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A POISSON ALLOCATION OF OPTIMAL TAIL

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The allocation problem for a d -dimensional Poisson point process is to find a way to partition the space to parts of equal size, and to assign the parts to the configuration points in a measurable, “deterministic” (equivariant) way. The goal is to make the diameter R of the part assigned to a configuration point have fast decay. We present an algorithm for $d \geq 3$ that achieves an $O(\exp(-cR^d))$ tail, which is optimal up to c . This improves the best previously known allocation rule, the gravitational allocation, which has an $\exp(-R^{1+o(1)})$ tail. The construction is based on the Ajtai–Komlós–Tusnády algorithm and uses the Gale–Shapley–Hoffman–Holroyd–Peres stable marriage scheme (as applied to allocation problems).

1. Introduction. Consider the random discrete point set ω in \mathbb{R}^d given by the Poisson point process of intensity 1. We would like to find functions $\psi_\omega : \omega \rightarrow L^1(\mathbb{R}^d)$ that assign to each point of ω a set of measure 1, and such that ψ_ω is a measurable, equivariant function of ω . Then we call $\omega \mapsto \psi_\omega$ an allocation rule. See Definition 2 for more details.

We prove the existence of an allocation rule of the following (optimal) tail.

THEOREM 1.1. *For $d \geq 3$ there exist $c, b > 0$ and an allocation rule $\omega \mapsto \psi_\omega$ for the Poisson point process such that*

$$\mathbf{P}[\text{diam}(\{0\} \cup \psi_\omega(0)) > R | 0 \in \omega] \leq c \exp(-bR^d).$$

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The proof of Theorem 1.1 is constructive. The allocation rule presented here is built upon a generalization of the algorithm due to Ajtai, Komlós and Tusnády [1] and employs a local variant of the stable marriage allocation introduced by Hoffman, Holroyd and Peres [4], based on the “stable marriage” algorithm of Gale and Shapley.

Informally, in this allocation problem, we want to divide land between a set of farmers randomly scattered in space, and in such a way that each farmer knows the rule to determine his own land up to a small error, by looking at the locations of other farmers in a big enough neighborhood. This rule has to be the same for everybody. We want to find an allocation rule where the maximal distance a farmer has to walk (fly) between any two points of his land is minimal. That is, we want the tail

$$\mathbf{P}[\text{diam}(\{0\} \cup \psi_\omega(0)) > R | 0 \in \omega]$$

to decay as fast in R as possible, where $\psi_\omega(0)$ denotes the cell of 0 (conditioned on having a center in 0).

Now we give the precise definitions. In our previous, less formal definition, we wanted to partition the space and assign pieces to the centers. Here it will be more convenient to use the indicator functions of these pieces. Let Ω be the set of discrete sets of points in \mathbb{R}^d . The points of an $\omega \in \Omega$ are called *centers*. Let us fix an $\omega \in \Omega$.

DEFINITION 1 (Allocation). An *allocation* is a function $\psi_\omega : \omega \rightarrow L^1(\mathbb{R}^d)$ such that:

- (1) for every $\xi \in \omega$, $\psi_\omega(\xi)$ is a function with values from $\{0, 1\}$;
- (2) for Lebesgue almost every $x \in \mathbb{R}^d$, there is at most one $\xi \in \omega$ such that $\psi_\omega(\xi)(x) = 1$;
- (3) for every $\xi \in \omega$, $\int_{\mathbb{R}^d} \psi_\omega(\xi)(x) dx = 1$.

Without assumption (3) we call ψ_ω a *weak allocation*. We say that $\psi_\omega : \omega \rightarrow L^1(\mathbb{R}^d)$ is a *weak fractional allocation* if (1') and (2'), below, hold and a *fractional allocation* if (1'), (2') and (3) hold:

- (1') for every $\xi \in \omega$, $\psi_\omega(\xi)$ is a function with values from $[0, 1]$;
- (2') for Lebesgue almost every $x \in \mathbb{R}^d$, $\sum_{\xi \in \omega} \psi_\omega(\xi)(x) \leq 1$.

Sometimes it will be natural to think about a (fractional) allocation $\psi_\omega : \omega \rightarrow L^1(\mathbb{R}^d)$ as a family $\{\psi_\omega(\xi) : \xi \in \omega\}$ of functions.

DEFINITION 2 (Allocation scheme, allocation rule). Let \mathbf{P} be the law of the Poisson point process of intensity 1 in \mathbb{R}^d . An *allocation scheme* is a mapping $\omega \mapsto \psi_\omega$ that is defined for \mathbf{P} -almost every $\omega \in \Omega$, measurable (i.e., if (S, \mathcal{S}, μ) is the underlying probability space of the Poisson point process,

and $L^1(\mathbb{R}^d)$ is equipped with the Borel sets induced by the L^1 -distance, the mapping $s \mapsto (\omega(s), \psi_{\omega(s)})$ from $S \rightarrow \{(\omega, \phi) | \omega \in \Omega, \phi \in [L^1(\mathbb{R}^d)]^\omega\}$ is measurable with respect to \mathcal{S} , and such that almost surely ψ_ω is an allocation. If, furthermore, the mapping $\omega \mapsto \psi_\omega$ is translation-equivariant; that is, for any $\omega \in \Omega$, $\xi \in \omega$ and $x, y \in \mathbb{R}^d$ we have

$$\psi_{\omega+x}(\xi+x)(y+x) = \psi_\omega(\xi)(y),$$

then we call the allocation scheme an *allocation rule*. Define *weak*, *fractional* and *weak fractional allocation rules (schemes)* analogously.

Some use the term *allocation factors* of the point process for allocation rules. We mention that allocation rules satisfy something stronger than (2) from Definition 1; namely, for Lebesgue almost every $x \in \mathbb{R}^d$, there is *exactly one* $\xi \in \omega$ such that $\psi_\omega(\xi)(x) = 1$; see, for example, [7] for a proof of this simple statement.

Denote by 0 the origin in \mathbb{R}^d and by λ the Lebesgue measure in \mathbb{R}^d . The letters b and c will stand for positive real constants, whose value may change when employed in different statements. From now on, *we always assume that $d \geq 3$* .

We want to define an allocation rule in such a way that the probability $\mathbf{P}[\text{diam}(\psi_{\omega \cup \{0\}}(0) \cup \{0\}) \geq R | 0 \in \omega]$ decays as fast as possible. By translation invariance we could have taken any other fixed $\xi_0 \in \mathbb{R}^d$ instead of 0, and the tail would be the same.

Define $\Omega' := \{\omega \cup \{0\} : \omega \in \Omega\}$ and $\mathbf{P}' = [\cdot | 0 \in \omega]$ as the Palm version of the probability measure \mathbf{P} that defines the Poisson point process (and \mathbf{E}' the corresponding expectation). To facilitate readability we will tend to use the notation ω' for elements of Ω' . It is well known that $\mathbf{P}'[\{\omega \cup \{0\} : \omega \in E\}] = \mathbf{P}[E]$ for every measurable E in Ω ; see, for example, [10] for the proof of this statement and for other basic facts and definitions about the Palm version of point processes. Rephrasing the previous paragraph, our goal is to make $\mathbf{P}'[\text{diam}(\psi_{\omega'}(0) \cup \{0\}) > R]$ have fast decay.

In several papers ([4] and follow-up works) a slightly different problem is the focus of interest. Suppose we have the allocation rules $\omega \mapsto \psi_\omega$ and $\omega \mapsto \phi_\omega$. The rule $\omega \mapsto \psi_\omega$ defines a unique center $\xi_0 = \xi_0(\omega, \psi_\omega)$ with $0 \in \psi_\omega(\xi_0)$ almost surely. Then

$$(1.1) \quad \mathbf{P}[\text{diam}(\phi_{\omega-\xi_0}(0) \cup \{0\}) > R] = \mathbf{P}'[\text{diam}(\phi_{\omega'}(0) \cup \{0\}) > R],$$

where the equation follows from Theorem 13 in [7] (the claim that ξ_0 is a so-called nonrandomized extra head scheme).

The objective in the setup of the cited papers is again to obtain a rule with optimal tail bound, this time for the random variable $|\xi_0|$. Our setup is stronger than this one, meaning that for any allocation rule $\omega \mapsto \psi_\omega$ the

tail probability $\mathbf{P}[\text{diam}(\{0\} \cup \psi_\omega(0)) > R | 0 \in \omega]$ is greater than or equal to $\mathbf{P}[|\xi_0(\omega, \psi_\omega)| > R]$, hence any upper bound on the decay of the former (as in Theorem 1.1) implies an upper bound on the decay of the latter (as in [4]). To see this, set $\phi = \psi$ and apply (1.1), so that we have

$$\begin{aligned}
 \mathbf{P}[|\xi_0| > R] &\leq \mathbf{P}\left[\max_{x \in \psi_\omega(\xi_0)} |x - \xi_0| > R\right] \leq \mathbf{P}[\text{diam}(\psi_\omega(\xi_0) \cup \{\xi_0\}) > R] \\
 (1.2) \quad &= \mathbf{P}[\text{diam}(\psi_{\omega - \xi_0}(0) \cup \{0\}) > R] \\
 &= \mathbf{P}'[\text{diam}(\psi_{\omega'}(0) \cup \{0\}) > R].
 \end{aligned}$$

In fact, the current setup is strictly stronger: imagine an allocation rule when most cells consist of ball-like pieces of almost unit volume containing the corresponding center, and a small extra piece far away from this one. For such an allocation, the quantity on the left of (1.1) would decay quickly, while on the right we could have slow decay.

The relation between the two discussed tail events allows us to phrase a lower bound on $\mathbf{P}'[\text{diam}(\psi_{\omega'}(0) \cup \{0\}) > R]$ which every allocation rule has to satisfy. Clearly,

$$(1.3) \quad \mathbf{P}[|\xi_0| > R] \geq \mathbf{P}[B(0, R) \cap \omega = \emptyset] = \exp(-\lambda(B(0, R))) = \exp(-cR^d),$$

where $B(0, R)$ is the Euclidean ball around 0 with radius R , and $c > 0$ is a constant only depending on the dimension d . Therefore it follows through (1.2) for any allocation rule $\omega \mapsto \psi_\omega$ that

$$\exp(-cR^d) \leq \mathbf{P}'[\text{diam}(\psi_{\omega'}(0) \cup \{0\}) > R],$$

which explains the claim regarding the optimality of our construction.

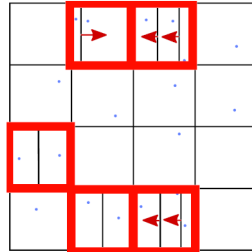
The allocation problem was first studied in a finite setup, where finitely many points are distributed uniformly and independently in a box. Here, of course, the requirement of equivariance is meaningless. We will present later a variant of the algorithm by Ajtai, Komlós and Tusnády [1], which was a crucial component of several later methods for the finite problem. For n uniformly independently distributed points in a cube of volume n , it was proved [1] that the average diameter of an allocation cell is $\log^{1/2} n$ for $d = 2$ and finite for $d \geq 3$, and precise rates of decay were determined subsequently; see [11] for details and the sharpest results. Interest in the infinite setup originated from the fact that an allocation rule gives rise to a shift-coupling between a point process and its Palm version; see, for example, [5, 7]. In [7], Holroyd and Peres studied the problem of how to find the optimal tail of an allocation rule. In the same paper, they presented a *randomized* invariant allocation rule of optimal tail decay. (Randomized allocation rules are defined similarly to allocation rules in Definition 2, but the use of extra randomness is allowed; i.e., the allocation scheme is not necessarily a deterministic

function of the point configuration.) Let us mention that several related optimization problems are much easier to handle for the randomized variant, such as in the case of matching schemes of point processes or coin flips. Deterministic constructions have been an area of active research ever since, and allocation rules that satisfy additional conditions have been subjects of recent analysis (e.g., stability [4], connectedness [9]). The best previously known allocation rule for $d \geq 3$ was the *gravitational allocation*, investigated in [2, 3], where the tail is $\mathbf{P}[|\xi_0(\omega)| > R] = \exp(-R^{1+o(1)})$. In [7], the assumption $d \geq 3$ is necessary for an exponential tail, where $\mathbf{E}[|\xi_0(\omega)|^{d/2}] = \infty$ is proved for $d = 1, 2$. The best currently known upper bounds for the tails in the case of $d = 1$ and $d = 2$ were presented in [6], showing $\mathbf{P}[|\xi_0(\omega)| > R] \leq cR^{-1/2}$ and $\mathbf{P}[|\xi_0(\omega)| > R] \leq cR^{-0.496\dots}$, respectively, and they were achieved by the stable allocation of Hoffman, Holroyd and Peres [4]. Briefly, the reason for the drastic change of behavior from dimension 3 is that here the isoperimetric function of \mathbb{R}^d becomes larger in magnitude than the deviation of the number of points in a ball. For the existence of optimal allocations with respect to other quantities (e.g., the *average* distance of a center from the points of the cell) and connections to optimal transport, see [8].

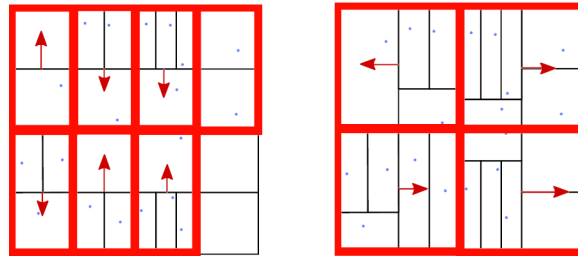
1.1. *Construction.* Let us present the (surprisingly simple) construction for Theorem 1.1 briefly, before going into the details. First, for any $v \in \mathbb{R}^d$ we will define a sequence of weak allocation schemes associated to it. These will not be equivariant yet. The construction will be based on a straightforward generalization of the algorithm of Ajtai, Komlós and Tusnády (*AKT algorithm*) [1], which assigns a piece of unit volume to each of n points in a box B , in such a way that the pieces partition B . Furthermore, if the points are scattered uniformly and independently (which is the same as the restriction of the Poisson point process to B , conditioned on there being n points in B), then the average diameter of the cells has asymptotically the same tail behavior as in Theorem 1.1 (if the volume of B and the number of random points in it are asymptotically the same). We extend the method to the Poisson point process in \mathbb{R}^d by subdividing \mathbb{R}^d to the cubes of size 2^n in $v + 2^n\mathbb{Z}^d + [0, 2^n)^d$, and applying the AKT algorithm to each of the cubes. The result is a weak allocation scheme for each n and v , which we call $\text{AKT}_{\omega, v, n}$, and denote the cell of $\xi \in \omega$ under this scheme by $f_{v, n}^{\omega, \xi}$. The algorithm for given n will be called the $\text{AKT}(v)$ algorithm run up to stage n . The details of the AKT algorithm are discussed in Section 2. See also Figure 1.

Having defined for every $v \in \mathbb{R}^d$ a sequence of weak allocation schemes dependent on v , we next want to remove this dependence and construct a sequence of weak allocations whose elements are equivariant (i.e., weak allocation rules). We will see that for every $\xi \in \omega$

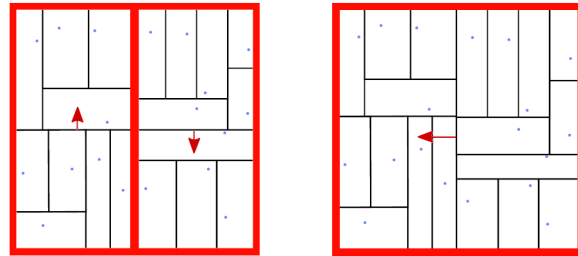
$$(1.4) \quad f_{v, n}^{\omega, \xi} = f_{u, n}^{\omega, \xi} \quad \text{whenever } u - v \in 2^n\mathbb{Z}^d.$$



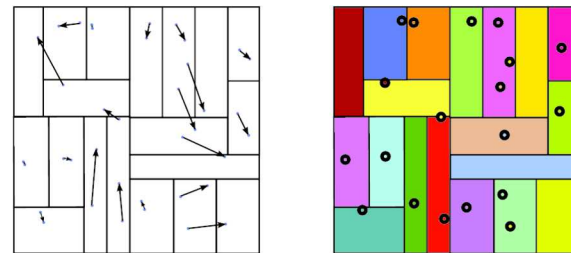
(a) The initial stage with centers



(b) The two steps of the first stage



(c) The two steps of the second stage



(d) The position of centers and final auxiliary points, the resulting local allocation

FIG. 1. AKT local allocation between $v + [0, 2^2]^d$ and the centers in $d = 2$ (v is the bottom left corner). Transformations take place inside the highlighted cuboids; the bisector walls touched by the arrows are moved in the indicated direction. (The length of an arrow may not be proportional to the length of the shift, for better perspicuity.)

Now, define

$$(1.5) \quad f_n^{\omega, \xi} := \frac{1}{2^{nd}} \int_{[0, 2^n]^d} f_{v, n}^{\omega, \xi} dv,$$

a function from \mathbb{R}^d to $[0, 1]$. This is well defined (the integral exists) by Lemma 3.2 below. It is equivariant by (1.4), so it is a fractional weak allocation rule for each $n \in \mathbb{N}$. Now we want to get rid of “weakness.” We will prove that with probability 1 all the $f_n^{\omega, \xi}$ have an L^1 limit ($\xi \in \omega$),

$$(1.6) \quad f^{\omega, \xi} := \lim_{n \rightarrow \infty} f_n^{\omega, \xi},$$

and that it is a function of integral 1. From the properties of the AKT construction (Lemma 2.2 below) we will show that the diameter of the support of this limit function has the tail that we want. We will conclude that the map $\eta: \omega \mapsto f^\omega$ with $f^\omega: \xi \mapsto f^{\omega, \xi}$ ($\xi \in \omega \in \Omega$) defines a fractional allocation rule with the desired tail.

Finally, we will define an allocation $\omega \mapsto \psi_\omega$ with $\psi_\omega: \xi \mapsto \psi_\omega(\xi)$ ($\xi \in \omega$) from the above fractional allocation. It will be such that the support of $\psi_\omega(\xi)$ is contained in the support of $f^{\omega, \xi}$. This step (Lemma 3.8) will be based on the fact that almost every point of \mathbb{R}^d is contained in the support of $f^{\omega, \xi}$ for only finitely many ξ .

In the next section we give a summary of the AKT construction, presenting the generalized version that we are using. Section 3 continues with the sequence of invariant weak allocations, the limiting fractional allocation and finally, the allocation rule that satisfies Theorem 1.1. This section ends with some concluding remarks. In Section 4 we give the necessary bounds for the concentration of the cell diameter, which come by technical modifications of the similar, usual bounds for the AKT method.

2. The generalized AKT algorithm, bounds. The AKT method was developed by Ajtai, Komlós and Tusnády in [1], and it outputs a local allocation (see definition below) between a finite cube and the i.i.d. uniform random points lying in it. We require the following notion.

DEFINITION 3. Let C be a measurable bounded subset of \mathbb{R}^d and ω_C a finite, nonempty subset of C . Call $\psi: \omega_C \rightarrow L^1(C)$ a *local allocation* between C and ω_C if:

- (1) for every $\xi \in \omega_C$, $\psi(\xi)$ takes values from $\{0, 1\}$;
- (2) for Lebesgue almost every $x \in C$ there is exactly one $\xi \in \omega_C$ such that $\psi(\xi)(x) = 1$;
- (3) for every $\xi \in \omega_C$, $\int_C \psi(\xi)(x) dx = \frac{1}{|\omega_C|} \lambda(C)$.

Fix $\omega \in \Omega$, and assume for simplicity that for any two points of ω , their respective coordinates are pairwise different (which has probability 1 if they are distributed according to the Poisson point process).

We now present the local allocation given by the AKT scheme between the cube $C = v + [0, 2^N]^d$ and $\omega_C := \omega \cap C$, where $v \in \mathbb{R}^d$ and $N \geq 1$. See Figure 1 for an illustration. We define the method recursively with respect to N .

For $N = 0$ and $|\omega_C| = k > 0$, let us fix some arbitrary way of dividing a unit cube containing k points into k connected parts with equal measure and assigning them to the points ξ . (For $k = 0$, we do not do anything.) For example, separate the centers by $k - 1$ hyperplanes orthogonal to the first axis such that the i th hyperplane separates i points of ω_C from the other $k - i$, and so that the hyperplane is at equal distance to the closest points, thus cutting C into k cuboids. Transform the cuboids by pushing the separating walls (defined by the hyperplanes) in the direction of the first axis until the volumes of the new cuboids are all equal. Each of the transformed cuboids contains the image of exactly one $\xi \in \omega_C$ by the transformation; call this image $\text{aux}(\xi, v, 0)$ and the transformed cuboid $C_{v,0}^{\omega,\xi}$. We call this stage of the construction the *initial stage*, and $C_{v,0}^{\omega,\xi}$ the *initial cell* of ξ .

Suppose that for every nonnegative integer $n < N$, a local allocation between $C = v + [0, 2^n]^d$ and $\omega \cap C$, if nonempty, has been defined for any $v \in \mathbb{R}^d$. For a $\xi \in \omega \cap (v + [0, 2^n]^d)$ let $C_{v,n}^{\omega,\xi}$ be the cell assigned to ξ by this local allocation, and suppose that an ‘‘auxiliary point’’ $\text{aux}(\xi, v, n) \in C_{v,n}^{\omega,\xi}$ is also given for ξ . Let $\text{Aux}_{v,n}(C) := \{\text{aux}(\xi, v, n) : \xi \in C \cap \omega\}$. Define now a local allocation between $C = v + [0, 2^N]^d$ and $\omega \cap C$ in d steps as follows.

For every $\xi \in C \cap \omega$, let $v' \in v + 2^{N-1}\{0, 1\}^d$ be such that $\xi \in \omega \cap (v' + [0, 2^{N-1}]^d)$, and consider the local allocation between $v' + [0, 2^{N-1}]^d$ and $\omega \cap (v' + [0, 2^{N-1}]^d)$. Then $\text{aux}(\xi, v', N - 1)$ and $C_{v',N-1}^{\omega,\xi}$ are the auxiliary point and the cell, respectively, for ξ by this local allocation. We will define $\text{aux}(\xi, v, N)$ and $C_{v,N}^{\omega,\xi}$ in d steps. Let $\hat{C}_{\xi,0} := C_{v',N-1}^{\omega,\xi}$ and $\text{aux}_{\xi,0} := \text{aux}(\xi, v', N - 1)$.

Let \mathcal{D}_i be the following collection of cuboids. Each element of \mathcal{D}_i will be a translation of the cuboid $[0, 2^{N-1}]^{d-i} \times [0, 2^N]^i$, and the elements of \mathcal{D}_i are such that their disjoint union is C . Formally, $\mathcal{D}_i = \{([0, 2^{N-1}]^{d-i} \times [0, 2^N]^i) + v + (k_1, \dots, k_{d-i}, 0, \dots, 0)2^{N-1} : (k_1, \dots, k_{d-i}, 0, \dots, 0) \in \{0, 1\}^d\}$. For $i = 1, \dots, d$, do the following. For each $K \in \mathcal{D}_i$, consider the hyperplane orthogonal to the $(d - i + 1)$ th axis that splits K into two congruent parts. Let K_1 and K_2 be these two congruent parts. Consider cuboids K'_1 and K'_2 that have the following properties:

- (i) they also partition K ;

- (ii) $\overline{K'_1} \cap \overline{K'_2}$ is parallel to $\overline{K_1} \cap \overline{K_2}$ (where \overline{X} denotes the closure of a set $X \subset \mathbb{R}^d$);
- (iii) $\frac{\lambda(K'_1)}{\lambda(K'_2)} = \frac{|K_1 \cap \omega|}{|K_2 \cap \omega|}$. If $|K_1 \cap \omega| = 0$ ($|K_2 \cap \omega| = 0$), then K'_1 (K_2) is a degenerate cuboid.

One can obtain K'_1 from K_1 by an affine transformation. For each $\xi \in K_1$ (which is equivalent to $\text{aux}_{\xi, i-1} \in K_1$), let the image of $\hat{C}_{\xi, i-1}$ under this translation be called $\hat{C}_{\xi, i}$, and let the image of $\text{aux}_{\xi, i-1}$ be $\text{aux}_{\xi, i}$. Proceed similarly for K'_2 : it is the image of K_2 by an affine transformation, and for $\xi \in K_2$ we define $\text{aux}_{\xi, i}$ and $\hat{C}_{\xi, i}$ as the images of $\text{aux}_{\xi, i-1}$ and $\hat{C}_{\xi, i-1}$ under this transformation, respectively. At the end of the cycle, $i = d$. Define $C_{v, N}^{\omega, \xi} := \hat{C}_{\xi, d}$ and $\text{aux}(\xi, v, N) := \text{aux}_{\xi, d}$.

Let us fix an arbitrary $v \in \mathbb{R}^d$. Up to this point we only defined the auxiliary points and cells $(\text{aux}(\xi, v, N), C_{v, N}^{\omega, \xi})$ for any $\xi \in \omega \cap (v + [0, 2^N]^d)$. Now we will extend the definition to all configuration points $\xi \in \omega$. Define $\text{aux}(\xi, v, N)$ as the point $\text{aux}(\xi, v', N)$, where v' is the unique element of $v + 2^N \mathbb{Z}^d$ so that $\xi \in v' + [0, 2^N]^d$. Similarly, for any $v \in \mathbb{R}^d$ define $C_{v, N}^{\omega, \xi}$ as the cell $C_{v', N}^{\omega, \xi}$, where v' is the unique element of $v + 2^N \mathbb{Z}^d$ so that $\xi \in v' + [0, 2^N]^d$. We remark that auxiliary points and corresponding cells can also be defined for noncenters, as their images by the respective affine transformations of the cells as above. This is the point in the construction where the statement of (1.4) becomes evident, as $f_{v, n}^{\omega, \xi}$ will be defined as the indicator function of $C_{v, n}^{\omega, \xi}$.

For each fixed N , we have defined the algorithm recursively, but we will prefer to think about it as an algorithm running through N stages, where in the i th stage $C_{v, i}^{\omega, \xi}$ and $\text{aux}(\xi, v, i)$ are constructed for each $\xi \in \omega$. Each stage consists of d steps, one for each axis.

Next we present the AKT weak allocation scheme in an infinite setup. (This will not yet be a weak allocation rule because equivariance fails.) Let ω now be distributed according to the Poisson point process. For each $v \in \mathbb{R}^d$ and $n \geq 1$ we define $\text{AKT}_{\omega, v, n}$ to be the weak allocation, whose restriction to C ($C \in v + 2^n \mathbb{Z}^d + [0, 2^n]^d$ with $C \cap \omega \neq \emptyset$) is the local allocation between C and $C \cap \omega$ given by the previous method. We call the algorithm that produces $\text{AKT}_{\omega, v, n}$ $\text{AKT}(v)$ run up to stage n , or simply $\text{AKT}(v)$. Note that for any $n < n'$ and $i \leq n$ the transformations taking place in the i th step of the algorithm are the same for $\text{AKT}(v)$ run up to stage n and for $\text{AKT}(v)$ run up to stage n' . Hence the latter can be thought of as a continuation of the former.

For a simpler discussion, condition now on $0 \in \omega$, and take $\xi = 0$.

LEMMA 2.1. *For \mathbf{P}' -almost all $\omega' \in \Omega'$, the sequence of cells $\{C_{v,n}^{\omega',0}\}_{n \geq 1}$ of 0 resulting from $\text{AKT}_{\omega',v,n}$ satisfies*

$$\lim_{n \rightarrow \infty} \lambda(C_{v,n}^{\omega',0}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_{v,n}^{\omega',0}(x) dx = 1,$$

where convergence is uniform in v .

LEMMA 2.2. *There exist $c, b > 0$ and an increasing family of events $(E_R)_{R > 0}$ such that:*

- (i) $1 - \mathbf{P}'[E_R] < c \exp(-bR^d)$;
- (ii) for every $\omega' \in E_R$ and $v \in \mathbb{R}^d$, the diameter of the cell $C_{v,n}^{\omega',0} \cup \{0\}$ of 0 in the $\text{AKT}(v)$ run up to stage n is at most cR ;
- (iii) there are constants $c_n(R)$ such that the series $\sum_{n=1}^{\infty} c_n(R)$ is summable, and for every $\omega' \in E_R$ and $v \in \mathbb{R}^d$ the bound

$$\|f_{v,n}^{\omega',0} - f_{v,n+1}^{\omega',0}\|_1 = \lambda(C_{v,n}^{\omega',0} \Delta C_{v,n+1}^{\omega',0}) < c_n(R)$$

holds for every n .

The proof follows from the usual analysis of the AKT algorithm, with a slight modification needed because of uniformity in v ; for details, see Section 4.

3. The allocation rule. In this section we will provide the necessary details for the construction that was sketched in Section 1.1 and prove that it is indeed well defined. Simultaneously we will execute the analysis on the tail behavior of the diameter of certain cells in order to verify Theorem 1.1. Recall that $f_{v,n}^{\omega,\xi}$ was defined as the indicator function of the cell of ξ resulting from $\text{AKT}(v)$ run up to stage n , where $\xi \in \omega \in \Omega$.

The next lemma is straightforward from the definitions.

LEMMA 3.1. *For \mathbf{P} -almost every $\omega \in \Omega$, for almost every $v \in \mathbb{R}^d$ and every $n \geq 0$, the map $f_{v,n}^\omega : \omega \rightarrow L^1(\mathbb{R}^d)$, $f_{v,n}^\omega(\xi) = f_{v,n}^{\omega,\xi}$ is a weak allocation.*

Of course, the map $\omega \mapsto f_{v,n}^\omega$ in the lemma is far from being equivariant; hence it does not define a weak allocation rule.

LEMMA 3.2. *For \mathbf{P}' -almost every $\omega' \in \Omega'$ and fixed $n \geq 1$, the map $v \mapsto f_{v,n}^{\omega',0}$ is L^1 -continuous in $v \in [0, 2^n]^d$, except for possibly v in the union of countably many hyperplanes.*

PROOF. The function $f_{v,n}^{\omega',0}$ changes continuously in v , except for the case when a hyperplane H of the form $H = v + \{(x_1, \dots, x_{j-1}, k, x_{j+1}, \dots, x_d) : x_i \in \mathbb{R}\}$, $j \in \{1, \dots, d\}$, $k \in \{1, \dots, 2^n\}$ contains some point of ω' . To see this, let v and v' be two points such that each point of ω' is on the same side of the respective pairs of hyperplanes above. Recall from Section 2 that the constructed cells do not vary from the viewpoint of the reference points in this case. More precisely, the indicator function $f_{v,n}^{\omega',0}$ is the translate of $f_{v',n}^{\omega',0}$ by $v - v'$.

The above containment property is equivalent to the condition that v is in $\bigcup_{\xi \in \omega' \cap [-2^n, 2^n]^d} \bigcup_{j=1}^d \bigcup_{k=1}^{2^n} -\xi + \{(x_1, \dots, x_{j-1}, k, x_{j+1}, \dots, x_d) : x_i \in \mathbb{R}^d\}$. This is a countable union of sets of measure 0 with probability 1; hence the points of discontinuity form a set of measure 0. \square

Define [as in (1.5)] the averaging function $f_n^{\omega,\xi} = \frac{1}{2^{nd}} \int_{[0,2^n]^d} f_{v,n}^{\omega,\xi} dv$ for $\omega \in \Omega$, $\xi \in \omega$. By the previous lemma, the integral in the definition exists.

PROPOSITION 3.3. *For \mathbf{P} -almost every $\omega \in \Omega$, for every $\xi \in \omega$, the L^1 limit $f^{\omega,\xi}$ of $f_n^{\omega,\xi}$ exists as $n \rightarrow \infty$, it is a function with values in $[0, 1]$, it has integral 1, and satisfies $\mathbf{P}[\text{diam}(\{\xi\} \cup \text{supp}(f^{\omega,\xi})) > R | \xi \in \omega] \leq c \exp(-bR^d)$ with some $c, b > 0$.*

PROOF. Since the measure \mathbf{P} is equivariant, it is enough to prove the claim for the Palm version: for \mathbf{P}' -almost every $\omega' \in \Omega'$ and with $\xi = 0$. Recall that E_R is a monotone increasing family of events that exhausts a subset of \mathbf{P}' -measure 1 in Ω' , and that the function $f^{\omega',0}$, if exists, does not depend on R . Fix R , let E_R be as in Lemma 2.2 and assume that $\omega' \in E_R$.

For an arbitrary $m, n \in \mathbb{Z}^+$, $m \leq n$, $u \in 2^{2m}\{0, 1, \dots, 2^{n-m} - 1\}^d$, let

$$g_n^{m,u,\omega'} = g_n^{m,u} := \frac{1}{2^{md}} \int_{u+[0,2^m]^d} f_{v,n}^{\omega',0} dv.$$

In particular, $f_n^{\omega',0} = g_n^{n,u}$. Then

$$(3.1) \quad f_n^{\omega',0} = \frac{1}{2^{(n-m)d}} \sum_{u \in 2^{2m}\{0,1,\dots,2^{n-m}-1\}^d} g_n^{m,u}.$$

If $u, u' \in 2^{2m} \cdot \{0, 1, \dots, 2^{n-m} - 1\}^d$, then $g_m^{m,u} = g_m^{m,u'}$ because the sequence of dyadic partitions used in the construction for such a u and u' is the same up to stage m , also showing (1.4). We have

$$\begin{aligned} \|g_n^{m,u} - g_m^{m,u}\|_1 &= \left\| 2^{-md} \int_{u+[0,2^m]^d} f_{v,n}^{\omega',0} dv - 2^{-md} \int_{u+[0,2^m]^d} f_{v,m}^{\omega',0} dv \right\|_1 \\ &\leq 2^{-md} \int_{u+[0,2^m]^d} \|f_{v,n}^{\omega',0} - f_{v,m}^{\omega',0}\|_1 dv, \end{aligned}$$

and similarly for u' .

By Lemma 2.2(iii), the right-hand side of this inequality is bounded from above by $\sum_{i=m}^{\infty} c_i(R)$.

Hence by the triangle inequality we obtain

$$(3.2) \quad \begin{aligned} \|g_n^{m,u} - g_n^{m,u'}\|_1 &\leq \|g_n^{m,u} - g_m^{m,u}\|_1 + \|g_m^{m,u} - g_m^{m,u'}\|_1 + \|g_n^{m,u'} - g_m^{m,u'}\|_1 \\ &\leq 2 \sum_{i=m}^{\infty} c_i(R). \end{aligned}$$

Using (3.1) this implies

$$(3.3) \quad \begin{aligned} \|f_n^{\omega',0} - g_n^{m,0}\|_1 &\leq \sum_{u \in 2^m \{0,1,\dots,2^{n-m}-1\}^d} \frac{1}{2^{(n-m)d}} \|g_n^{m,u} - g_n^{m,0}\|_1 \\ &\leq 2 \sum_{i=m}^{\infty} c_i(R). \end{aligned}$$

On the other hand, for any $m, n \in \mathbb{Z}^+$, $m \leq n$,

$$\begin{aligned} \|f_n^{\omega',0} - f_m^{\omega',0}\|_1 &\leq \|f_n^{\omega',0} - g_n^{m,0}\|_1 + \|g_n^{m,0} - f_m^{\omega',0}\|_1 \\ &= \|f_n^{\omega',0} - g_n^{m,0}\|_1 + \|g_n^{m,0} - g_m^{m,0}\|_1. \end{aligned}$$

The first term on the right is $\leq 2 \sum_{i=m}^{\infty} c_i(R)$ by (3.3), and the second term is $\leq \sum_{i=m}^n c_i(R)$ by Lemma 2.2(iii). We conclude that $(f_n^{\omega',0})$ is a Cauchy sequence, and so there is a limit $f^{\omega',0}$ in L^1 . The fact that $f^{\omega',0}$ takes values in $[0,1]$ follows directly from the same fact about $f_n^{\omega',0}$. By Lemma 2.1 it is easy to see that $\int f_n^{\omega',0} \rightarrow 1$, and by Lemma 2.2 the support of each $f_n^{\omega',0}$ is within radius cR around 0; hence the dominated convergence theorem implies that $\int f^{\omega',0} = 1$. The bound on the tail probability of the support is the consequence of Lemma 2.2(i). The above hold for every R and $\mathbf{P}'[\cup E_R] = 1$; hence the proposition follows. \square

The next proposition implies the variant of Theorem 1.1 for fractional allocation rules instead of allocation rules; see Definition 1.

PROPOSITION 3.4. *The map $\eta: \omega \mapsto f^\omega$ with $f^\omega: \omega \rightarrow L^1(\mathbb{R}^d)$, $\xi \mapsto f^{\omega,\xi}$ is a fractional allocation rule. It satisfies*

$$\mathbf{P}[\text{diam}(\{0\} \cup \text{supp } f^{\omega,0}) > R | 0 \in \omega] \leq c \exp(-bR^d)$$

for some c and $b > 0$.

PROOF. By Proposition 3.3 we have that $\eta: \omega \mapsto f^\omega$ with $f^\omega: \omega \rightarrow L^1(\mathbb{R}^d)$, $\xi \mapsto f^{\omega', \xi}$ satisfies (1') and (3) in the definition of a fractional allocation rule and similarly for the claim about the support of $f^{\omega, 0}$. Measurability and equivariance are clear from the construction. So it only remains to prove (2').

By Lemma 3.1, $\xi \mapsto f_{v,n}^{\omega, \xi}$ defined on ω is a weak allocation; in particular, (2) from Definition 1 holds. Hence for almost every $x \in \mathbb{R}^d$,

$$\begin{aligned} \sum_{\xi \in \omega} f_n^{\omega, \xi}(x) &= \sum_{\xi \in \omega} 2^{-nd} \left(\int_{[0, 2^n]^d} f_{v,n}^{\omega, \xi} dv \right)(x) \\ &= 2^{-nd} \int_{[0, 2^n]^d} \sum_{\xi \in \omega} f_{v,n}^{\omega, \xi}(x) dv \leq 2^{-nd} \int_{[0, 2^n]^d} 1 dv = 1, \end{aligned}$$

showing that $\{f_n^{\omega, \xi}: \xi \in \omega\}$ satisfies (2').

Finally, this implies $\sum_{\xi \in \omega} \lim_n f_n^{\omega, \xi}(x) = \lim_n \sum_{\xi \in \omega} f_n^{\omega, \xi}(x) \leq 1$ (applying the dominated convergence theorem for every compact subset of \mathbb{R}^d). \square

The following theorem is a special case of the Campbell–Mecke formula. We will use it in the proof of Lemma 3.6.

THEOREM 3.5 [10]. *For any integrable $f: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^+$,*

$$\mathbf{E} \left[\sum_{x \in \omega} f(x, \omega) \right] = \int_{\mathbb{R}^d} \mathbf{E}[f(x, \omega) | x \in \omega] dx,$$

where \mathbf{E} is expectation with respect to the Poisson point process of unit intensity.

LEMMA 3.6. **P**-almost surely for almost every $x \in \mathbb{R}^d$, x is contained in $\text{supp}(f^{\omega, \xi})$ for only finitely many ξ 's.

PROOF. Let $z \in \mathbb{Z}^d$, and denote by Y_z the random variable that is the number of centers ξ of ω such that the intersection of $\text{supp}(f^{\omega, \xi})$ and $z + [0, 1]^d$ is nonempty. Then

$$\mathbf{E}Y_z \leq \mathbf{E} \left[\sum_{\xi \in \omega} A(\xi, \omega) \right],$$

where $A(x, \omega) = 1$ if $x \in \omega$ and $\text{diam}(\{x\} \cup \text{supp} f^{\omega, x}) > |x - z| - \sqrt{d}$, and $A(x, \omega) = 0$ otherwise. Using Proposition 3.4 and Theorem 3.5 we have

$$\begin{aligned} \mathbf{E}Y_z &\leq \int_{\mathbb{R}^d} \mathbf{P}[\text{diam}(\{x\} \cup \text{supp} f^{\omega, x}) > |x - z| - \sqrt{d} | x \in \omega] dx \\ &\leq \int_{\mathbb{R}^d} c \exp(-b(|x| - \sqrt{d})^d) dx < \infty. \end{aligned}$$

Hence $\mathbf{P}[Y_z = \infty] = 0$, and also $\mathbf{P}[\bigcup_{z \in \mathbb{Z}^d} \{Y_z = \infty\}] = 0$, which implies the statement. \square

A direct consequence is the following:

LEMMA 3.7. *For \mathbf{P} -almost every ω one can partition \mathbb{R}^d to countably many measurable sets of the form $S = \bigcap_{i=1}^k \text{supp}(f^{\omega, \xi_i})$ (with some $\xi_1, \dots, \xi_k \in \omega$ and $k \in \mathbb{N}$).*

The previous lemma will enable us to define an allocation rule from our fractional allocation rule, in such a way that the cell allocated to a center $\xi \in \omega$ is contained in $\text{supp}(f^{\omega, \xi})$. Namely, for each set S as in Lemma 3.7, we will partition S into measurable pieces S_1, \dots, S_k such that $\lambda(S_i) = \int_S f^{\omega, \xi_i}(x) dx$. We will do it in a way such that S and the f^{ω, ξ_i} ($i = 1, \dots, k$) determine the pieces S_i in some previously fixed (deterministic) way, and such that the pieces change continuously with S (in terms of Hausdorff distance between sets, say). The central issue is to obtain a partition and an association to the centers that is translation equivariant. A method to do so was suggested to us by Yuval Peres, replacing the original, less elegant proof for the following lemma:

LEMMA 3.8. *Let $\eta: \omega \mapsto f^\omega$ be a fractional allocation rule that satisfies Lemma 3.7. Then there is an allocation rule $\psi: \omega \mapsto \psi_\omega$ such that for every $\xi \in \omega$ we have $\text{supp}(\psi_\omega(\xi)) \subset \text{supp}(f^{\omega, \xi})$.*

PROOF. For each set of the form $S = \bigcap_{i=1}^k \text{supp}(f^{\omega, \xi_i})$ as in Lemma 3.7, let $\alpha = \{\xi_1, \dots, \xi_k\}$ and $c_i := \int_S (f^{\omega, \xi_i})(x) dx$, $i = 1, \dots, k$. So $\sum_{i=1}^k c_i$ is the Lebesgue measure of S . Apply a version of the site-optimal Gale and Shapley algorithm (see [4] for a more detailed description) within S to partition it into $\mathcal{S} = (S_1, \dots, S_k)$ (up to a remainder set of measure zero) as follows with S_i corresponding to ξ_i . First we put all the points of S that are equidistant to any pair of centers from α into the set W . We note that W has measure zero. Now we will stage-wise define a series of disjoint subsets of $S \setminus W$ and auxiliary sets corresponding to the centers in α . Let $A_1(\xi_1), \dots, A_1(\xi_k)$ be the intersections of the Voronoi cells of α with $S \setminus W$, and set $R_0(\xi_i) = \emptyset$ for all $i \in [k]$. Suppose that we have already constructed the sets in $\{A_l(\xi_i), R_{l-1}(\xi_i) : i \in [k], l \leq n\}$. The disjoint sets $A_{n+1}(\xi_1), \dots, A_{n+1}(\xi_k)$ will be obtained as follows. We define for all n and $\xi_i \in \alpha$ the rejection radius $r_n(\xi_i) = \inf\{r \geq 0 : \lambda(A_n(\xi_i) \cap B(\xi_i, r)) \geq c_i\}$, and the rejection sets $R_n(\xi_i) = A_n(\xi_i) \setminus B(\xi_i, r_n(\xi_i))$. Using this notation we define $A_{n+1}(\xi_i) = (A_n(\xi_i) \cap B(\xi_i, r_n(\xi_i))) \cup N(\xi_i)$, where $x \in N(\xi_i)$, if and only

if $x \notin \bigcup_{j=1}^n R_j(\xi_i)$, $x \in \bigcup_{j=1}^k R_n(\xi_j)$, and if for any other $l \in [k]$, it holds that $x \notin \bigcup_{j=1}^n R_j(\xi_l)$, then $|x - \xi_i| < |x - \xi_l|$.

With the aid of these sets we are able to define \mathcal{S} , let $S_i = \bigcup_{n \geq 1} \bigcap_{k \geq n} (A_k(\xi_i) \cap B(\xi_i, r_k(\xi_i)))$. It is clear that the map $\xi_i \mapsto S_i$ is equivariant under translations, that is, for any $y \in \mathbb{R}^d$ we have $\xi_i + y \mapsto S_i + y$ when we run the algorithm for $S + y$ and $\alpha + y$. It is also true that the S_i 's are pairwise disjoint, because the sets $A_n(\xi_1), \dots, A_n(\xi_k)$ were also pairwise disjoint by definition for all $n \geq 1$. It remains to show that they provide a partition of $S \setminus W$ up to a set of measure zero. First note that for each i we have $\lambda(S_i) = \liminf_{n \rightarrow \infty} \lambda(A_n(\xi_i) \cap B(\xi_i, r_n(\xi_i))) = \limsup_{n \rightarrow \infty} \lambda(A_n(\xi_i) \cap B(\xi_i, r_n(\xi_i))) \leq c_i$. If there exists an $x \notin \bigcup_{i \in [k]} S_i$, then $x \in \bigcap_{i \in [k]} \bigcup_{n \geq 1} R_n(\xi_i)$, which implies that $\bigcup_{n \geq 1} R_n(\xi_i)$ is nonempty for each $i \in [k]$, so therefore $\lambda(S_i) = c_i$. It follows that $\lambda(S \setminus (\bigcup_{i \in [k]} S_i)) = 0$.

We also provide the intuitive picture for better understanding. Let each ξ_i start growing a ball around itself at linear speed, simultaneously. At time t let each ξ_i capture all points of S that its growing ball meets, as long as no other ball has captured the point at some earlier time. However, when the total measure of points in S that ξ_i has captured by time t is equal to c_i , stop growing its ball; that is, let $P_i(t')$ be equal to $P_i(t)$ as long as $t' > t$ [where $P_i(t)$ is the set of points in S that have been captured by ξ_i by time t]. It is clear that by some time t the Lebesgue measure of $P_i(t)$ is equal to c_i for each i . Define this $P_i(t)$ to be the part of the cell of ξ within S . One can also show that the correspondence between α and \mathcal{S} is stable; see [4] for more details.

Doing this subdivision for each of the countably many S , we get the cell of ξ_i assigned to it by our allocation. Measurability and invariance follow from our method and the assumptions. \square

We finish by noting that Proposition 3.4 and Lemma 3.8 imply Theorem 1.1.

If we wanted the cells assigned by our allocation rule to be connected and to contain the corresponding center, this could be done by growing ‘‘tendrils’’ that connect the pieces of each cell C of the original allocation, and the center for C . By taking care to preserve measurability and equivariance, we believe that this can be done, but we omit the details here.

REMARK 3.9. We have only worked out the allocation rule for the translation-equivariant case. However, one can make it isometry-equivariant. Besides parameter v that determined a translate of $2^n \mathbb{Z}^d$ in the definition of the function $f_{v,n}^{\omega,\xi}$, we need to introduce a parameter $\theta \in \{x \in \mathbb{R}^d, |x| = 1\}$ to determine a rotation of \mathbb{Z}^d . When we integrate through v in the definition of $f_n^{\omega,\xi}$, we then have to integrate with respect to θ as well; otherwise every part of the proof extends to this modified setup automatically.

REMARK 3.10. An intuitive interpretation of the AKT algorithm comes from thinking of the centers as gas particles. Then the procedure systematically equates pressures between neighboring cubes in the dyadic subdivision. For the sake of analysis it was easier to allow cells around particles to expand in the directions of the axes, but a more “canonical” version would be obtained without this artifact. That is, put one particle of gas in each center of the point process, and start growing cells around them (small balls at the beginning), whose pressures would be proportional to the volume of the cell and at normal direction to the surface. When two cells meet, the pressure differences between them would tend to equate, and in the limit all the cells would have the same volumes. The tail behavior of their diameters should be as good as that of our fractional allocation rule (and perhaps better if we take constant factors into account). On an even more speculative note, we mention that the above procedure looks like a modification of the stable allocation rule: do not fix centers, and let the growing cells “push” each other while occupying yet unoccupied territories.

QUESTION 3.11. Can one make the above heuristics precise in order to obtain a canonical allocation rule of optimal tail?

4. Proofs of Lemmas 2.1 and 2.2. The cuboid $\text{AKT}_{\omega,v,n}(0)$ is a result of nd affine transformations (in n stages), not taking into account in how many steps the initial cell of 0 is constructed. Hence we can bound the diameter of $C_{v,n}^{\omega,0} \cup \{0\}$ by first bounding $|\text{aux}(0, v, n)|$ and then bounding the sum of the lengths of the edges of $C_{v,n}^{\omega,0}$ (which bounds the diameter of $C_{v,n}^{\omega,0}$). In each stage there is at most 1 step along each of the d axes. Also, the sizes of the steps along different axes are independent as random variables. Therefore, if we wish to obtain an upper bound on the total movement of a point x during the shifts, steps along different axes can be treated separately.

The next lemma is standard, and we prove it only for completeness.

LEMMA 4.1. *Let X be a random variable with Poisson distribution of mean λ . If $0 \leq \rho \leq 2$, then*

$$\mathbf{P}[|X - \lambda| > \lambda\rho] < 2 \exp\left(-\frac{\lambda\rho^2}{4}\right).$$

PROOF. Note that the moment generating function of X is

$$M(t) = \mathbf{E}[\exp(tX)] = \exp(\lambda(e^t - 1)), \quad t \in \mathbb{R}.$$

For one side

$$\begin{aligned} \mathbf{P}[X - \lambda > \lambda\rho] &= \mathbf{P}[\exp(tX) > \exp(\lambda t(\rho + 1))] \\ &< \frac{\mathbf{E}[\exp(tX)]}{\exp(\lambda t(\rho + 1))} = \frac{\exp(\lambda(e^t - 1))}{\exp(\lambda t(\rho + 1))}, \end{aligned}$$

where $t > 0$, and we use Markov's inequality. Now, for $0 < t < 1$ we have $e^t < 1 + t + t^2$, so

$$\mathbf{P}[X - \lambda > \lambda\rho] < \frac{\mathbf{E}[\exp(tX)]}{\exp(\lambda t(\rho + 1))} = \exp(\lambda(t^2 + t) - \lambda t(\rho + 1)).$$

The last expression is minimized by $t = \rho/2$, so

$$\mathbf{P}[X - \lambda > \lambda\rho] < \exp\left(-\frac{\rho^2\lambda}{4}\right).$$

For the other bound,

$$\begin{aligned} \mathbf{P}[\lambda - X > \lambda\rho] &= \mathbf{P}[\exp(-tX) > \exp(-\lambda t(1 - \rho))] \\ &< \frac{\mathbf{E}[\exp(-Xt)]}{\exp(-\lambda t(1 - \rho))} = \frac{\exp(\lambda(e^{-t} - 1))}{\exp(-\lambda t(1 - \rho))}, \end{aligned}$$

where $t > 0$, and we use again Markov's inequality. Now, for $0 < t$ we have $e^{-t} < 1 - t + t^2/2$, so

$$\mathbf{P}[\lambda - X > \lambda\rho] < \frac{\exp(\lambda(e^{-t} - 1))}{\exp(-\lambda t(1 - \rho))} = \exp(\lambda(t^2/2 - t) + \lambda t(1 - \rho)).$$

The last expression is minimized by $t = \rho$, so

$$\mathbf{P}[\lambda - X > \lambda\rho] < \exp\left(-\frac{\rho^2\lambda}{2}\right). \quad \square$$

For a measurable subset $B \subset \mathbb{R}^d$, let $N(B)$ denote the number of centers of the Poisson point process in B . Let $l(C_{v,n}^{\omega,\xi}) \in \mathbb{R}^d$ denote the vector, whose i th coordinate is the length of an edge parallel to the i th axis of the cuboid $C_{v,n}^{\omega,\xi}$.

The next lemma summarizes all the needed consequences of the concentration of the number of centers in a fixed set. Namely, the discrepancy of this number determines the distribution of how much a center is moved (through its auxiliary points) and a cuboid deformed during the AKT(v) procedure, and these two give bounds on the distance of the center from the resulting cell and the diameter of the cell, respectively.

LEMMA 4.2. *There exist $c, b > 0$ and an increasing family of events $(E_R)_{R>0}$ such that:*

- (i) $1 - \mathbf{P}'[E_R] < c \exp(-bR^d)$;
- (ii) *there exist $c'_i = c'_i(R)$ ($i \in \{0, 1, \dots\}$) such that for every $\omega' \in E_R$ and every $v \in \mathbb{R}^d$, one has $|\text{aux}(0, v, i) - \text{aux}(0, v, i - 1)| \leq c'_i$ and $\sum_{i=0}^{\infty} c'_i < cR$;*

(iii) *there exist $e_i = e_i(R)$ ($i \in \{0, 1, \dots\}$) such that for every $\omega' \in E_R$ and every $v \in \mathbb{R}^d$, one has $|l(C_{v,i}^{\omega',0}) - l(C_{v,i-1}^{\omega',0})|_\infty < e_i$ and such that $\sum_{i=0}^\infty e_i < cR$.*

PROOF. Our analysis will loosely follow the argument of Talagrand and Yukich [11], although we are able to use less sophisticated methods because the sizes of the induced displacements occurring in further stages of the AKT algorithm decay much more rapidly in $d \geq 3$ than in $d = 2$. On the other hand, we have to achieve uniform bounds with respect to $v \in \mathbb{R}^d$.

It is enough to prove the lemma only for $R > R_0$ for some suitably chosen $R_0 > 0$. Then the lemma will follow for every $R > 0$, perhaps with different constants.

Fix $R > R_0$; R_0 will be determined later.

In any given stage the shift of the auxiliary point of 0 in the direction of the i th axis only depends on the number of auxiliary points in the currently considered cuboids. Moreover, the length of the shift only depends on the i th coordinates of the auxiliary points in the cube. Hence the shifts in different directions are independent. Therefore it will be enough to bound the shifts along the first axis.

We set $r_0(R) = \lceil \log_2 R \rceil - d - 1$ for each $R > 0$. That is, $\frac{R}{2^{d+1}} \leq 2^{r_0} < \frac{R}{2^d}$.

We will define the event $A_R = A_{R,1}$ in terms of bounds on the number of centers in certain cuboids. For each $n \geq r_0$ consider cuboids that satisfy the following three conditions: the cuboid is the translate of $[0, 2^{n-1} - 2^{-n-1}) \times [0, 2^n - 2^{-n}) \times \dots \times [0, 2^n - 2^{-n})$, has a corner in $2^{-n}\mathbb{Z}^d$, and either contains 0, or one of its translates by $\pm(2^{n-1} - 2^{-n-1}, 0, \dots, 0)$ does. Let \mathcal{G}_1^n denote the set of these objects. Similarly, for each $n \geq r_0$ consider cuboids that satisfy the following three conditions: the cuboid is the translate of $[0, 2^{n-1} + 2^{-n-1}) \times [0, 2^n + 2^{-n}) \times \dots \times [0, 2^n + 2^{-n})$, has a corner in $2^{-n}\mathbb{Z}^d$ and either it contains 0 or one of its translates by $\pm(2^{n-1} + 2^{-n-1}, 0, \dots, 0)$ does. Let \mathcal{G}_2^n denote their set. Let us set $\rho_n = \rho_n(R) = 2^{-(5n/4)-2d}R^{5/4}$. For each $n \geq r_0 = r_0(R)$ and $Q \in \mathcal{G}_1^n \cup \mathcal{G}_2^n$, define the event B_Q , that $|N(Q \setminus \{0\}) - \lambda(Q)| < \lambda(Q)\rho_n$. Let $A_R = \bigcap_{n=r_0}^\infty \bigcap_{Q \in \mathcal{G}_1^n \cup \mathcal{G}_2^n} B_Q$. Note that $r_0(R)$ and $\rho_n(R)$ for any fixed n are increasing in R , and therefore $(A_R)_{R>0}$ is an increasing family of events.

It is straightforward from the defining formula, that $\rho_{r_0} \geq \rho_n$ for any $n \geq r_0$. Also,

$$(4.1) \quad \rho_{r_0} = 2^{-r_0(5/4)-2d}R^{5/4} \leq \left(\frac{2^{d+1}}{R}\right)^{5/4} 2^{-2d}R^{5/4} = 2^{(5/4)-(3d/4)} \leq \frac{1}{2}.$$

Furthermore,

$$(4.2) \quad \sum_{n=r_0}^\infty \rho_n = \rho_{r_0} \sum_{n=r_0}^\infty \frac{\rho_n}{\rho_{r_0}} \leq \frac{1}{2} \sum_{i=0}^\infty (2^{-5/4})^i = \frac{1}{2 - 2^{-1/4}} < 1.$$

First, we establish an upper bound on $(1 - \mathbf{P}'[A_R])$. The distribution of $N(Q \setminus \{0\})$ according to the Palm version of the Poisson point process is Poisson with mean $\lambda(Q)$. Thus we can use Lemma 4.1 [noting that $\rho_n \leq \frac{1}{2}$ for all $n \geq r_0$ and that $\lambda(Q) < 2^{(n-1)d}$ for $Q \in \mathcal{G}_1^n \cup \mathcal{G}_2^n$] and a simple union bound to get

$$\begin{aligned} 1 - \mathbf{P}'[A_R] &\leq \sum_{n=r_0}^{\infty} \sum_{Q \in \mathcal{G}_1^n \cup \mathcal{G}_2^n} (1 - \mathbf{P}'[B_Q]) \\ &< \sum_{n=r_0}^{\infty} (|\mathcal{G}_1^n| + |\mathcal{G}_2^n|) 2 \exp\left(-\frac{\rho_n^2 2^{(n-1)d}}{4}\right). \end{aligned}$$

Using $|\mathcal{G}_1^n| = 3(2^{2n} - 1)^d$ and $|\mathcal{G}_2^n| = 3(2^{2n} + 1)^d$, we conclude that

$$\begin{aligned} (4.3) \quad 1 - \mathbf{P}'[A_R] &< \sum_{n=r_0}^{\infty} 2^{(2n+1)d+4} \exp(-2^{(n-1)d} 2^{-((5n)/2)-4d-2} R^{5/2}) \\ &= \sum_{n=r_0}^{\infty} 2^{2nd+d+4} \exp(-2^{n(d-5/2)-5d-2} R^{5/2}). \end{aligned}$$

Denote the i th term in the sum by a_i . Observe that

$$(4.4) \quad a_{r_0} < R^{2d} 2^{-2d^2+d+4} \exp(-R^d 2^{-d^2-(7/2)d+1/2}),$$

(by $R/2^{d+1} \leq 2^{r_0} \leq R/2^d$), and that

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= 2^{2d} \exp(2^{n(d-5/2)-5d-2} R^{5/2} (1 - 2^{(d-5/2)})) \\ &\leq 2^{2d} \exp(-2^{(n-r_0)(d-5/2)-d^2-(7/2)d+1/2} R^d (2^{(d-5/2)} - 1)) \\ &\leq 2^{2d} \exp(-R^d 2^{-d^2-(7/2)d+1/2} (2^{(d-5/2)} - 1)). \end{aligned}$$

Hence there exists a constant $c' > 0$ such that for R_0 chosen large enough, for every $R > R_0$,

$$1 - \mathbf{P}'[A_R] < \exp(-c' R^d).$$

Now let us assume that $n \geq r_0$ throughout the following computation. Let W be an arbitrary translate of $[0, 2^n]^d$ containing 0, with U its left half and V its right. Then, when conditioned on A_R (and on 0 being a center), the following is true:

$$(4.5) \quad \left| \frac{N(U) - N(V)}{N(U) + N(V)} \right| < 4\rho_n.$$

We can show this by making an easy observation: there are $U', V' \in \mathcal{G}_1^n$ and $U'', V'' \in \mathcal{G}_2^n$, so that $U' \subset U \subset U''$ and $V' \subset V \subset V''$. Note that

$$(4.6) \quad \frac{N(U) - N(V)}{N(U) + N(V)} \leq \frac{N(U \setminus \{0\}) - N(V \setminus \{0\})}{N(U \setminus \{0\}) + N(V \setminus \{0\})} + \frac{1}{N(U \setminus \{0\}) + N(V \setminus \{0\})}.$$

On A_R we have

$$(4.7) \quad \begin{aligned} & \frac{N(U \setminus \{0\}) - N(V \setminus \{0\})}{N(U \setminus \{0\}) + N(V \setminus \{0\})} \\ & \leq \frac{N(U'' \setminus \{0\}) - N(V' \setminus \{0\})}{N(U' \setminus \{0\}) + N(V' \setminus \{0\})} \\ & < \frac{1/2(2^n + 2^{-n})^d(1 + \rho_n) - 1/2(2^n - 2^{-n})^d(1 - \rho_n)}{(2^n - 2^{-n})^d(1 - \rho_n)} \\ & = \frac{1}{2} \left[\frac{1 + \rho_n}{1 - \rho_n} \left(\frac{2^n + 2^{-n}}{2^n - 2^{-n}} \right)^d - 1 \right], \end{aligned}$$

where the first inequality holds by monotonicity of $N(\cdot)$ and the second by the definition of A_R . To further estimate (4.7), use that

$$\frac{1 + \rho_n}{1 - \rho_n} = 1 + \frac{2\rho_n}{1 - \rho_n} \leq 1 + 4\rho_n,$$

(recalling $\rho_n \leq \frac{1}{2}$ for $n \geq r_0$) and that

$$\left(\frac{2^n + 2^{-n}}{2^n - 2^{-n}} \right)^d = \left(1 + \frac{2}{2^{2n} - 1} \right)^d < \left(1 + \frac{1}{2^{2n-2}} \right)^d < 1 + 2^d \frac{1}{2^{2n-2}}.$$

We obtain from the two previous expressions and (4.7) that

$$(4.8) \quad \frac{N(U \setminus \{0\}) - N(V \setminus \{0\})}{N(U \setminus \{0\}) + N(V \setminus \{0\})} < \frac{1}{2} \left(4\rho_n + \frac{1}{2^{2n-2-d}} + \frac{1}{2^{2n-2-d}} 4\rho_n \right),$$

which is an upper bound for the first term on the right-hand side of (4.6). The second term on the right-hand side of (4.6) is (rather roughly) bounded by

$$(4.9) \quad \frac{1}{N(U \setminus \{0\}) + N(V \setminus \{0\})} < \frac{1}{(2^n - 2^{-n})^d} \frac{1}{1 - \rho_n} < \frac{1}{2^{2n}}.$$

If $R > R_0(d)$ for an R_0 chosen suitably large, it follows that $\rho_n = 2^{-(5n)/4-2d} R^{5/4}$ is greater than 2^{-2n+2} for $n \geq r_0$. Therefore by adding up bounds (4.8) and (4.9), we show that for $R > R_0$,

$$\frac{N(U) - N(V)}{N(U) + N(V)} < 4\rho_n.$$

Symmetry of A_R in U and V implies

$$\left| \frac{N(U) - N(V)}{N(U) + N(V)} \right| < 4\rho_n.$$

Having estimated the discrepancy of points in the two halves of cubes, we are now ready to give upper bounds on the total shift during the AKT procedure. Let v' be an arbitrary point in \mathbb{R}^d , and let $D_n = (\text{aux}(0, v', n))_1 - (\text{aux}(0, v', n-1))_1$, the signed amount by which the auxiliary point of 0 is translated along the first axis in the n th stage of $\text{AKT}(v')$. By the observation that until the n th stage, every displacement takes place inside a cube of sidelength 2^n , we trivially have

$$\left| \sum_{n=1}^{r_0-1} D_n \right| \leq 2^{r_0-1} < \frac{R}{2^{d+1}} < \frac{R}{2d}.$$

For $n \geq r_0$, $D_n = (2^{n-1} - h_n) \frac{N(U_n) - N(V_n)}{N(U_n) + N(V_n)}$, where Q_n is the element of $v' + 2^n\mathbb{Z} + [0, 2^n)^d$ containing 0, with U_n its left half and V_n its right, and $0 \leq h_n \leq 2^{n-1}$ is the distance of $\text{aux}(0, v', n-1)$ to the hyperplane separating U_n and V_n . Conditioned on A_R (and 0 being a center) we then have

$$\begin{aligned} \left| \sum_{n=r_0}^{\infty} D_n \right| &\leq \sum_{n=r_0}^{\infty} \left| (2^{n-1} - h_n) \frac{N(U_n) - N(V_n)}{N(U_n) + N(V_n)} \right| \\ &< \sum_{n=r_0}^{\infty} 2^{n+1} \rho_n = \sum_{n=r_0}^{\infty} 2^{-n/4-2d+1} R^{5/4} \end{aligned} \tag{4.10}$$

$$\begin{aligned} &= R^{5/4} 2^{-r_0/4-2d+1} \sum_{n=0}^{\infty} 2^{-n/4} \\ &\leq R 2^{-(7/4)d+5/4} \frac{1}{1-2^{-1/4}} < \frac{R}{2d}, \end{aligned} \tag{4.11}$$

since $\frac{R}{2^{d+1}} \leq 2^{r_0}$ by the choice of r_0 . Thus we have that, conditioned on the event A_R and 0 being a center, for every $v' \in \mathbb{R}^d$ the total shift of 0 along the first axis, $|(\text{aux}(0, v', n))_1|$, is at most $\frac{R}{d}$ for every $n \geq 1$ when the $\text{AKT}(v')$ is run up to stage n . Furthermore, we have seen in (4.10) that on A_R the length $|(\text{aux}(0, v', n))_1 - (\text{aux}(0, v', n-1))_1|$ is at most $b'_n(R) := 2^{-n/4-2d+1} R^{5/4}$ whenever $n \geq r_0(R)$.

For $1 < i \leq d$ one can define the events $A_{R,i}$ similarly to $A_{R,1}$, to establish tail bounds of the lengths of the shifts along the i th axis. We define the subevents of $A_{R,i}$ analogously to the subevents of $A_{R,1}$, conditioning on the relative deviation (with respect to the expectation) of the number of configuration points being between factors $(1 - \rho_n)$ and $(1 + \rho_n)$ in certain

cuboids, with n depending on the side length of the cuboid. Thus we arrive at the same failure probability bound as in (4.11).

Now define

$$(4.12) \quad E_R := \bigcap_{i=1}^d A_{R,i}.$$

This event satisfies conditions (i) and (ii) of the lemma: (i) holds by a union bound on the complement events. On the other hand, (ii) is true by (4.11) summed up for each of the d directions (using the triangle inequality), with $c'_n(R) = db'_n(R)$. Furthermore, $(E_R)_{R>0}$ is an increasing family of events, since for each i , the families $(A_{R,i})_{R>0}$ are as well.

Now we turn to the proof of (iii).

It is clear that $|l(C_{v,n}^{\omega',0})|_\infty \leq 2^{r_0-1} < \frac{R}{2^{d+1}}$ for $n < r_0$ because transformations of the cell only happen inside a cube of sidelength 2^{r_0-1} .

For $n \geq r_0$,

$$|l(C_{v,n}^{\omega',0})|_\infty < |l(C_{v,n-1}^{\omega',0})|_\infty (1 + 4\rho_n)$$

because, conditioned on E_R , in each step of the n th stage, at most one sidelength can be stretched to at most $(1 + 4\rho_n)$ times its current size, using (4.5). This implies by (4.2) that

$$\begin{aligned} |l(C_{v,n}^{\omega',0})|_\infty &< |l(C_{v,r_0-1}^{\omega',0})|_\infty \prod_{i=r_0}^{\infty} (1 + 4\rho_i) \leq \frac{R}{2^{d+1}} \exp\left(\sum_{i=r_0}^{\infty} 4\rho_i\right) \\ &< \frac{R}{2^{d+1}} \exp(4). \end{aligned}$$

On E_R , each component of $l(C_{v,n+1}^{\omega',0})$ is between multiplicative factor $(1 - 4\rho_{n+1})$ and $(1 + 4\rho_{n+1})$ of the respective component of $l(C_{v,n}^{\omega',0})$. This implies with the previous computation that

$$|l(C_{v,n+1}^{\omega',0}) - l(C_{v,n}^{\omega',0})|_\infty < 4\rho_{n+1} |l(C_{v,n+1}^{\omega',0})|_\infty < 4\rho_{n+1} \frac{R}{2^{d+1}} \exp(4).$$

The $\{\rho_n\}$ series is summable [see (4.2)] and $\sum_{i=r_0}^{\infty} \rho_i \frac{\exp(4)}{2^{d-1}} R < \frac{\exp(4)}{2^{d-1}} R$, which proves the claim. \square

PROOF OF LEMMA 2.1. What we need to prove is that the measure of the cell assigned to 0 by $\text{AKT}(v)$ run up to stage n tends to 1 with n . This is again a simple consequence of the fact that the number of Poisson points in a large cube is concentrated around the volume of the cube and that for any v and n the weak allocation $\text{AKT}_{\omega',v,n}$ is the composition of local allocations between the classes of the dyadic partition and the centers lying

in them. The only extra technicality comes from the fact that we want to prove convergence of the cell volumes uniformly in v , but this can be checked in the same way as in the proof of Lemma 4.2, so we only sketch it here. For $R > 0$, let the events E_R be the same as in that lemma. In particular, for any $\omega' \in E_R$ and $v \in \mathbb{R}^d$ we have that $(2^n - 2^{-n})^d(1 - \rho_n(R)) < N(Q_n \setminus \{0\}) < (2^n + 2^{-n})^d(1 + \rho_n(R))$, where Q_n is the element of $v + 2^n\mathbb{Z}^d + [0, 2^n)^d$ which contains 0. Since $C_{v,n}^{\omega',0}$ is a cell of a local allocation between Q_n and $\omega' \cap Q_n$, $\lambda(C_{v,n}^{\omega',0}) = \frac{2^{nd}}{N(Q_n)} \rightarrow 1$ as $n \rightarrow \infty$, uniformly in v . The events E_R exhaust a set of measure 1 in Ω' , so the claim of the lemma follows. \square

PROOF OF LEMMA 2.2. We take the same events E_R as in Lemma 4.2, and therefore (i) is satisfied. Furthermore, $(E_R)_{R>0}$ is an increasing family.

To show (ii), use the following upper bound together with (ii) and (iii) of Lemma 4.2:

$$\text{diam}(C_{v,n}^{\omega',0} \cup \{0\}) \leq |\text{aux}(0, v, n)|_1 + d|l(C_{v,n}^{\omega',0})|_\infty.$$

To verify that (iii) holds, let us fix $n \geq r_0(R)$. Consider the transformations of the cell of 0 that occur during the steps of the n th stage. There are d steps, and they all can be treated similarly. Therefore we only consider the step along the first axis. Let A denote the cell before this step, and B thereafter. We introduce an auxiliary cell C : if $w \in \mathbb{R}^d$ is the shift vector of the auxiliary point of 0 in this step, then let $C = A + w$. First, on E_R we obtain an upper bound on $\lambda(A\Delta C)$.

$$\lambda(A\Delta C) \leq 2|w||l(A)|_\infty^{d-1} \leq 2c'_n(R)R^{d-1},$$

where we use (ii) and (iii) from Lemma 4.2. Now we bound $\lambda(C\Delta B)$ using (iii) from Lemma 4.2:

$$\lambda(C\Delta B) \leq |l(A)|_\infty^{d-1}|l(A) - l(B)|_\infty \leq R^{d-1}e_n(R).$$

If we consider the two series whose summands are the respective right-hand sides of the two previous highlighted formulas, then both of them are absolutely summable by Lemma 4.2. We obtain $c_n(R) := d(R^{d-1}(2c'_n(R) + e_n(R)))$. \square

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REFERENCES

- [1] AJTAI, M., KOMLÓS, J. and TUSNÁDY, G. (1984). On optimal matchings. *Combinatorica* 4 259–264. [MR0779885](#)

- [2] CHATTERJEE, S., PELED, R., PERES, Y. and ROMIK, D. (2010). Gravitational allocation to Poisson points. *Ann. of Math. (2)* **172** 617–671. [MR2680428](#)
- [3] CHATTERJEE, S., PELED, R., PERES, Y. and ROMIK, D. (2010). Phase transitions in gravitational allocation. *Geom. Funct. Anal.* **20** 870–917. [MR2729280](#)
- [4] HOFFMAN, C., HOLROYD, A. E. and PERES, Y. (2006). A stable marriage of Poisson and Lebesgue. *Ann. Probab.* **34** 1241–1272. [MR2257646](#)
- [5] HOLROYD, A. E. and LIGGETT, T. M. (2001). How to find an extra head: Optimal random shifts of Bernoulli and Poisson random fields. *Ann. Probab.* **29** 1405–1425. [MR1880225](#)
- [6] HOLROYD, A. E., PEMANTLE, R., PERES, Y. and SCHRAMM, O. (2009). Poisson matching. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 266–287. [MR2500239](#)
- [7] HOLROYD, A. E. and PERES, Y. (2005). Extra heads and invariant allocations. *Ann. Probab.* **33** 31–52. [MR2118858](#)
- [8] HUESMANN, M. and STURM, K.-T. (2013). Optimal transport from Lebesgue to Poisson. *Ann. Probab.* **41** 2426–2478. [MR3112922](#)
- [9] KRIKUN, M. (2007). Connected allocation to Poisson points in \mathbb{R}^2 . *Electron. Commun. Probab.* **12** 140–145. [MR2318161](#)
- [10] STOYAN, D., KENDALL, W. S. and MECKE, J. (1987). *Stochastic Geometry and Its Applications*. Akademie-Verlag, Berlin. [MR0879119](#)
- [11] TALAGRAND, M. and YUKICH, J. E. (1993). The integrability of the square exponential transportation cost. *Ann. Appl. Probab.* **3** 1100–1111. [MR1241036](#)

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