Existence of solutions in non-convex dynamic programming and optimal investment*

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Abstract

We establish the existence of minimizers in a rather general setting of dynamic stochastic optimization in finite discrete time without assuming either convexity or coercivity of the objective function. We apply this to prove the existence of optimal investment strategies for non-concave utility maximization problems in financial market models with frictions, a first result of its kind. The proofs are based on the dynamic programming principle whose validity is established under quite general assumptions.

Keywords. non-convex optimization; dynamic programming; non-concave utility functions; market frictions; illiquidity

1 Introduction

In classical optimal investment problems rational agents maximize their expected utility, which is usually assumed the be a concave function of their terminal wealth. Concavity is justified by the *risk aversion* of the given agent; see e.g. [4], [25] or [19, Chapter 2]. However, recently there has been growing interest in non-concave utilities as well. For instance, the alternative theory of [45, 23] considered so-called "S-shaped" utilities which are convex (risk-seeking) up to a certain wealth level and concave (risk-averse) above it. They also argued that investors distort objective probabilities in their decision-making procedures.

In order to tackle this lack of concavity, there is a need for new mathematical tools. The first contribution of the present paper is to prove a fairly general dynamic programming principle for discrete time, multistep stochastic

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optimization problems with a not necessarily concave objective function (Theorem 3 below). We adopt the general stochastic dynamic programming format of [40] and [18], which extends more familiar stochastic control problems in discrete time. We extend the existence results of [40] and [18] by relaxing their assumptions on compactness and convexity. We follow the arguments of [30] where it was assumed that the objective is given in terms of a convex integrand that has an integrable lower bound. In the context of utility maximization, the boundedness means that the utility functions are bounded from above but we pose no restrictions on its domain. The unbounded case is left for future research. It has been argued that being bounded above is a rather natural assumption on a utility function; see e.g. [1, 42, 26].

Our main existence result, Theorem 3 below, is a direct extension of the existence result contained in [30, Theorem 2]. It was shown in [30] how the abstract results on stochastic dynamic programming quickly yield extensions of some fundamental results in financial mathematics to nonlinear market models with illiquidity effects and portfolio constraints. Similarly, the main result of this paper gives existence results on optimal investment in nonlinear market models for nonconcave utility functions. The financial applications are given in Sections 4 and 5.

It is clear that a mere existence result is not of much practical significance if nothing else is known of the solutions of an optimization problem. Existence is, however, a basic first step in the analysis and in the general class of problems considered here, a highly nontrivial question already. The techniques used in existence proofs often provide tools and estimates that later prove useful in solving problems in more concrete problem classes. Furthermore, existence results together with counterexamples delineate what kind of assumptions (on the utility and on the underlying market model) are necessary to have a well-posed problem.

There exists wide literature on existence results on optimal investment beyond the classical setting of concave utilities and perfectly liquid financial markets. The rest of this section gives an overview of the relevant literature in order to put our financial contributions in perspective.

Existence results for optimal strategies in general, semimartingale models of frictionless markets were obtained in [24, 43] for concave utility functions, see also the references therein for earlier developments. Subsequently, models with transaction costs also received a treatment, still in the case of concave utilities, see [6, 8]. Here we do not review the plethora of papers in more specific model classes.

Studies on non-concave utilities are less abundant. One-step models of frictionless markets were considered in [21, 5]. Multistep models posed various challenges: in the presence of probability distortions weak convergence techniques had to be applied, [10, 34] and, as shown in [31], in this case the domain of optimization may fail to be closed. With no distortions, the optimization problems could be treated by dynamic programming but the absence of concavity requires more involved arguments, see [31, 11, 12]. The case of bounded above utilities was treated in [31]. Possibly unbounded utilities appear in [11] and in [12]. The former paper treats utility functions defined on $\mathbb{R}$ while the latter considers utilities on $\mathbb{R}_+$. Certain recursive utility specifications figure in [2, 16] but they are very different in spirit from all the other works cited. Due to the mathematical difficulties, continuous-time studies focused mainly on the case of complete markets where every contingent claim can be replicated; see [9, 3, 22, 7, 13, 36, 32, 33].

All the articles in the previous paragraph assumed frictionless trading in the given financial market. A more realistic description of the trading mechanism must account for illiquidity effects and/or transaction costs as well. To our knowledge, all previous existence results on optimal investment under such market frictions assume a concave utility function; see e.g. [20, 29, 15] and the references therein.

Theorem 4 below provides an existence result for optimal investment in discrete-time illiquid markets and with bounded above, but not necessarily concave utilities defined on $\mathbb{R}$. This seems to be the first result involving non-concave utilities and markets with frictions at the same time. Further extension is given in Theorem 5 which extends the market model by allowing for general convex trading costs and portfolio constraints. In particular, the model does not assume the existence of a cash-account (a perfectly liquid numeraire asset) a priori. Moreover, we allow for intertemporal random endowments/liabilities.

2 Dynamic programming

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a complete filtered probability space and let h be a proper *normal $\mathcal{F}$-integrand* on $\mathbb{R}^n$, i.e. an extended real-valued $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$-measurable function such that $h(\cdot, \omega)$ is lower semicontinuous (lsc) for P-almost every $\omega \in \Omega$. A normal integrand may be interpreted as a "random lsc function". Accordingly, properties of normal integrands are interpreted in the P-almost sure sense. For example, a normal integrand h is convex, positively homogeneous, positive on a set $C \subseteq \mathbb{R}^n, \ldots$ if there is an $A \in \mathcal{F}$ with $P(A) = 1$ such that $h(\cdot, \omega)$ is convex, positively homogeneous, positive on $C, \ldots$ for all $\omega \in A$. This is consistent with the convention of interpreting inequalities etc. for random variables in the P-almost sure sense. Indeed, random variables may be viewed as normal integrands which do not depend on x.

We will make extensive use of the theory of normal integrands and measurable set-valued mappings found in [41, Chapter 14]. In [41], a normal integrand is defined as a function f whose *epigraphical mapping* $\omega \mapsto \operatorname{epi} f := \{(x, \alpha) \,|\, f(x, \omega) \leq \alpha\}$ is $\mathcal{F}$-measurable and closed-valued. Recall that a set-valued mapping $S : \Omega \rightrightarrows \mathbb{R}^n$ is $\mathcal{F}$-measurable if $\{\omega \in \Omega \,|\, S(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$ for all open $U \subseteq \mathbb{R}^n$. Given that $\mathcal{F}$ is complete, our definition of a normal integrand is consistent with that of [41]. Indeed, by [41, Corollary 14.34], the epi-graphical mapping of an extended real-valued function f is $\mathcal{F}$-measurable and closed-valued if and only if f is $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$-measurable and $f(\cdot, \omega)$ is lsc for all $\omega \in \Omega$. Moreover, if the latter property holds P-almost surely, one can redefine $f(\cdot, \omega)$ on a P-null set so that it is lsc for all $\omega \in \Omega$.

Given a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, we will denote the set of $\mathcal{G}$-measurable $\mathbb{R}^d$-valued random variables by $L^0(\Omega, \mathcal{G}, P; \mathbb{R}^d)$, or simply by $L^0(\mathcal{G})$ when d is clear from the context. The set of integrable $\mathbb{R}$-valued random variables will be denoted by L^1.

We will study the dynamic stochastic optimization problem

$$\text{minimize} \quad Eh(x) := \int h(x(\omega), \omega) dP(\omega) \quad \text{over } x \in \mathcal{N}, \tag{P}$$

where $\mathcal{N} := \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\}$ for given integers n_t such that $n_0 + \cdots n_T = n$. Here and in what follows, we define the expectation of an extended real-valued random variable as $+\infty$ unless its positive part is integrable. For simplicity, we assume throughout the article that there is an $m \in L^1$ such that $h \geq m$. We also assume that (P) is feasible in the sense that $\inf_{x \in \mathcal{N}} Eh(x) < \infty$. In particular, h is then *proper* in the sense that the functions $h(\cdot, \omega)$ are not identically $+\infty$ and they do not take on the value $-\infty$.

Given a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the integrable lower bound m implies, by Theorem 2.1 and Corollary 2.2 of [14], the existence of a normal $\mathcal{G}$-integrand $E^{\mathcal{G}} h$ such that $E^{\mathcal{G}} h \geq E^{\mathcal{G}} m$ and

$$(E^{\mathcal{G}} h)(x(\cdot), \cdot) = E^{\mathcal{G}} h(x(\cdot), \cdot) \quad P\text{-a.s.}$$

for all $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$. The normal integrand $E^{\mathcal{G}} h$ is called the *conditional expectation* of h. It is a proper normal integrand as soon as $h(x(\cdot), \cdot)$ is integrable for some $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$.

We will use the notation $E_t = E^{\mathcal{F}_t}$ and $x^t = (x_0, \ldots, x_t)$ and define extended real-valued functions $h_t, \tilde{h}_t : \mathbb{R}^{n_0 + \cdots + n_t} \times \Omega \to \overline{\mathbb{R}}$ recursively for $t = T, \ldots, 0$ by the general dynamic programming recursion

$$\begin{aligned} \tilde{h}_T &= h, \\ h_t &= E_t \tilde{h}_t, \\ \tilde{h}_{t-1}(x^{t-1}, \omega) &= \inf_{x_t \in \mathbb{R}^{n_t}} h_t(x^{t-1}, x_t, \omega). \end{aligned} \tag{1}$$

This generalization of the classical dynamic programming recursion appeared first in Rockafellar and Wets [40] and Evstigneev [18]. In order to guarantee that the above recursion is well-defined and that optimal solutions exist, we will need to impose appropriate growth conditions on the functions h_t.

Following [41], we say that a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is *level-bounded in x locally uniformly in u* if for each $\bar{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ the set $\{(x, u) \,|\, u \in U, f(x, u) \leq \alpha\}$ is bounded for some neighborhood U of $\bar{u}$.

Theorem 1. *Assume that, for $t = 0, \ldots, T$, the function $(x^{t-1}, x_t) \mapsto h_t(x^{t-1}, x_t, \omega)$ is level-bounded in x_t locally uniformly in x^{t-1} for P-almost every ω whenever h_t is well-defined and proper. Then h_t is a well-defined proper normal integrand for all $t = 0, \ldots, T$ and*

$$Eh_t(x^t) \geq \inf_{x' \in \mathcal{N}} Eh(x') \quad t = 0, \ldots, T \quad \forall x \in \mathcal{N}. \tag{2}$$

Optimal solutions $x \in \mathcal{N}$ exist and they are characterized by the condition

$$x_t(\omega) \in \operatorname*{argmin}_{x_t} h_t(x^{t-1}(\omega), x_t, \omega) \quad P\text{-}a.s. \quad t = 0, \dots, T,$$

which is equivalent to having equalities in (2).

Proof. Clearly, $\tilde{h}_T$ is a well-defined proper normal integrand. As noted above, the lower bound $\tilde{h}_T \geq m$ implies that h_T is a well-defined proper normal integrand with $h_T \geq E_T m$. Assume now that h_t is well-defined proper normal integrand with an integrable lower bound m_t. By [41, Proposition 14.47], the level boundedness condition implies that $\tilde{h}_{t-1}$ is a normal integrand. It is also clear that $\tilde{h}_{t-1} \geq m_t$ so that h_{t-1} is well-defined proper normal integrand with the lower bound $E_{t-1}m_t$. The first claim now follows by induction.

Given $x \in \mathcal{N}$, we have

$$Eh_t(x^t) \geq E\tilde{h}_{t-1}(x^{t-1}) = Eh_{t-1}(x^{t-1}) \quad t = 1, \dots, T,$$

by definition of $\tilde{h}_{t-1}$ and h_{t-1}, so

$$Eh(x) = Eh_T(x^T) \geq \dots \geq Eh_t(x^t) \geq \dots \geq Eh_0(x^0) \geq E \inf_{x_0 \in \mathbb{R}^{n_0}} h_0(x_0).$$

On the other hand, by [41, Theorem 1.17], the infimum in $\tilde{h}_{t-1}(x^{t-1}(\omega)) = \inf_x h_t(x^{t-1}(\omega), x, \omega)$ is attained P-almost surely. By [41, Proposition 14.45(c)], the function $f(x, \omega) := h_t(x^{t-1}(\omega), x, \omega)$ is an $\mathcal{F}_t$-measurable normal integrand so, by [41, Theorem 14.37], $h_t(x^{t-1}(\omega), \cdot, \omega)$ admits an $\mathcal{F}_t$-measurable ω-wise minimizer. By induction, there exists an $x \in \mathcal{N}$ such that the above inequalities hold as equalities. □

The above result extends Theorems 1 and 2 of [18] where it was assumed that the sets $\{x \in \mathbb{R}^n \,|\, h(x, \omega) \leq \alpha\}$ are compact for every $\omega \in \Omega$ and $\alpha \in \mathbb{R}$. Indeed, by [18, Theorem 5], this compactness condition is inherited by $\tilde{h}_t$ and h_t, which clearly implies the uniform level-boundedness assumption in Theorem 1. On the other hand, the compactness condition often fails in models of financial mathematics where mere no-arbitrage conditions have been found sufficient.

3 Asymptotic analysis of the existence condition

We now come to the main result of the paper which is a nonconvex extension of [30, Theorem 2], which in turn extends well-known results in financial mathematics on the existence of optimal trading strategies under the no-arbitrage condition. Applications to optimal investment with nonconvex utilities will be given in Sections 4 and 5 below.

We start by recalling some more terminology from variational analysis; see [41]. The *indicator function* of a set C is the extended real-valued function δ_C defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise. Given a function g

on $\mathbb{R}^n$, the set $\operatorname{dom} g := \{x \in \mathbb{R}^n \mid g(x) < \infty\}$ is called the *effective domain* of g. The *horizon function* of a proper function g is the lsc positively homogeneous function g^∞ given by

$$g^\infty(x) := \lim_{\alpha \nearrow \infty} \inf_{\substack{\gamma > \alpha \\ x' \in \mathbb{B}(x, 1/\alpha)}} \frac{g(\gamma x')}{\gamma},$$

where $\mathbb{B}(x, 1/\alpha)$ denotes the closed ball of radius $1/\alpha$ around x; see [41, Theorem 3.21].

As noted on page 89 of [41], the horizon function is not affected if we replace g by $x \mapsto g(x + \bar{x}) + c$, where $\bar{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Thus, for any $\bar{x} \in \operatorname{dom} g$

$$g^\infty(x) = \lim_{\alpha \nearrow \infty} \inf_{\substack{\gamma > \alpha \\ x' \in \mathbb{B}(x, 1/\alpha)}} \frac{g(\gamma x' + \bar{x}) - g(\bar{x})}{\gamma}.$$

It follows that

$$g^\infty(x) \le \liminf_{\alpha \nearrow \infty} \frac{g(\alpha x + \bar{x}) - g(\bar{x})}{\alpha}. \tag{3}$$

for all $\bar{x} \in \operatorname{dom} g$. We will say that a function g is *asymptotically regular* if (3) holds as an equality for all $\bar{x} \in \operatorname{dom} g$. This definition is motivated by the fact that horizon functions of asymptotically regular functions obey some convenient calculus rules similar to those of the recession function in the convex case; see the appendix. Recall that the *recession function* $0^+ g$ of a proper convex lsc function is given by

$$(0^+ g)(x) = \lim_{\alpha \nearrow \infty} \frac{g(\alpha x + \bar{x}) - g(\bar{x})}{\alpha}$$

for every $\bar{x} \in \operatorname{dom} g$; see [38, Theorem 8.5].

The regularity property is also preserved under such operations. Primary examples of asymptotically regular functions are convex lsc functions as well as lsc functions on the real line.

Example 1. *By [41, Theorem 3.21], lsc proper convex functions are asymptotically regular and their horizon functions coincide with recession functions as defined in [38]. All proper lsc functions on the real line are regular as well. Indeed, for $w > 0$ (analogously for $w < 0$), we see from the definition that $g^\infty(w) = (g + \delta_{\mathbb{R}_+})^\infty(w)$, so the positive homogeneity of g^∞ and an expression similar to [41, Theorem 3.26] give*

$$g^\infty(x) = x g^\infty(1) = x \liminf_{\alpha \nearrow \infty} \frac{g(\alpha)}{\alpha} = \liminf_{\alpha \nearrow \infty} \frac{g(\alpha x)}{\alpha}.$$

Applying this to the translated function $g_{\bar{x}}(x) := g(x + \bar{x})$ and using the fact that $g_{\bar{x}}^\infty = g^\infty$ proves the claim.

We now return to problem (P) and note that, given a proper normal integrand h, the function h^∞ defined by $h^\infty(\cdot, \omega) := h(\cdot, \omega)^\infty$ is a proper normal integrand, by [41, Exercise 14.54].

The following gives a sufficient condition for Theorem 1.

Lemma 2. *If* $\{x \in \mathcal{N} \mid h^\infty(x) \le 0\} = \{0\}$*, then the assumption of Theorem 1 is satisfied.*

Proof. We proceed by induction. Assume first that h_t is a well-defined proper normal integrand such that $\{x^t \in \mathcal{N}_t \,|\, h^\infty_t(x^t) \le 0\} = \{0\}$, where $\mathcal{N}_t := \{x^t \,|\, x_s \in L^0(\mathcal{F}_s)\ s = 0, \dots, t\}$. This implies $h^\infty_t(0, x_t) > 0$ for all $x_t \ne 0$ since otherwise there would exist a nonzero $x_t \in L^0(\mathcal{F}_t)$ with $h^\infty_t(0, x_t) \le 0$. By [41, Theorem 3.31], both h_t and h^∞_t (since the horizon function of h^∞_t is h^∞_t itself) are level-bounded in x_t locally uniformly in x^{t-1} and

$$\tilde{h}^\infty_{t-1}(x^{t-1}, \omega) = \inf_{x_t \in \mathbb{R}^{n_t}} h^\infty_t(x^{t-1}, x_t, \omega).$$

Just like in the proof of Theorem 1, $\tilde{h}_{t-1}$, h_{t-1}, $\tilde{h}^\infty_{t-1}$ and h^∞_{t-1} are then well-defined proper normal integrands and for every $x \in \mathcal{N}$ there is an $\mathcal{F}_t$-measurable x_t such that

$$\tilde{h}^\infty_{t-1}(x^{t-1}(\omega), \omega) = h^\infty_t(x^{t-1}(\omega), x_t(\omega), \omega)$$

for P-almost every ω. Thus,

$$\begin{aligned}
&\{x^{t-1} \in \mathcal{N}_{t-1} \,|\, h^\infty_{t-1}(x^{t-1}(\omega), \omega) \le 0 \text{ a.s.}\}\\
&\subseteq \{x^{t-1} \in \mathcal{N}_{t-1} \,|\, \tilde{h}^\infty_{t-1}(x^{t-1}(\omega), \omega) \le 0 \text{ a.s.}\}\\
&= \{x^{t-1} \in \mathcal{N}_{t-1} \,|\, \exists x_t \in L^0(\mathcal{F}_t) :\ h^\infty_t(x^{t-1}(\omega), x_t(\omega), \omega) \le 0 \text{ a.s.}\}
\end{aligned}$$

where the inclusion follows from Lemma 6 in appendix. The last expression equals $\{0\}$ by the induction hypothesis. It now suffices to note that

$$\{x \in \mathcal{N} \,|\, h^\infty_T(x) \le 0\} \subseteq \{x \in \mathcal{N} \,|\, h^\infty(x) \le 0\},$$

by Lemma 6 in the appendix. □

Recall that a set-valued mapping $S : \Omega \rightrightarrows \mathbb{R}^n$ is *measurable* if $S^{-1}(O) \in \mathcal{F}$ for every open $O \subset \mathbb{R}^n$. Here $S^{-1}(O) := \{\omega \in \Omega \,|\, S(\omega) \cap O \ne \emptyset\}$ is the inverse image of O.

We are now ready to prove the main result of this paper. It extends the existence result from [30, Theorem 9] to nonconvex dynamic programming.

Theorem 3. *Assume that there is a measurable set-valued mapping* $N : \Omega \rightrightarrows \mathbb{R}^n$ *such that* $N(\omega)$ *is linear for each* ω*,*

$$\{x \in \mathcal{N} \mid h^\infty(x) \le 0\} = \{x \in \mathcal{N} \,|\, x \in N\},$$

and that $Eh(x + x') = Eh(x)$ *for all* $x, x' \in \mathcal{N}$ *with* $x' \in N$ *almost surely. Then optimal solutions to* (P) *exist.*

Proof. By [27, Lemma 5.3], there exist $\mathcal{F}_t$-measurable set-valued mappings N_t such that $N_t(\omega)$ are linear and $x_t \in L^0(\mathcal{F}_t; N_t)$ if and only if $\tilde{x}_t = x_t$ for some $\tilde{x} \in \mathcal{N}$ with $\tilde{x} \in N$ and $\tilde{x}^{t-1} = 0$. Let

$$\bar{h}(x, \omega) = h(x, \omega) + \delta_{\Gamma(\omega)}(x),$$

where $\Gamma = N_0^\perp \times \cdots \times N_T^\perp$ and $N_t^\perp(\omega)$ denotes the orthogonal complement of $N_t(\omega)$. By [41, Exercise 14.12 and Proposition 14.11], Γ is measurable so, by [41, Example 14.32 and Proposition 14.44], $\bar{h}$ is a normal integrand.

Let us show that for every $x \in \mathcal{N}$, there exists $\bar{x} \in \mathcal{N}$ such that

$$Eh(x) = E\bar{h}(\bar{x}). \tag{4}$$

Let $\bar{x}_0$ be the projection of x_0 to $N_0^\perp$. Since x_0 and N_0 are $\mathcal{F}_0$-measurable, $\bar{x}_0$ is $\mathcal{F}_0$-measurable [41, Exercise 14.17]. By definition of N_0, there exists $\tilde{x} \in \mathcal{N}$ with $\tilde{x} \in N$ and $\tilde{x}_0 = -(x_0 - \bar{x}_0) \in N_0$. By assumption, $Eh(x) = Eh(x + \tilde{x})$. Moreover, $x_0 + \tilde{x}_0 = \bar{x}_0 \in N_0^\perp$. We may repeat the argument for $t = 1, \dots, T$ to construct $\bar{x} \in \mathcal{N}$ with the claimed properties. Since $\bar{h} \geq h$ and (4) holds, we have that minimizers of $E\bar{h}$ minimize Eh. By Theorem 1, it thus suffices to show that $\bar{h}$ satisfies the condition in Lemma 2.

Clearly, $\delta_\Gamma^\infty = \delta_\Gamma$. By Lemma 7 below, $\bar{h}^\infty \geq h^\infty + \delta_\Gamma$, so

$$\{x \in \mathcal{N} \mid \bar{h}^\infty(x) \leq 0\} \subseteq \{x \in \mathcal{N} \mid h^\infty(x) \leq 0,\ x \in \Gamma\}.$$

An element x of the set on the right has both $x_0 \in N_0$ and $x_0 \in N_0^\perp$ and thus, $x_0 = 0$. We then have $x_1 \in N_1$ and, similarly, $x_1 = 0$. Repeating the argument for $t = 2, \dots, T$, we get $x = 0$. □

If h is a convex normal integrand such that $\{x \in \mathcal{N} \mid h^\infty(x) \leq 0\}$ is a linear space, the condition of Theorem 3 is satisfied with

$$N(\omega) = \{x \in \mathbb{R}^n \mid h^\infty(x, \omega) \leq 0,\ h^\infty(-x, \omega) \leq 0\}.$$

Indeed, this set is linear and, by [38, Corollary 8.6.1], $h(x + x', \omega) = h(x, \omega)$ for all $x' \in N(\omega)$. We thus recover the existence result in [30, Theorem 2] (recall that the horizon function of a proper lsc convex function coincides with its recession function). Applications to nonconvex problems will be given in Sections 4 and 5 below.

4 Optimal investment under market frictions

This section applies Theorem 3 to the problem of optimal investment in illiquid financial markets. We consider the discrete-time version of the model in [20]; see also [17].

Let Z_t, $t = 0, \dots, T$ be an adapted sequence of $(d-1)$-dimensional random variables representing the marginal price of $d - 1$ risky assets in an economy. We imagine that if "very small" amounts of asset i were traded then this would take place at the price Z_t^i at time t. We assume that the riskless asset in this economy has a price identically 1 at all times.

As in Carassus and Rásonyi [11], we model trading strategies by predictable processes $\phi = (\phi_t)_{t=1}^T$, where ϕ_t denotes the portfolio of risky assets held over $(t-1, t]$. Thus $\Delta\phi_t = \phi_t - \phi_{t-1}$ is the portfolio of risky assets bought at time

$t-1$ and $\phi_t = \phi_0 + \sum_{i=1}^t \Delta\phi_i$. In perfectly liquid markets, the corresponding "value process" starting at initial capital x is given by

$$V_t^x = x + \sum_{i=1}^t \phi_i \cdot \Delta Z_i.$$

In order to model illiquidity effects, we first rewrite the above as

$$V_t^x = x - \sum_{i=1}^t \Delta\phi_i \cdot Z_{i-1} + \phi_t \cdot Z_t,$$

with the convention $\phi_0 = 0$. As usual, the last term is interpreted as the liquidation value one would obtain by liquidating the portfolio at time t. Under illiquidity, it is more meaningful to track the position on the cash account without assuming liquidation at every t. We denote the cash position held over $(t-1, t]$ by X_t^0.

If illiquidity costs at time t are given by an $\mathcal{F}_t$-normal integrand $G_t : \mathbb{R}^{d-1} \times \Omega \to \mathbb{R}_+$, we have that the change in the cash position at time $t-1$ is

$$\Delta X_t^0(\phi) = -\Delta\phi_t \cdot Z_{t-1} - G_{t-1}(\Delta\phi_t)$$

(recall that $\Delta\phi_t$ is the portfolio of risky assets bought at time $t-1$). Summing up, we get

$$X_t^0(\phi) := X_0^0 - \sum_{i=1}^t \Delta\phi_i \cdot Z_{i-1} - \sum_{i=1}^t G_{i-1}(\Delta\phi_i).$$

The "liquidation value" of the portfolio at time T is given by

$$X_{T+1}^0(\phi) := X_0^0 - \sum_{i=1}^{T+1} \Delta\phi_i \cdot Z_{i-1} - \sum_{i=1}^{T+1} G_{i-1}(\Delta\phi_i),$$

where $\phi_{T+1} := 0$. Note that the $\Delta\phi_i$ are control variables here while X_t^0 is the controlled process.

We assume that the functions G_t are convex in the first argument and

$$\lim_{\alpha\to\infty} \frac{G_t(\alpha x, \omega)}{\alpha} \geq -Z_t(\omega) \cdot x, \quad \forall x \in \mathbb{R}^{d-1}, \tag{5}$$

$$\lim_{\alpha\to\infty} \frac{G_t(\alpha x, \omega)}{\alpha} > -Z_t(\omega) \cdot x, \quad \forall x \notin \mathbb{R}_-^{d-1}. \tag{6}$$

These conditions hold in particular if liquidity costs are *superlinear* in the volume; see Guasoni and Rásonyi [20]. The above condition allows also for *free disposal* of all securities in the sense that the total cost $S_t(x,\omega) := G_t(x,\omega) + Z_t(\omega) \cdot x$ is nondecreasing with respect to the partial order induced by $\mathbb{R}_-^{d-1}$. This is quite a natural assumption e.g. in most securities markets where unit prices are always nonnegative.

We will consider an optimal investment problem of an agent whose risk preferences are described by a possibly nonconcave utility function $u : \mathbb{R} \to \mathbb{R}$ with $u(0) = 0$. More precisely, we will assume that u is nondecreasing, upper semicontinuous, bounded from above and that

$$\limsup_{\alpha\to\infty} \frac{u(\alpha w, \omega)}{\alpha} < 0 \quad \forall w < 0. \tag{7}$$

If u is nondecreasing and it is concave on both $(-\infty, v)$ and $[v, \infty)$ for some $v \in \mathbb{R}$ then (7) clearly holds; see [9] for such a setting.

An application of Theorem 3 yields the following existence result; see Example 4 below for the proof.

Theorem 4. *Let G_t be convex normal integrands, satisfying* (5), (6) *and* $G_t(0, \omega) = 0$ *almost surely. Let u be nondecreasing, upper semicontinuous and bounded from above, satisfying* (7) *and* $u(0) = 0$. *For an investor with initial capital $X_0^0 = w$ and zero initial stock position $\phi_0^j = 0$, $j = 1, \dots, d-1$, there exists an optimal strategy ϕ^* with*

$$\sup_{\phi} Eu(X^0_{T+1}(\phi)) = Eu(X^0_{T+1}(\phi^*)).$$

Remark 1. A similar result has been obtained in Theorem 5.1 of [20], in a continuous-time setting. However, in the discrete-time case, Theorem 4 above goes much further. In [20], u was assumed concave while we do not need this assumption here. Also, in [20], $|G_t(x)|$ was assumed to dominate a positive multiple of a power function $|x|^\alpha$ with $\alpha > 1$ while here we only need (5) and (6). One can also allow a random endowment in the problem without any additional work. This and other extensions will be considered in the following section.

5 Extension to markets with portfolio constraints

This section extends the analysis of the previous section to market models with general convex trading costs and convex portfolio constraints. In particular, we do not assume the existence of a cash account a priori. As in [29], we assume that trading costs are given by an adapted sequence $S = (S_t)_{t=0}^T$ of convex $\mathcal{F}_t$-normal integrands on $\mathbb{R}^d$ such that $S_t(0, \omega) = 0$. We also allow for portfolio constraints given by an adapted sequence $D = (D_t)_{t=0}^T$ of closed convex sets in $\mathbb{R}^d$, each containing the origin. We assume that $D_T = \{0\}$, i.e. that the agent liquidates her portfolio at the terminal date.

In a market without perfectly liquid assets it is important to distinguish between payments at different points in time. We will describe the agent's preferences over sequences of payments by a normal integrand V on $\mathbb{R}^{T+1}$. More precisely, the agent prefers to make an adapted sequence $c^1 = (c^1_t)_{t=0}^T$ of payments over another c^2 if

$$EV(c^1) < EV(c^2)$$

while she is indifferent between the two if the expectations are equal. A possible choice would be $V(c,\omega) = -\sum_{t=0}^{T} u(-c_t)$ for (possibly nonconcave) utility functions u_t. We allow $V(\cdot,\omega)$ to be nonconvex but will require the following.

Assumption 1. There is an $m \in L^1$ such that $V(\cdot,\omega) \geq m(\omega)$, the functions $V(\cdot,\omega)$ are asymptotically regular, nonincreasing in directions of $\mathbb{R}^{T+1}_-$, $V(0,\omega) = 0$ and

$$V^\infty(y,\omega) \leq 0 \iff y \in \mathbb{R}^{T+1}_-$$

for P-almost every ω.

Remark 2 (Inada condition)**.** *In Assumption 1, the asymptotic regularity condition holds in particular when*

$$V^\infty(\cdot,\omega) = \delta_{\mathbb{R}^{T+1}_-} \quad \forall \omega \in \Omega. \tag{8}$$

Indeed, by (3)*,*

$$V^\infty(y,\omega) \leq \liminf_{\alpha \nearrow \infty} \frac{V(\alpha y + \bar{y},\omega) - V(\bar{y},\omega)}{\alpha},$$

for all $\bar{y} \in \operatorname{dom} V(\cdot,\omega)$, while $V \geq m$ implies $V^\infty \geq 0$. It thus suffices to note that the last expression cannot be positive for $y \in \mathbb{R}^{T+1}_-$ when $V(\cdot,\omega)$ is nonincreasing in the directions of $\mathbb{R}^{T+1}_-$.

Condition (8) *can be seen as an extension of the classical* Inada condition*. Indeed, a differentiable concave function u on $\mathbb{R}$ satisfies the Inada condition if its derivative approaches $0/+\infty$ when its argument approaches $+\infty/-\infty$. This implies that the recession function of $V(y) := -u(-y)$ equals $\delta_{\mathbb{R}_-}$. Indeed, the recession function of V can be expressed as*

$$V^\infty(y) := \begin{cases} v^+ y & \text{if } y \geq 0, \\ v^- y & \text{if } y \leq 0, \end{cases}$$

where $v^+ = \sup_y V'(y)$ and $v^- = \inf_y V'(y)$. This follows by elementary arguments from the definition of the recession function.

Given an adapted sequence c, consider the problem

$$\text{minimize} \quad EV\left(S(\Delta z) + c\right) \quad \text{over} \quad z \in \mathcal{N}_D, \tag{9}$$

where $\mathcal{N}_D := \{z \in \mathcal{N} \mid z_t \in D_t \ \forall t\}$ denotes the set of feasible trading strategies, $z_{-1} := 0$ and $S(\Delta z)$ denotes the adapted process $(S_t(\Delta z_t(\omega),\omega))_{t=0}^T$ of trading costs. Here z_t denotes the portfolio of assets held over $(t, t+1]$ (In the notation of the previous section $z_t = (X^0_{t+1}, \phi_{t+1})$). Recall that $D_T = \{0\}$ so the agent is required to liquidate his positions at time T. The sequence $c = (c_t)_{t=0}^T$ is interpreted as a financial liability that may involve payments possibly at every t. We allow c_t to take arbitrary real-values so it may describe endowments as well as liabilities. Problem (9) can be interpreted as an *asset-liability management* problem where one looks for trading strategies z whose proceeds cover the liability c as well as possible as measured by EV. In the convex case, problem (9) was studied in [29], where existence of solutions was derived from the results of [30].

Example 2. *Problems where the portfolios are required to be self-financing (as in Section 4) fit* (9) *with*

$$V(y,\omega) = \begin{cases} V_T(y_T,\omega) & \text{if } y_t \le 0 \text{ for } t < T, \\ +\infty & \text{otherwise,} \end{cases} \tag{10}$$

where V_T is a normal integrand on $\mathbb{R}$. Problem (9) *can then be written with explicit budget constraints as*

$$\begin{array}{lll} \text{minimize} & EV_T(S_T(\Delta z_T) + c_T) \quad \text{over} \quad z \in \mathcal{N}_D \\ \text{subject to} & S_t(\Delta z_t) + c_t \le 0, \quad t = 0,\dots,T-1. \end{array}$$

Function (10) *satisfies Assumption 1 as soon as $V_T \ge m$, $V_T(\cdot,\omega)$ is nondecreasing, $V_T(0,\omega) = 0$ and*

$$\liminf_{\alpha\nearrow\infty} \frac{V_T(\alpha y_T,\omega)}{\alpha} > 0 \quad \forall y_T > 0 \tag{11}$$

almost surely. Indeed, $V(\cdot,\omega)$ is now the sum of the indicator function $\delta_{\mathbb{R}^T_-\times\mathbb{R}}$ and $g_2(y,\omega) := V_T(y_T,\omega)$. Being an lsc proper function on the real line, $V_T(\cdot,\omega)$ is asymptotically regular (see Example 1), so Lemmas 8 and 7 imply that V is asymptotically regular as well and $V^\infty = \delta_{\mathbb{R}^T_-\times\mathbb{R}} + g_2^\infty$. Condition (11) *implies $V_T^\infty(y_T,\omega) > 0$ for $y_T > 0$. On the other hand, since V_T is nondecreasing, $V_T^\infty(y_T,\omega) \le 0$ for $y_T \le 0$ and thus, $V^\infty(y,\omega) \le 0 \iff y \in \mathbb{R}^{T+1}_-$.*

The existence result below involves an auxiliary market model given by

$$S_t^\infty(x,\omega) = \sup_{\alpha>0} \frac{S_t(\alpha x,\omega)}{\alpha},$$
$$D_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha D_t(\omega).$$

By [41, Theorem 3.21], $S_t^\infty(\cdot,\omega)$ is the horizon function of $S_t(\cdot,\omega)$ while by [41, Theorem 3.6], $D_t^\infty(\omega)$ coincides with the *horizon cone* of $D_t(\omega)$ defined in [41, Section 3.B]. Note that in models with proportional transaction costs, S is sublinear so that $S^\infty = S$. Similarly, when the constraints are conical we simply have $D^\infty = D$. By [41, Exercise 14.21], D_t^∞ is $\mathcal{F}_t$-measurable closed convex cone and, by [41, Exercise 14.54], S_t^∞ is $\mathcal{F}_t$-measurable normal integrand sublinear in x.

Theorem 5. *Assume that $\{z \in \mathcal{N}_{D^\infty} \mid S^\infty(\Delta z) \le 0\}$ is a linear space and that V satisfies Assumption 1. Then the infimum in* (9) *is attained.*

Proof. In order to apply Theorem 3, we write (9) as

$$\begin{array}{ll} \text{minimize} & EV(y) \quad \text{over} \quad z \in \mathcal{N}_D,\ y \in \mathcal{N} \\ \text{subject to} & S(\Delta z) + c \leq y, \end{array}$$

where y_t can be interpreted as the total expenditure at time t. This fits (P) with $x = (z, y)$ and $h(x, \omega) = V(y, \omega) + \delta_{C(\omega)}(x)$, where

$$C(\omega) = \{x \,|\, S_t(\Delta z_t, \omega) + c_t(\omega) \leq y_t,\ z_t \in D_t(\omega)\ \forall t\}.$$

The convexity of S and D imply the convexity of $C(\omega)$ so, by Lemma 7, asymptotic regularity of V implies $h^\infty(x, \omega) = V^\infty(y, \omega) + \delta^\infty_{C(\omega)}(x)$. Since $\delta^\infty_{C(\omega)} = \delta_{C^\infty(\omega)}$, [41, Proposition 3.9] and [41, Exercise 3.24] give

$$h^\infty(x, \omega) = \begin{cases} V^\infty(y, \omega) & \text{if } S^\infty_t(\Delta z_t, \omega) \leq y_t,\ z_t \in D^\infty_t(\omega)\ \forall t, \\ +\infty & \text{otherwise.} \end{cases}$$

Having assumed that $V^\infty(y, \omega) \leq 0$ if and only if $y \in \mathbb{R}^{T+1}_-$, we get

$$\begin{aligned} \{x \in \mathcal{N} |\, h^\infty(x) \leq 0\} &= \{x \in \mathcal{N} |\, V^\infty(y) \leq 0,\ z \in D^\infty,\ S^\infty(\Delta z) \leq y\} \\ &= \{x \in \mathcal{N} |\, y \leq 0,\ z \in D^\infty,\ S^\infty(\Delta z) \leq y\} \\ &= \{x \in \mathcal{N} |\, y = 0,\ z \in D^\infty,\ S^\infty(\Delta z) \leq 0\}, \end{aligned}$$

(the inequalities and the inclusions are required to hold almost surely for every $t = 0, \dots, T$) where the last equality follows from the fact that $-S^\infty(-\Delta z) \leq S^\infty(\Delta z)$ (because $S^\infty_t(\cdot, \omega)$ is sublinear) and the assumption that $\{z \in \mathcal{N}_{D^\infty} \,|\, S^\infty(\Delta z) \leq 0\}$ is linear. Defining

$$L(\omega) = \{x \in \mathbb{R}^n \,|\, y = 0,\ z_t \in D^\infty_t(\omega),\ S^\infty_t(\Delta z_t, \omega) \leq 0\ \forall t\}$$

we thus have that the conditions of Theorem 3 are satisfied with $N(\omega) = L(\omega) \cap [-L(\omega)]$. □

The following example specializes Theorem 5 to optimization of terminal utility and market models with a cash account.

Example 3. *Consider again the setting of Example 2 and assume that there is a perfectly liquid asset, say asset* 0*, so that, denoting* $z = (z^0, \tilde{z})$*,*

$$\begin{aligned} S_t(z, \omega) &= z^0 + \tilde{S}_t(\tilde{z}, \omega) \quad t = 0, \dots, T, \\ D_t(\omega) &= \mathbb{R} \times \tilde{D}_t(\omega) \quad t = 0, \dots, T-1, \end{aligned}$$

while still $D_T = \{0\}$*. Here* $\tilde{S}_t$ *and* $\tilde{D}_t$ *are* $\mathcal{F}_t$*-measurable normal integrands and set-valued mappings, respectively, on* $\mathbb{R}^{d-1}$ *such that* $\tilde{D}_T = \{0\}$*. We can then substitute out the "cash variable"* z^0 *from the problem of Example 2. Indeed,* *given an adapted $\tilde{z}$ it is optimal (since V_T is nondecreasing) to choose z^0 so that the budget constraints are satisfied as equalities. It follows that*

$$z^0_{T-1} = -\sum_{t=0}^{T-1} \tilde{S}_t(\Delta\tilde{z}_t) - \sum_{t=0}^{T-1} c_t,$$

so the problem can be written as (recall again that $D_T = \{0\}$)

$$\text{minimize} \qquad EV_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{z}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad z \in \mathcal{N}_D. \tag{12}$$

Since $S^\infty_t(z,\omega) = z^0 + \tilde{S}^\infty_t(\tilde{z},\omega)$ for $t = 0,\dots,T$ and $D^\infty_t(\omega) = \mathbb{R}\times\tilde{D}^\infty_t(\omega)$ for $t = 0,\dots,T-1$, the linearity condition of Theorem 5 means that

$$\{z \in \mathcal{N} \mid \Delta z^0 + \tilde{S}^\infty(\Delta\tilde{z}) \le 0,\ \tilde{z} \in \tilde{D}^\infty,\ z^0_T = 0\} \tag{13}$$

(the inequality and the inclusion are required to hold almost surely for every $t = 0,\dots,T$) is a linear space. This holds, in particular, if

$$\tilde{S}^\infty_t \ge 0 \quad \textit{and} \quad \tilde{S}_t(\tilde{z})^\infty > 0,\ \forall\tilde{z} \notin \mathbb{R}^{d-1}_-, \tag{14}$$

for $t = 0,\dots,T$. Indeed, the first inequality implies $\Delta z^0 \le 0$ and then $z^0 = 0$ since $z^0_{-1} = 0$, by assumption. Then, the second inequality implies $\Delta\tilde{z}_t \le 0$. Since $z_{-1} = 0$ and $D_T = \{0\}$, by assumption, this can only hold if $z = 0$.

The proof of Theorem 4 is now a simple application of the above example.

Example 4 (Proof of Theorem 4)**.** *Consider Example 3 with $V_T(c,\omega) = -u(-c,\omega)$, $\tilde{S}_t(\tilde{z},\omega) = Z_t(\omega)\cdot\tilde{z} + G_t(\tilde{z},\omega)$, $\tilde{D}_t := \mathbb{R}^{d-1}$, $c_0 = -w$ and $c_t = 0$ for $t = 1,\dots,T$. We can then write problem* (12) *as*

$$\text{maximize} \qquad Eu\left(w - \sum_{t=0}^{T}[Z_t\cdot\Delta\tilde{z}_t + G_t(\Delta\tilde{z}_t)]\right) \quad \text{over} \quad z \in \mathcal{N}_D.$$

This is exactly the problem formulated in Section 4 where the notation $\phi_t = \tilde{z}_{t-1}$ was used. Conditions (11) *and* (14) *now become the conditions on G and u given in Section 4. Indeed, since $\tilde{S}_t(\cdot,\omega)$ are convex,* (14) *becomes* (5) *and* (6)*; see Example 1.*

The linearity condition in Theorem 5 is a generalization of the no-arbitrage condition in classical perfectly liquid markets. Indeed, when $S_t(x) = s_t\cdot x$ and $D_t \equiv \mathbb{R}^d$, the linearity condition means that any $x \in \mathcal{N}_D$ with $s_t\cdot\Delta x_t \le 0$ satisfies $s_t\cdot\Delta x_t = 0$, that is, there is no arbitrage. In nonlinear unconstrained models, it becomes the *robust no-arbitrage condition* introduced by Schachermayer [44]; see [28, Section 4] for details. The linearity condition in Theorem 5 may very well hold even if the model allows for arbitrage. One has $\{z \in \mathcal{N}_{D^\infty} \mid S^\infty(\Delta z) \le 0\} = \{0\}$, for example, when S is such that $S^\infty_t(z,\omega) > 0$ for all $z \notin \mathbb{R}^d_-$. Indeed, $S^\infty(\Delta z) \le 0$ then implies $\Delta z_t \le 0$ componentwise, which must hold as an equality since, by assumption, $x_{-1} = 0$ and $D_T = \{0\}$. Such a condition holds e.g. in limit order markets where the limit order books always have finite depth. Further conditions are given in [28, 29].

Appendix

The following lemma was used in the proof of Lemma 2.

Lemma 6. *We have* $E^{\mathcal{G}}h^\infty \le (E^{\mathcal{G}}h)^\infty$ *and*

$$\{x \in L^0(\mathcal{G}) \,|\, (E^{\mathcal{G}}h)^\infty(x) \le 0\} \subseteq \{x \in L^0(\mathcal{G}) \,|\, h^\infty(x) \le 0\}.$$

Proof. Let $x \in L^0(\mathcal{G})$ and $A \in \mathcal{G}$. We have that

$$\begin{aligned}
E[1_A h^\infty(x)] &= E[1_A \lim_{\alpha \nearrow \infty} \inf_{\substack{\gamma > \alpha \\ x' \in \mathbb{B}(x,1/\alpha)}} \frac{h(\gamma x')}{\gamma}] \\
&= \lim_{\alpha \nearrow \infty} E[1_A \inf_{\substack{\gamma > \alpha \\ x' \in \mathbb{B}(x,1/\alpha)}} \frac{h(\gamma x')}{\gamma}] \\
&\le \lim_{\alpha \nearrow \infty} \inf_{\substack{\gamma \in L^0(\mathcal{G};[\alpha,\infty)) \\ x' \in L^0(\mathcal{G};\mathbb{B}(x,1/\alpha))}} E[1_A \frac{h(\gamma x')}{\gamma}] \\
&= \lim_{\alpha \nearrow \infty} \inf_{\substack{\gamma \in L^0(\mathcal{G};[\alpha,\infty)) \\ x' \in L^0(\mathcal{G};\mathbb{B}(x,1/\alpha))}} E[1_A \frac{(E^{\mathcal{G}}h)(\gamma x')}{\gamma}] \\
&= \lim_{\alpha \nearrow \infty} E[1_A \inf_{\substack{\gamma > \alpha \\ x' \in \mathbb{B}(x,1/\alpha)}} \frac{(E^{\mathcal{G}}h)(\gamma x')}{\gamma}] \\
&= E[1_A (E^{\mathcal{G}}h)^\infty(x)],
\end{aligned}$$

which gives $E^{\mathcal{G}}h^\infty \le (E^{\mathcal{G}}h)^\infty$ since $x \in L^0(\mathcal{G})$ and $A \in \mathcal{G}$ were arbitrary. Here the second and the last equality follow from monotone convergence, and the fourth follows from the interchange rule [39, Theorem 3A].

To prove the second claim, let $x \in L^0(\mathcal{G})$ such that $(E^{\mathcal{G}}h)^\infty(x) \le 0$. By the first claim, $E^{\mathcal{G}}h^\infty(x) \le 0$ almost surely so, by the definition of a conditional integrand,

$$(E^{\mathcal{G}}h^\infty)(x) = E^{\mathcal{G}}h^\infty(x).$$

Since $h^\infty \ge 0$, we have $h^\infty(x) \le 0$ almost surely if and only if $E^{\mathcal{G}}h^\infty(x) \le 0$ almost surely. □

For lsc proper convex functions, one has $(g_1 + g_2)^\infty = g_1^\infty + g_2^\infty$ whenever $\operatorname{dom} g_1 \cap \operatorname{dom} g_2 \ne \emptyset$. More generally, we have the following.

Lemma 7. *Let* g_1 *and* g_2 *be proper lsc functions with proper horizon functions such that* $\operatorname{dom} g_1 \cap \operatorname{dom} g_2 \ne \emptyset$. *Then* $(g_1 + g_2)^\infty \ge g_1^\infty + g_2^\infty$. *If* g_1 *is convex and* g_2 *is asymptotically regular, then* $g_1 + g_2$ *is asymptotically regular and*

$$(g_1 + g_2)^\infty = g_1^\infty + g_2^\infty.$$

Proof. We always have

$$
\begin{aligned}
(g_1+g_2)^\infty(x) &= \lim_{\alpha\nearrow\infty} \inf_{\substack{\gamma>\alpha\\ x'\in\mathbb{B}(x,1/\alpha)}} [g_1(\gamma x')/\gamma + g_2(\gamma x')/\gamma] \\
&\geq \lim_{\alpha\nearrow\infty} \left[\inf_{\substack{\gamma>\alpha\\ x'\in\mathbb{B}(x,1/\alpha)}} g_1(\gamma x')/\gamma + \inf_{\substack{\gamma>\alpha\\ x'\in\mathbb{B}(x,1/\alpha)}} g_2(\gamma x')/\gamma \right] \\
&= g_1^\infty(x) + g_2^\infty(x).
\end{aligned}
$$

Given $\bar{x} \in \operatorname{dom} g_1 \cap \operatorname{dom} g_2$, (3) yields

$$
\begin{aligned}
(g_1+g_2)^\infty(x) &\leq \liminf_{\alpha\nearrow\infty} \frac{g_1(\alpha x+\bar{x}) + g_2(\alpha x+\bar{x}) - g_1(\bar{x}) - g_2(\bar{x})}{\alpha} \\
&\leq \sup_{\alpha>0} \frac{g_1(\alpha x+\bar{x}) - g_1(\bar{x})}{\alpha} + \liminf_{\alpha\nearrow\infty} \frac{g_2(\alpha x+\bar{x}) - g_2(\bar{x})}{\alpha} \\
&= g_1^\infty(x) + g_2^\infty(x),
\end{aligned}
$$

where the last equation follows from convexity of g_1 (see Example 1) and asymptotic regularity of g_2. The above inequalities also imply that g_1+g_2 is asymptotically regular. □

We also have the following result on product spaces. The proof is almost identical to that of Lemma 7 and is omitted.

Lemma 8. *Let g_1 and g_2 be proper lsc functions with proper horizon functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let $g(x_1,x_2) = g_1(x_1) + g_2(x_2)$. Then $g^\infty(x_1,x_2) \geq g_1^\infty(x_1) + g_2^\infty(x_2)$. If g_1 is convex and g_2 is asymptotically regular, then g is asymptotically regular and*

$$g^\infty(x_1,x_2) = g_1^\infty(x_1) + g_2^\infty(x_2).$$

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