are no edges between U_1 and U_2 in G. For $i = 1, 2$, let μ_i be a probability measure on Ω_i. We are interested in the maximum of $\mu_1(U_1)\mu_2(U_2)$.

Example 10.2. Let n, k, l, t be positive integers, and let $\Omega_1 = \binom{[n]}{k}$ and $\Omega_2 = \binom{[n]}{l}$. For $x \in \Omega_1$ and $y \in \Omega_2$ let $x \sim y$ if $|x \cap y| < t$. Let $\mu_i(z) = 1/|\Omega_i|$ for every $z \in \Omega_i$. In this case to determine the maximum of $\mu_1(U_1)\mu_2(U_2)$ for cross independent $U_1 \subset \Omega_1$ and $U_2 \subset \Omega_2$ is equivalent to determine the maximum of the product of sizes $|\mathcal{A}||\mathcal{B}|$ for cross t-intersecting families $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{l}$.

We are going to explain how to encode a problem such as Example 10.2 as a positive semidefinite problem. Let $\mathbb{R}^{\Omega \times \Omega}$ be the set of real matrices with rows and columns indexed by Ω, and let $\mathbb{R}^{\Omega}$ be the set of real column vectors with coordinates indexed by Ω. The sets $\mathbb{R}^{\Omega_i\times\Omega_j}$ and $\mathbb{R}^{\Omega_i}$ are similarly defined. Let $\mathrm{S}\mathbb{R}^{\Omega\times\Omega}$ be the set of symmetric matrices in $\mathbb{R}^{\Omega\times\Omega}$.

Now, suppose that $U_1 \subset \Omega_1, U_2 \subset \Omega_2$ are cross independent in G. Let $\boldsymbol{x}_i \in \mathbb{R}^{\Omega_i}$ be the characteristic vector of U_i, and let $\boldsymbol{x} := (\boldsymbol{x}_1/\sqrt{\mu_1(U_1)}, \boldsymbol{x}_2/\sqrt{\mu_2(U_2)}) \in \mathbb{R}^{\Omega}$ be a column vector. Define a matrix X representing U_1, U_2 by

$$X := \boldsymbol{x}\,\boldsymbol{x}^{\mathsf{T}} \in \mathrm{S}\mathbb{R}^{\Omega\times\Omega}.$$

Then $X \succeq 0$ and $X \geq 0$. Let $J_{ij} \in \mathbb{R}^{\Omega_i\times\Omega_j}$ be the all ones matrix, and let $\Delta_i \in \mathbb{R}^{\Omega_i\times\Omega_i}$ be the diagonal matrix whose (x,x)-entry is $\mu_i(\{x\})$ for $x \in \Omega_i$. Then it follows that

$$\frac{1}{2}\begin{bmatrix} 0 & \Delta_1 J_{12}\Delta_2 \\ \Delta_2 J_{21}\Delta_1 & 0 \end{bmatrix} \bullet X = \sqrt{\mu_1(U_1)\mu_2(U_2)}, \tag{44}$$

$$\begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_2 \end{bmatrix} \bullet X = 1. \tag{45}$$

For $x \in \Omega_i, y \in \Omega_j$, let $E_{xy} \in \mathbb{R}^{\Omega_i\times\Omega_j}$ be the matrix with a 1 in the (x,y)-entry and 0 elsewhere. Then,

$$\begin{bmatrix} 0 & E_{xy} \\ E_{yx} & 0 \end{bmatrix} \bullet X = 0 \quad \text{for} \quad x \in \Omega_1,\ y \in \Omega_2,\ x \sim y, \tag{46}$$

So X is a feasible solution to the following SDP problem in primal form:

$$\begin{aligned} \text{(P):}\quad & \text{maximize} && \text{the LHS of (44)} \\ & \text{subject to} && \text{(45), (46)}, \\ &&& X \succeq 0,\ X \geq 0, \end{aligned}$$

where $X \in S\mathbb{R}^{\Omega\times\Omega}$ is the variable, and $X \geq 0$ means that X is nonnegative.

The corresponding problem in dual form is

$$\begin{aligned} \text{(D):}\quad & \text{minimize} && \alpha+\beta \\ & \text{subject to} && \begin{bmatrix} \alpha\Delta_1 & 0 \\ 0 & \beta\Delta_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & \Delta_1 J_{12}\Delta_2 \\ \Delta_2 J_{21}\Delta_1 & 0 \end{bmatrix} \\ &&& \quad = S + Z + \sum_{x\sim y}\gamma_{xy}\begin{bmatrix} 0 & E_{xy} \\ E_{yx} & 0 \end{bmatrix}, \\ &&& S \succeq 0,\ Z \geq 0, \end{aligned}$$

where $\alpha, \beta, \gamma_{xy} \in \mathbb{R}$, and $S, Z \in S\mathbb{R}^{\Omega\times\Omega}$ are the variables. Then one can routinely verify the weak duality: if X is feasible in (P) and $(\alpha, \beta, \gamma_{xy}, S, Z)$ is feasible in (D), then $\Delta J \Delta \bullet X \leq \alpha + \beta$. This immediately yields the following.

Theorem 10.2 ([139])**.** *Let G be a bipartite graph with bipartition $V(G) = \Omega_1 \sqcup \Omega_2$, and let μ_i be a probability measure on Ω_i for $i = 1, 2$. Suppose that $U_1 \subset \Omega_1, U_2 \subset \Omega_2$ are cross independent in G. If $(\alpha, \beta, \gamma_{xy}, S, Z)$ is feasible in (D), then*

$$\mu_1(U_1)\mu_2(U_2) \leq (\alpha+\beta)^2.$$

We will present some applications of Theorem 10.2. We start with extending Corollary 10.1 for bipartite graphs. For this we need some preparation.

Let A_{12} be the bipartite adjacency matrix of G with rows indexed by $\Omega_1 = \{x_1, \ldots, x_m\}$ and columns indexed by $\Omega_2 = \{y_1, \ldots, y_n\}$, that is, (x, y)-entry of A_{12}

is 1 if $x \sim y$ and 0 otherwise, and let $A_{21} := A_{12}^{\mathsf{T}}$. Then the adjacency matrix A of G is

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}.$$

Definition 10.1. We say that G is *biregular with respect to* μ_1, μ_2 if the following conditions are satisfied.

(P1) For $i = 1, 2$ there exists $V_i \in \mathbb{R}^{\Omega_i \times \Omega_i}$ whose first column is the all ones vector.
(P2) There exists $D_{12} \in \mathbb{R}^{\Omega_1 \times \Omega_2}$ and $D_{21} := D_{12}^{\mathsf{T}}$ which have (the same) diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{m,n\}} \geq 0$ and 0 elsewhere.
(P3) By letting

$$V := \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}, \quad D := \begin{bmatrix} 0 & D_{12} \\ D_{21} & 0 \end{bmatrix}, \quad \Delta := \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix},$$

these matrices satisfy

$$AV = VD, \quad V^{\mathsf{T}} \Delta V = I, \quad (\Delta A)^{\mathsf{T}} = \Delta A.$$

Call the σ_i's *singular values* of G with respect to μ_1, μ_2.

If G is biregular with respect to μ_1, μ_2, then it follows that

$$V^{\mathsf{T}} \Delta A V = D, \quad V_1^{\mathsf{T}} \Delta_1 J_{12} \Delta_2 V_2 = E_{11}. \tag{47}$$

In fact, for the latter, $V^{\mathsf{T}} \Delta V = I$ implies $\sum_l (\Delta_1)_l (V_1)_{li} (V_1)_{lj} = \delta_{ij}$, and using (P1) with $j = 1$ it follows $\sum_l (\Delta_1)_l (V_1)_{li} = \delta_{i1}$, so (i,j)-entry of $(V_1^{\mathsf{T}} \Delta_1 J_{12} \Delta_2 V_2)$ is

$$\sum_k (V_1)_{ki} \sum_l (\Delta_1)_k (\Delta_2)_l (V_2)_{lj} = \sum_k (\Delta_1)_k (V_1)_{ki} \sum_l (\Delta_2)_l (V_2)_{lj} = \delta_{i1} \delta_{j1},$$

as needed. If we compare (P3) and (47) with (40), (41) and (42), then we see that this bipartite graph version of SDP is a natural generalization of Example 10.1.

Corollary 10.2. *Let G be a bipartite graph with bipartition $\Omega_1 \sqcup \Omega_2$, and for $i = 1, 2$ let μ_i be a probability measure on Ω_i. Suppose that G is biregular with respect to μ_1, μ_2 with largest two singular values $\sigma_1 > \sigma_2 > 0$. If $U_1 \subset \Omega_1$ and $U_2 \subset \Omega_2$ are cross independent, then*

$$\sqrt{\mu_1(U_1)\mu_2(U_2)} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2},$$

Proof. We construct a feasible solution to (D) with objective value $\theta := \frac{\sigma_2}{\sigma_1 + \sigma_2}$. To this end we let $\alpha = \beta = \theta/2$, $Z = 0$, and we need to show that by choosing γ suitably it follows that

$$S := \frac{\theta}{2} \Delta - \frac{1}{2} \begin{bmatrix} 0 & \Delta_1 J_{12} \Delta_2 \\ \Delta_2 J_{21} \Delta_1 & 0 \end{bmatrix} + \gamma \Delta A \succeq 0,$$

where A is the adjacency matrix of G and Δ is defined in (P3).

Using the biregular property it follows

$$V^{\mathsf{T}} S V = \frac{\theta}{2} I - \frac{1}{2} \begin{bmatrix} 0 & E_{11} \\ E_{11} & 0 \end{bmatrix} + \gamma D \cong \bigoplus_i S^{(i)},$$

where

$$S^{(1)} := \frac{1}{2}\begin{bmatrix} \theta & 2\gamma\sigma_1 - 1 \\ 2\gamma\sigma_1 - 1 & \theta \end{bmatrix}, \quad S^{(i)} := \frac{1}{2}\begin{bmatrix} \theta & 2\gamma\sigma_i \\ 2\gamma\sigma_i & \theta \end{bmatrix} \ (i \geq 2).$$

To show $S \succeq 0$ it suffices to show $V^\mathsf{T} S V \succeq 0$, or $S^{(i)} \succeq 0$ for all i. This can be done by choosing γ so that $\det(S^{(i)}) \geq 0$, e.g., $\gamma = \frac{1}{2(\sigma_1+\sigma_2)}$. □

Corollary 10.2 is a cheap application of Theorem 10.2, but if $\mu_1 = \mu_2$ (and $\Omega_1 = \Omega_2$), then it provides a sharp upper bound for the product of measures in some cases, see e.g., [145, 146]. The cases when $\mu_1 \neq \mu_2$ are more difficult to deal with. It was only 2014 when Suda and Tanaka came up with the following vector space version of the Erdős–Ko–Rado theorem for cross intersecting families by making the best use of Theorem 10.2.

Theorem 10.3 (Suda–Tanaka [138])**.** *Let V be an n-dimensional vector space over $\mathbb{F}_q$. For $i = 1, 2$, let $n \geq 2k_i$ and $U_i \subset \begin{bmatrix} V \\ k_i \end{bmatrix}$. If $\dim(x \cap y) \geq 1$ for all $x \in U_1$ and $y \in U_2$, then*

$$|U_1||U_2| \leq \begin{bmatrix} n-1 \\ k_1 - 1 \end{bmatrix} \begin{bmatrix} n-1 \\ k_2 - 1 \end{bmatrix}.$$

They constructed a feasible solution to the dual problem (D) as a one-parameter family, where the use of $Z > 0$ is inevitable for the optimality. The same proof can be applied to obtain the corresponding result for families of subsets (and it is even easier). We also get a bipartite graph version of Example 10.1.

Example 10.3. Let G be a bipartite graph with bipartition $\Omega_1 \sqcup \Omega_2$ with $\Omega_i = 2^{[n]}$, and $x \sim y$ iff $x \cap y = \emptyset$ for $x \in \Omega_1$ and $y \in \Omega_2$. For $i = 1, 2$ let $p_i \in (0, 1/2]$ and let $\mu_i : \Omega_i \to [0, 1]$ be the product measure. If $U_1 \subset \Omega_1$ and $U_2 \subset \Omega_2$ are cross independent (in other words, $U_1, U_2 \subset 2^{[n]}$ are cross intersecting), then

$$\mu_1(U)\mu_2(U_2) \leq p_1 p_2.$$

Sketch of Proof. We construct an optimal feasible solution to (D) and apply Theorem 10.2. The construction is very similar to that of Example 10.1. Assume that $p_1 \geq p_2$. For the case $n = 1$ let $c_i := \sqrt{p_i/q_i}$, and define

$$A_{ij} := \begin{bmatrix} 1 - \frac{p_j}{q_i} & \frac{p_j}{q_i} \\ 1 & 0 \end{bmatrix}, \quad D_{ij} := \begin{bmatrix} 1 & 0 \\ 0 & -c_i c_j \end{bmatrix}, \quad V_i := \begin{bmatrix} 1 & c_i \\ 1 & -\frac{1}{c_i} \end{bmatrix}, \quad \Delta_i := \begin{bmatrix} q_i & 0 \\ 0 & p_i \end{bmatrix}.$$

Then we have $V_i^\mathsf{T}(\Delta_i A_{ij})V_j = D_{ij}$ and $(\Delta_i A_{ij})^\mathsf{T} = \Delta_j A_{ji}$. We choose

$$\alpha = \beta = \sqrt{p_1 p_2}/2, \quad \sum_{x \sim y} \gamma_{xy} E_{xy} = \eta \Delta_1 A_{12}, \quad Z = \begin{bmatrix} \epsilon_1 \Delta_1 A_{11} & 0 \\ 0 & \epsilon_2 \Delta_2 A_{22} \end{bmatrix},$$

where ϵ_2 will be used as a parameter. After some computation one can verify that $S \succeq 0$, where

$$\begin{bmatrix} \alpha\Delta_1 & 0 \\ 0 & \beta\Delta_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & \Delta_1 J_{12} \Delta_2 \\ \Delta_2 J_{21} \Delta_1 & 0 \end{bmatrix} = S + Z + \begin{bmatrix} 0 & \eta\Delta_1 A_{12} \\ \eta\Delta_2 A_{21} & 0 \end{bmatrix},$$

if and only if

$$\epsilon_1 = \frac{p_2}{p_1}\epsilon_2 + \frac{(p_1 - p_2)p_2}{2\sqrt{p_1 p_2}}, \quad \eta = \frac{p_2}{\sqrt{p_1 p_2}}\epsilon_2 + \frac{q_2}{2}, \quad 0 \leq \epsilon_2 \leq \sqrt{p_1 p_2}/2.$$

In this case we get a feasible solution of (D) with objective value $\sqrt{p_1 p_2}$, and this completes the proof for the case $n = 1$. Finally the general case $n \geq 2$ follows from this with the tensor product trick as in Example 10.1. □

We present one more application which extends Example 10.3. Let us define a probability measure μ on $\Omega = 2^{[n]}$ with respect to a probability vector $\boldsymbol{p} = (p^{(1)}, p^{(2)}, \ldots, p^{(n)}) \in (0,1)^n$ as follows: For $U \subset \Omega$ let

$$\mu(U) := \sum_{x \in U} \prod_{l \in x} p^{(l)} \prod_{k \in [n] \setminus x} (1 - p^{(k)}).$$

This measure was introduced by Fishburn, Frankl, Freed, Lagarias, and Odlyzko in [46], where they considered the maximum measure for intersecting families. In [139] the following extension of their result to cross intersecting families is obtained based on Theorem 10.2.

Theorem 10.4 ([139]). *For $i = 1, 2$, let μ_i be a probability measure on $\Omega_i = 2^{[n]}$ with respect to a probability vector $\boldsymbol{p}_i = (p_i^{(1)}, \ldots, p_i^{(n)})$, and let $U_i \subset \Omega_i$. Suppose that U_1 and U_2 are cross intersecting.*

(i) *If $p_i^{(1)} = \max\{p_i^{(l)} : 1 \leq l \leq n\}$ for $i = 1, 2$, and $p_i^{(l)} \leq 1/2$ for all $i = 1, 2$, $2 \leq l \leq n$, then $\mu_1(U_1)\mu_2(U_2) \leq p_1^{(1)} p_2^{(1)}$.*

(ii) *If $p_1^{(1)} p_2^{(1)} = \max\{p_1^{(l)} p_2^{(l)} : 1 \leq l \leq n\}$, and $p_i^{(l)} \leq 1/3$ for all $i = 1, 2$, $1 \leq l \leq n$, then $\mu_1(U_1)\mu_2(U_2) \leq p_1^{(1)} p_2^{(1)}$.*

We remark that we do not require $p_i^{(1)} \leq 1/2$ in (i).

Borg considered a problem concerning cross intersecting integer sequences in [15], and obtained similar results to Theorem 10.4 using shifting technique under assumption that $p_i^{(1)} \geq p_i^{(2)} \geq \cdots \geq p_i^{(n)}$.

Conjecture 10.1 ([139]). *(ii) of Theorem 10.4 is still valid if we replace the condition $p_i^{(l)} \leq 1/3$ with $p_i^{(l)} \leq 1/2$.*

It is interesting that if this conjecture is true then each family in the optimal case is intersecting, but not necessarily measure maximal.

Another interesting problem is to obtain cross t-intersecting version of the Erdős–Ko–Rado Theorem (or its measure version).

Problem 10.1. *For $i = 1, 2$, let μ_i be a probability measure on $\Omega_i = 2^{[n]}$ with respect to a probability vector $\boldsymbol{p}_i$. Determine or estimate $\max \mu_1(U_1)\mu_2(U_2)$ where $U_1 \subset \Omega_1$ and $U_2 \subset \Omega_2$ run over all cross t-intersecting families.*

See [16, 70, 122, 127, 145, 146] for some related results.

Finally we mention that Borg established the following striking result using a purely combinatorial argument very recently.

Theorem 10.5 (Borg [17]). *Let $7 \le t \le a \le b$ and $n \ge (t+2)(b-t)+a-1$. If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross t-intersecting, then $|\mathcal{A}||\mathcal{B}| \le \binom{n-t}{a-t}\binom{n-t}{b-t}$.*

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