Links between generalized Montréal-functors

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Abstract

Let o be the ring of integers in a finite extension $K/\mathbb{Q}_p$ and $G = \mathbf{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ with connected centre and split Borel $\mathbf{B} = \mathbf{TN}$. We show that Breuil's [2] pseudocompact (φ, Γ)-module $D^\vee_\xi(\pi)$ attached to a smooth o-torsion representation π of $B = \mathbf{B}(\mathbb{Q}_p)$ is isomorphic to the pseudocompact completion of the basechange $\mathcal{O}_\mathcal{E} \otimes_{\Lambda(N_0),\ell} \widetilde{D_{SV}}(\pi)$ to Fontaine's ring (via a Whittaker functional $\ell \colon N_0 = \mathbf{N}(\mathbb{Z}_p) \to \mathbb{Z}_p$) of the étale hull $\widetilde{D_{SV}}(\pi)$ of $D_{SV}(\pi)$ defined by Schneider and Vigneras [9]. Moreover, we construct a G-equivariant map from the Pontryagin dual $\pi^\vee$ to the global sections $\mathfrak{Y}(G/B)$ of the G-equivariant sheaf $\mathfrak{Y}$ on G/B attached to a noncommutative multivariable version $D^\vee_{\xi,\ell,\infty}(\pi)$ of Breuil's $D^\vee_\xi(\pi)$ whenever π comes as the restriction to B of a smooth, admissible representation of G of finite length.

Contents

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1 Introduction

1.1 Notations

Let $G = \mathbf{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ with connected centre and a fixed split Borel subgroup $\mathbf{B} = \mathbf{TN}$. Put $B := \mathbf{B}(\mathbb{Q}_p)$, $T := \mathbf{T}(\mathbb{Q}_p)$, and $N := \mathbf{N}(\mathbb{Q}_p)$. We denote by Φ_+ the set of roots of T in N, by $\Delta \subset \Phi_+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi_+$, a $\mathbb{Q}_p$-homomorphism onto the root subgroup N_α of N such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbb{Q}_p$ and $t \in T(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha\in\Phi_+} u_\alpha(\mathbb{Z}_p)$ is a subgroup of $N(\mathbb{Q}_p)$. We put $N_{\alpha,0} := u_\alpha(\mathbb{Z}_p)$ for the image of u_α on $\mathbb{Z}_p$. We denote by T_+ the monoid of dominant elements t in $T(\mathbb{Q}_p)$ such that $\mathrm{val}_p(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by T_{++} the subset of strictly dominant elements, i.e. $\mathrm{val}_p(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $B_+ = N_0T_+, B_0 = N_0T_0$. The natural conjugation action of T_+ on N_0 extends to an action on the Iwasawa o-algebra $\Lambda(N_0) = o[[N_0]]$. For $t \in T_+$ we denote this action of t on $\Lambda(N_0)$ by φ_t. The map $\varphi_t \colon \Lambda(N_0) \to \Lambda(N_0)$ is an injective ring homomorphism with a distinguished left inverse $\psi_t \colon \Lambda(N_0) \to \Lambda(N_0)$ satisfying $\psi_t \circ \varphi_t = \mathrm{id}_{\Lambda(N_0)}$ and $\psi_t(u\varphi_t(\lambda)) = \psi_t(\varphi_t(\lambda)u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ and $\lambda \in \Lambda(N_0)$.

Each simple root α gives a $\mathbb{Q}_p$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section u_α. We denote by $\ell_\alpha : N_0 \to \mathbb{Z}_p$, resp. $\iota_\alpha \colon \mathbb{Z}_p \to N_0$, the restriction of x_α, resp. u_α, to N_0, resp. $\mathbb{Z}_p$.

Since the centre of G is assumed to be connected, there exists a cocharacter $\xi \colon \mathbb{Q}_p^\times \to T$ such that $\alpha \circ \xi$ is the identity on $\mathbb{Q}_p^\times$ for each $\alpha \in \Delta$. We put $\Gamma := \xi(\mathbb{Z}_p^\times) \leq T$ and often denote the action of $s := \xi(p)$ by $\varphi = \varphi_s$.

By a smooth o-torsion representation π of G (resp. of $B = \mathbf{B}(\mathbb{Q}_p)$) we mean a torsion o-module π together with a smooth (ie. stabilizers are open) and linear action of the group G (resp. of B).

For example, $\mathbf{G} = \mathrm{GL}_n$, B is the subgroup of upper triangular matrices, N consists of the strictly upper triangular matrices (1 on the diagonal), T is the diagonal subgroup, $N_0 = \mathbf{N}(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_it_{i+1}^{-1}$, x_{α_i} sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(\cdot)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient $\cdot$ and 0 everywhere else.

Let $\ell \colon N_0 \to \mathbb{Z}_p$ (for now) any surjective group homomorphism and denote by $H_0 \lhd N_0$ the kernel of ℓ. The ring $\Lambda_\ell(N_0)$, denoted by $\Lambda_{H_0}(N_0)$ in [9], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\mathrm{id}}(N_0^{(2)})$ where $N_0^{(2)}$ is the $\mathbb{Z}_p$-points of the unipotent radical of a split Borel subgroup in GL_2. We refer the reader to [9] for the proofs of some of the following claims.

The maximal ideal $\mathcal{M}(H_0)$ of the completed group o-algebra $\Lambda(H_0) = o[[H_0]]$ is generated by ϖ and by the kernel of the augmentation map $o[[H_0]] \to o$.

The ring $\Lambda_\ell(N_0)$ is the $\mathcal{M}(H_0)$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in the ideal $\mathcal{M}(H_0)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n\in\mathbb{Z}} a_nX^n$ over $\Lambda(H_0)$ in the variable X with $\lim_{n\to-\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_\ell(N_0) \cong \varprojlim_{n,k} \Lambda(N_0/H_k)[1/X]/\varpi^n$ for $H_k \lhd N_0$ normal subgroups contained and open in H_0 satisfying $\bigcap_{k\geq 0} H_k = \{1\}$ see also [13].

For a finite index subgroup $\mathcal{G}_2$ in a group $\mathcal{G}_1$ we denote by $J(\mathcal{G}_1/\mathcal{G}_2) \subset \mathcal{G}_1$ a (fixed) set of representatives of the left cosets in $\mathcal{G}_1/\mathcal{G}_2$.

1.2 General overview

By now the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is very well understood through the work of Colmez [3], [4] and others (see [1] for an overview). To review Colmez's work let $K/\mathbb{Q}_p$ be a finite extension with ring of integers o, uniformizer ϖ and residue field k. The starting point is Fontaine's [8] theorem that the category of o-torsion Galois representations of $\mathbb{Q}_p$ is equivalent to the category of torsion (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}} = \varprojlim_h o/\varpi^h((X))$. One of Colmez's breakthroughs was that he managed to relate p-adic (and mod p) representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to (φ,Γ)-modules, too. The so-called "Montréal-functor" associates to a smooth o-torsion representation π of the standard Borel subgroup $B_2(\mathbb{Q}_p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ a torsion (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$. There are two different approaches to generalize this functor to reductive groups G other than $\mathrm{GL}_2(\mathbb{Q}_p)$. We briefly recall these "generalized Montréal functors" here.

The approach by Schneider and Vigneras [9] starts with the set $\mathcal{B}_+(\pi)$ of generating B_+-subrepresentations $W \leq \pi$. The Pontryagin dual $W^\vee = \mathrm{Hom}_o(W, K/o)$ of each W admits a natural action of the inverse monoid B_+^{-1}. Moreover, the action of $N_0 \leq B_+^{-1}$ on $W^\vee$ extends to an action of the Iwasawa algebra $\Lambda(N_0) = o[[N_0]]$. For $W_1, W_2 \in \mathcal{B}_+(\pi)$ we also have $W_1 \cap W_2 \in \mathcal{B}_+(\pi)$ (Lemma 2.2 in [9]) therefore we may take the inductive limit $D_{SV}(\pi) := \varinjlim_{W\in\mathcal{B}_+(\pi)} W^\vee$. In general, $D_{SV}(\pi)$ does not have good properties: for instance it may not admit a canonical right inverse of the T_+-action making $D_{SV}(\pi)$ an étale T_+-module over $\Lambda(N_0)$. However, by taking a resolution of π by compactly induced representations of B, one may consider the derived functors D^i_{SV} of D_{SV} for $i \geq 0$ producing étale T_+-modules $D^i_{SV}(\pi)$ over $\Lambda(N_0)$. Note that the functor D_{SV} is neither left- nor right exact, but exact in the middle. The fundamental open question of [9] whether the topological localizations $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)} D^i_{SV}(\pi)$ are finitely generated over $\Lambda_\ell(N_0)$ in case when π comes as a restriction of a smooth admissible representation of G of finite length. One can pass to usual 1-variable étale (φ,Γ)-modules—still not necessarily finitely generated—over $\mathcal{O}_{\mathcal{E}}$ via the map $\ell\colon \Lambda_\ell(N_0) \to \mathcal{O}_{\mathcal{E}}$ which step is an equivalence of categories for finitely generated étale (φ,Γ)-modules (Thm. 8.20 in [10]).

More recently, Breuil [2] managed to find a different approach, producing a pseudocompact (ie. projective limit of finitely generated) (φ,Γ)-module $D^\vee_\xi(\pi)$ over $\mathcal{O}_{\mathcal{E}}$ when π is killed by a power ϖ^h of the uniformizer ϖ. In [2] (and also in [9]) ℓ is a *generic* Whittaker functional, namely ℓ is chosen to be the composite map

$$\ell\colon N_0 \to N_0/(N_0\cap[N,N]) \cong \prod_{\alpha\in\Delta} N_{\alpha,0} \overset{\sum_{\alpha\in\Delta} u_\alpha^{-1}}{\longrightarrow} \mathbb{Z}_p \ .$$

Breuil passes right away to the space of H_0-invariants π^{H_0} of π where H_0 is the kernel of the group homomorphism $\ell\colon N_0 \to \mathbb{Z}_p$. By the assumption that π is smooth, the invariant subspace π^{H_0} has the structure of a module over the Iwasawa algebra $\Lambda(N_0/H_0)/\varpi^h \cong o/\varpi^h[[X]]$. Moreover, it admits a semilinear action of F which is the Hecke action of $s := \xi(p)$: For any $m \in \pi^{H_0}$ we define

$$F(m) := \mathrm{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u\in J(H_0/sH_0s^{-1})} usm \ .$$

So π^{H_0} is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX = (sXs^{-1})F = ((X+1)^p-1)F$). We consider those (i) finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$-submodules $M \subset \pi^{H_0}$ that are (ii) invariant under the action of Γ and are (iii) *admissible* as a $\Lambda(N_0/H_0)/\varpi^h$-module, ie. the Pontryagin dual $M^\vee = \mathrm{Hom}_o(M, o/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition (iii) is equivalent to the usual admissibility condition in smooth representation theory, ie. that for any (or equivalently for a single) open subgroup $N' \leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over o. We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M \leq \pi^{H_0}$ satisfying $(i), (ii), (iii)$. Note that whenever M_1, M_2 are in $\mathcal{M}(\pi^{H_0})$ then so is $M_1 + M_2$. It is shown in [4] (see also [5] and Lemma 2.6 in [2]) that for $M \in \mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee[1/X]$ naturally admits a structure of an étale (φ, Γ)-module over $o/\varpi^h((X))$. Therefore Breuil [2] defines

$$D_\xi^\vee(\pi) := \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee[1/X] \ .$$

By construction this is a projective limit of usual (φ, Γ)-modules. Moreover, $D_\xi^\vee$ is right exact and compatible with parabolic induction [2]. It can be characterized by the following universal property: For any (finitely generated) étale (φ, Γ)-module over $o/\varpi^h((X)) \cong o/\varpi^h[[\mathbb{Z}_p]][([1]-1)^{-1}]$ (here $[1]$ is the image of the topological generator of $\mathbb{Z}_p$ in the Iwasawa algebra $o/\varpi^h[[\mathbb{Z}_p]]$) we may consider continuous $\Lambda(N_0)$-homomorphisms $\pi^\vee \to D$ via the map $\ell\colon N_0 \to \mathbb{Z}_p$ (in the weak topology of D and the compact topology of $\pi^\vee$). These all factor through $(\pi^\vee)_{H_0} \cong (\pi^{H_0})^\vee$. So we may require these maps be ψ_s- and Γ-equivariant where $\Gamma = \xi(\mathbb{Z}_p \setminus \{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s\colon (\pi^{H_0})^\vee \to (\pi^{H_0})^\vee$ is the dual of the Hecke-action $F\colon \pi^{H_0} \to \pi^{H_0}$ of s on π^{H_0}. Any such continuous ψ_s- and Γ-equivariant map f factors uniquely through $D_\xi^\vee(\pi)$. However, it is not known in general whether $D_\xi^\vee(\pi)$ is nonzero for smooth irreducible representations π of G (restricted to B).

The way Colmez goes back to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ requires the following construction. From any (φ, Γ)-module over $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$ and character $\delta\colon \mathbb{Q}_p^\times \to o^\times$ Colmez constructs a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf $\mathfrak{Y}\colon U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) of K-vectorspaces on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$. This sheaf has the following properties: (i) the centre of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via δ on $D \boxtimes_\delta \mathbb{P}^1$; (ii) we have $D \boxtimes_\delta \mathbb{Z}_p \cong D$ as a module over the monoid $\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$). Moreover, whenever D is 2-dimensional and δ is the character corresponding to the Galois representation of $\bigwedge^2 D$ via local class field theory then the G-representation of global sections $D \boxtimes_\delta \mathbb{P}^1$ admits a short exact sequence

$$0 \to \Pi(\check{D})^\vee \to D \boxtimes \mathbb{P}^1 \to \Pi(D) \to 0$$

where $\Pi(\cdot)$ denotes the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\check{D} = \mathrm{Hom}(D, \mathcal{E})$ is the dual (φ, Γ)-module.

In [10] the functor $D \mapsto \mathfrak{Y}$ is generalized to arbitrary $\mathbb{Q}_p$-split reductive groups G with connected centre. Assume that $\ell = \ell_\alpha : N_0 \to N_{\alpha,0} \cong \mathbb{Z}_p$ is the projection onto the root subgroup corresponding to a fixed simple root $\alpha \in \Delta$. Then we have an action of the monoid T_+ on the ring $\Lambda_\ell(N_0)$ as we have $tH_0t^{-1} \leq H_0$ for any $t \in T_+$. Let D be an étale (φ, Γ)-module finitely generated over $\mathcal{O}_{\mathcal{E}}$ and choose a character $\delta\colon \mathrm{Ker}(\alpha) \to o^\times$. Then we may let the monoid $\xi(\mathbb{Z}_p \setminus \{0\})\,\mathrm{Ker}(\alpha) \leq T$ (containing T_+) act on D via the character δ of $\mathrm{Ker}(\alpha)$

and via the natural action of $\mathbb{Z}_p \setminus \{0\} \cong \varphi^{\mathbb{N}_0} \times \Gamma$ on D. This way we also obtain a T_+-action on $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ making $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ an étale T_+-module over $\Lambda_\ell(N_0)$. In [10] a G-equivariant sheaf $\mathfrak{Y}$ on G/B is attached to D such that its sections on $\mathcal{C}_0 := N_0 w_0 B/B \subset G/B$ is B_+-equivariantly isomorphic to the étale T_+-module $(\Lambda_\ell(N_0) \otimes_{u_\alpha} D)^{bd}$ over $\Lambda(N_0)$ consisting of bounded elements in $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ (for a more detailed overview see section 4.1).

1.3 Summary of our results

Our first result is the construction of a noncommutative multivariable version of $D^\vee_\xi(\pi)$. Let π be a smooth o-torsion representation of B such that $\varpi^h \pi = 0$. The idea here is to take the invariants π^{H_k} for a family of open normal subgroups $H_k \leq H_0$ with $\bigcap_{k\geq 0} H_k = \{1\}$. Now Γ and the quotient group N_0/H_k act on π^{H_k} (we choose H_k so that it is normalized by both Γ and N_0). Further, we have a Hecke-action of s given by $F_k := \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot)$. As in [2] we consider the set $\mathcal{M}_k(\pi^{H_k})$ of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k. In section 2.1 we show that for any $M_k \in \mathcal{M}_k(\pi^{H_k})$ there is an étale (φ,Γ)-module structure on $M_k^\vee[1/X]$ over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$. So the projective limit

$$D^\vee_{\xi,\ell,\infty}(\pi) := \varprojlim_{k\geq 0} \varprojlim_{M_k \in \mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X]$$

is an étale (φ,Γ)-module over $\Lambda_\ell(N_0)/\varpi^h = \varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X]$. More-over, we also give a natural isomorphism $D^\vee_{\xi,\ell,\infty}(\pi)_{H_0} \cong D^\vee_\xi(\pi)$ showing that $D^\vee_{\xi,\ell,\infty}(\pi)$ corresponds to $D^\vee_\xi(\pi)$ via (the projective limit of) the equivalence of categories in Thm. 8.20 in [10]. Moreover, the natural map $\pi^\vee \to D^\vee_{\xi,\ell}(\pi)$ factors through the projection map $D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow D^\vee_{\xi,\ell}(\pi) = D^\vee_{\xi,\ell,\infty}(\pi)_{H_0}$. Note that this shows that $D^\vee_{\xi,\ell,\infty}(\pi)$ is naturally attached to π—not just simply via the equivalence of categories (loc. cit.)—in the sense that any ψ- and Γ-equivariant map from $\pi^\vee$ to an étale (φ,Γ)-module over $o/\varpi^h((X))$ factors uniquely through the corresponding multivariable (φ,Γ)-module. This fact is used crucially in the subsequent sections of this paper.

In section 2.2 we develop these ideas further and show that the natural map $\pi^\vee \to D^\vee_{\xi,\ell,\infty}(\pi)$ factors through the map $\pi^\vee \to D_{SV}(\pi)$. In fact, we show (Prop. 2.14) that $D^\vee_{\xi,\ell,\infty}(\pi)$ has the following universal property: Any continuous ψ_s- and Γ-equivariant map $f\colon D_{SV}(\pi) \to D$ into a finitely generated étale (φ,Γ)-module D over $\Lambda_\ell(N_0)$ factors uniquely through $\mathrm{pr} = \mathrm{pr}_\pi\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$. The association $\pi \mapsto \mathrm{pr}_\pi$ is a natural transformation between the functors D_{SV} and $D^\vee_{\xi,\ell,\infty}$. One application is that Breuil's functor $D^\vee_\xi$ vanishes on compactly induced representations of B (see Corollary 2.13).

In order to be able to compute $D^\vee_{\xi,\ell,\infty}(\pi)$ (hence also $D^\vee_\xi(\pi)$) from $D_{SV}(\pi)$ we introduce the notion of the *étale hull* of a $\Lambda(N_0)$-module with a ψ-action of T_+ (or of a submonoid $T_* \leq T_+$). Here a $\Lambda(N_0)$-module D with a ψ-action of T_+ is the analogue of a (ψ,Γ)-module over $o[[X]]$ in this multivariable noncommutative setting. The étale hull $\widetilde{D}$ of D (together with a canonical map $\iota\colon D \to \widetilde{D}$) is characterized by the universal property that any ψ-equivariant map $f\colon D \to D'$ into an étale T_+-module D' over $\Lambda(N_0)$ factors uniquely through ι. It can be constructed as a direct limit $\varinjlim_{t\in T_+} \varphi_t^* D$ where $\varphi_t^* D = \Lambda(N_0) \otimes_{\varphi_t,\Lambda(N_0)} D$ (Prop. 2.21). We show (Thm. 2.28 and the remark thereafter) that the pseudocompact completion of

$\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}\widetilde{D_{SV}}(\pi)$ is canonically isomorphic to $D^\vee_{\xi,\ell,\infty}(\pi)$ as they have the same universal property.

In order to go back to representations of G we need an étale action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$, not just of $\xi(\mathbb{Z}_p\setminus\{0\})$. This is only possible if $tH_0t^{-1}\leq H_0$ for all $t\in T_+$ which is not the case for generic ℓ. So in section 3 we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with an étale action of T_+ (extending that of $\xi(\mathbb{Z}_p\setminus\{0\})\leq T_+$) in case $\ell=\ell_\alpha$ is the projection of N_0 onto a root subgroup $N_{\alpha,0}\cong\mathbb{Z}_p$ for some simple root α in Δ. Moreover, we show (Prop. 3.8) that the map $\mathrm{pr}\colon D_{SV}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is ψ-equivariant for this extended action, too. Note that $D^\vee_{\xi,\ell,\infty}(\pi)$ may not be the projective limit of finitely generated étale T_+-modules over $\Lambda_\ell(N_0)$ as we do not necessarily have an action of T_+ on $M^\vee_\infty[1/X]$ for $M\in\mathcal{M}(\pi^{H_0})$, only on the projective limit. So the construction of a G-equivariant sheaf on G/B with sections on $\mathcal{C}_0=N_0w_0B/B\subset G/B$ isomorphic to a dense B_+-stable $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$ is not immediate from the work [10] as only the case of finitely generated modules over $\Lambda_\ell(N_0)$ is treated in there. However, as we point out in section 4.1 the most natural definition of bounded elements in $D^\vee_{\xi,\ell,\infty}(\pi)$ works: The $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as the union of ψ-invariant compact $\Lambda(N_0)$-submodules of $D^\vee_{\xi,\ell,\infty}(\pi)$. This section is devoted to showing that the image of $\widetilde{\mathrm{pr}}\colon\widetilde{D_{SV}}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ (Cor. 4.4) and that the constructions of [10] can be carried over to this situation (Prop. 4.7). We denote the resulting G-equivariant sheaf on G/B by $\mathfrak{Y}=\mathfrak{Y}_{\alpha,\pi}$.

Now consider the functors $(\cdot)^\vee\colon\pi\mapsto\pi^\vee$ and the composite

$$\mathfrak{Y}_{\alpha,\cdot}(G/B)\colon\pi\mapsto D^\vee_{\xi,\ell,\infty}(\pi)\mapsto\mathfrak{Y}_{\alpha,\pi}(G/B)$$

both sending smooth, admissible o/ϖ^h-representations of G of finite length to topological representations of G over o/ϖ^h. The main result of our paper (Thm. 4.17) is a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\mathfrak{Y}_{\alpha,\cdot}$. This generalizes Thm. IV.4.7 in [4]. The proof of this relies on the observation that the maps $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)^{bd}\to D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ in fact come from the G-action on $\pi^\vee$. More precisely, for any $g\in G$ and $W\in\mathcal{B}_+(\pi)$ we have maps

$$(g\cdot)\colon(g^{-1}W\cap W)^\vee\to(W\cap gW)^\vee$$

where both $(g^{-1}W\cap W)^\vee$ and $(W\cap gW)^\vee$ are naturally quotients of $W^\vee$. We show in (the proof of) Prop. 4.16 that these maps fit into a commutative diagram

$$\begin{array}{ccccc}
W^\vee & \longrightarrow & (g^{-1}W\cap W)^\vee & \xrightarrow{\ g\cdot\ } & (W\cap gW)^\vee\\
\downarrow{\scriptstyle \mathrm{pr}_W} & & \downarrow & & \downarrow\\
D^\vee_{\xi,\ell,\infty}(\pi)^{bd} & \longrightarrow & \mathrm{res}^{\mathcal{C}_0}_{g^{-1}\mathcal{C}_0\cap\mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}) & \xrightarrow{\ g\cdot\ } & \mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0\cap g\mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})
\end{array}$$

allowing us to construct the map $\beta_{G/B}$. The proof of Thm. 4.17 is similar to that of Thm. IV.4.7 in [4]. However, unlike that proof we do not need the full machinery of "standard presentations" in Ch. III.1 of [4] which is not available at the moment for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

2 Comparison of Breuil's functor with that of Schneider and Vigneras

2.1 A $\Lambda_\ell(N_0)$-variant of Breuil's functor

Our first goal is to associate a (φ,Γ)-module over $\Lambda_\ell(N_0)$ (not just over $\mathcal{O}_\mathcal{E}$) to a smooth o-torsion representation π of G in the spirit of [2] that corresponds to $D_\xi^\vee(\pi)$ via the equivalence of categories of [10] between (φ,Γ)-modules over $\mathcal{O}_\mathcal{E}$ and over $\Lambda_\ell(N_0)$.

Let H_k be the normal subgroup of N_0 generated by $s^kH_0s^{-k}$, ie. we put

$$H_k = \langle n_0 s^k H_0 s^{-k} n_0^{-1} \mid n_0 \in N_0\rangle \ .$$

H_k is an open subgroup of H_0 normal in N_0 and we have $\bigcap_{k\geq 0} H_k = \{1\}$. Denote by F_k the operator $\mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot)$ on π and consider the skew polynomial ring $\Lambda(N_0/H_k)/\varpi^h[F_k]$ where $F_k\lambda = (s\lambda s^{-1})F_k$ for any $\lambda\in\Lambda(N_0/H_k)/\varpi^h$. The set of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k is denoted by $\mathcal{M}_k(\pi^{H_k})$.

Lemma 2.1. *We have $F = F_0$ and $F_k\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) = \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ F_0$ as maps on π^{H_0}.*

Proof. We compute

$$\begin{aligned}F_k\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) = \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) =\\ \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ\mathrm{Tr}_{sH_ks^{-1}/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) =\\ \mathrm{Tr}_{H_k/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) =\\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ\mathrm{Tr}_{s^kH_0s^{-k}/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) =\\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ\mathrm{Tr}_{H_0/sH_0s^{-1}}\circ(s\cdot) =\\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ F_0\ . \quad\square\end{aligned}$$

□

Note that if $M\in\mathcal{M}(\pi^{H_0})$ then $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$ is a $s^kN_0s^{-k}H_k$-subrepresentation of π^{H_k}. So in view of the above Lemma we define M_k to be the N_0-subrepresentation of π^{H_k} generated by $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$, ie. $M_k := N_0\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$. By Lemma 2.1 M_k is a $\Lambda(N_0/H_k)/\varpi^h[F_k]$-submodule of π^{H_k}.

Lemma 2.2. *For any $M\in\mathcal{M}(\pi^{H_0})$ the N_0-subrepresentation M_k lies in $\mathcal{M}_k(\pi^{H_k})$.*

Proof. Let $\{m_1,\dots,m_r\}$ be a set of generators of M as a $\Lambda(N_0/H_0)/\varpi^h[F]$-module. We claim that the elements $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^km_i)$ $(i=1,\dots,r)$ generate M_k as a module over $\Lambda(N_0/H_k)/\varpi^h[F_k]$. Since both H_k and $s^kH_0s^{-k}$ are normalized by $s^kN_0s^{-k}$, for any $u\in N_0$ we have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kus^{-k}\cdot) = (s^kus^{-k}\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\ . \tag{1}$$

Therefore by continuity we also have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\lambda s^{-k}\cdot) = (s^k\lambda s^{-k}\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}$$

for any $\lambda \in \Lambda(N_0/H_0)/\varpi^h$. Now writing any $m \in M$ as $m = \sum_{j=1}^r \lambda_j F^{i_j} m_j$ we compute

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \sum_{j=1}^r \lambda_j F^{i_j} m_j) = \sum_{j=1}^r (s^k\lambda s^{-k}) F_k^{i_j} \,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^k m_j) \in$$
$$\in \sum_{j=1}^r \Lambda(N_0/H_k)/\varpi^h[F_k]\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^k m_j) \ .$$

For the stability under the action of Γ note that Γ normalizes both H_k and $s^kH_0s^{-k}$ and the elements in Γ commute with s.

Since M is admissible as an N_0-representation, s^kM is admissible as a representation of $s^kN_0s^{-k}$. Further by (1) the map $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}$ is $s^kN_0s^{-k}$-equivariant therefore its image is also admissible. Finally, M_k can be written as a finite sum

$$\sum_{u\in J(N_0/s^kN_0s^{-k}H_k)} u\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$$

of admissible representations of $s^kN_0s^{-k}$ therefore the statement. □ □

Lemma 2.3. *Fix a simple root $\alpha \in \Delta$ such that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$. Then for any $M \in \mathcal{M}(\pi^{H_0})$ the kernel of the trace map*

$$\mathrm{Tr}_{H_0/H_k} \colon Y_k := \sum_{u\in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})} u\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \to N_0F^k(M) \tag{2}$$

is finitely generated over o. In particular, the length of $Y_k^\vee[1/X]$ as a module over $o/\varpi^h((X))$ equals the length of $M^\vee[1/X]$.

Proof. Since any $u \in N_{\alpha,0} \leq N_0$ normalizes both H_0 and H_k and we have $N_{\alpha,0}H_0 = N_0$ by the assumption that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$, the image of the map (2) is indeed $N_0F^k(M)$. Moreover, by the proof of Lemma 2.6 in [2] the quotient $M/N_0F^k(M)$ is finitely generated over o. Therefore we have $M^\vee[1/X] \cong (N_0F^k(M))^\vee[1/X]$ as a module over $o/\varpi^h((X))$. In particular, their length are equal:

$$l := \mathrm{length}_{o/\varpi^h((X))} M^\vee[1/X] = \mathrm{length}_{o/\varpi^h((X))}(N_0F^k(M))^\vee[1/X] \ .$$

We compute

$$l = \mathrm{length}_{o/\varpi^h((X))} M^\vee[1/X] = \mathrm{length}_{o/\varpi^h((\varphi^k(X)))}(s^kM)^\vee[1/X] \geq$$
$$\geq \mathrm{length}_{o/\varpi^h((\varphi^k(X)))}(\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] =$$
$$= \mathrm{length}_{o/\varpi^h((X))}(o/\varpi^h[[X]] \otimes_{o/\varpi^h[[\varphi^k(X)]]} \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] \geq$$
$$\geq \mathrm{length}_{o/\varpi^h((X))} Y_k^\vee[1/X] \ .$$

By the existence of a surjective map (2) we must have equality in the above inequality everywhere. Therefore we have $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})^\vee[1/X] = 0$, which shows that $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})$ is finitely generated over o, because M is admissible, and so is $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k}) \leq M$. □ □

The kernel of the natural homomorphism

$$\Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_0)/\varpi \cong k[[X]]$$

is a nilpotent prime ideal in the ring $\Lambda(N_0/H_k)/\varpi^h$. We denote the localization at this ideal by $\Lambda(N_0/H_k)/\varpi^h[1/X]$. For the justification of this notation note that any element in $\Lambda(N_0/H_k)/\varpi^h[1/X]$ can uniquely be written as a formal Laurent-series $\sum_{n\gg-\infty} a_n X^n$ with coefficients a_n in the finite group ring $o/\varpi^h[H_0/H_k]$. Here X—by an abuse of notation—denotes the element $[u_0]-1$ for an element $u_0 \in N_{\alpha,0} \leq N_0$ with $\ell(u_0) = 1 \in \mathbb{Z}_p$. The ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ admits a conjugation action of the group Γ that commutes with the operator φ defined by $\varphi(\lambda) := s\lambda s^{-1}$ (for $\lambda \in \Lambda(N_0/H_k)/\varpi^h[1/X]$). A (φ,Γ)-module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a finitely generated module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ together with a semilinear commuting action of φ and Γ. Note that φ is no longer injective on the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ for $k \geq 1$, in particular it is not flat either. However, we still call a (φ,Γ)-module D_k over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ étale if the natural map

$$1\otimes\varphi\colon \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} D_k \to D_k$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules. For any $M \in \mathcal{M}(\pi^{H_0})$ we put

$$M_k^\vee[1/X] := \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k^\vee$$

where $(\cdot)^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_o(\cdot, K/o)$.

The group N_0/H_k acts by conjugation on the finite $H_0/H_k \lhd N_0/H_k$. Therefore the kernel of this action has finite index. In particular, there exists a positive integer r such that $s^r N_{\alpha,0} s^{-r} \leq N_0/H_k$ commutes with H_0/H_k. Therefore the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ is contained as a subring in $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Lemma 2.4. *As modules over the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ we have an isomorphism*

$$M_k^\vee[1/X] \to o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] \ .$$

In particular, $M_k^\vee[1/X]$ is induced as a representation of the finite group H_0/H_k, so the reduced (Tate-) cohomology groups $\tilde{H}^i(H', M_k^\vee[1/X])$ vanish for all subgroups $H' \leq H_0/H_k$ and $i \in \mathbb{Z}$.

Proof. By the definition of M_k we have a surjective $o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$-linear map

$$f\colon o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k \to M_k$$

sending $\lambda\otimes y$ to λy for $\lambda \in o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$ and $y \in Y_k$. By taking the Pontryagin dual of f and inverting X we obtain an injective $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$-homomorphism

$$f^\vee[1/X]\colon M_k^\vee[1/X] \to (o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^\vee[1/X] \cong$$
$$\cong o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} (Y_k^\vee[1/X]) \ .$$

On the other hand, by construction the action of the group H_0/H_k on the domain of f is via the action on the first term which is a regular left-translation action. Therefore the H_0/H_k-invariants can be computed as the image of the trace map:

$$(o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^{H_0/H_k} = (\sum_{h\in H_0/H_k} h) \otimes Y_k \ .$$

The composite of f with the bijection

$$(\sum_{h\in H_0/H_k} h)\otimes \mathrm{id}_{Y_k}\colon Y_k \overset{\sim}{\to} (\sum_{h\in H_0/H_k} h)\otimes Y_k$$

is the trace map on Y_k whose kernel is finitely generated over o by Lemma 2.3. In particular, the kernel of the restriction of f to the H_0/H_k-invariants is finitely generated over o. Dually, we find that $f^\vee[1/X]$ becomes surjective after taking H_0/H_k-coinvariants. Since $M_k^\vee[1/X]$ is a finite dimensional representation of the finite p-group H_0/H_k over the local artinian ring $o/\varpi^h((X))$ with residual characteristic p, the map $f^\vee[1/X]$ is in fact an isomorphism as its cokernel has trivial H_0/H_k-coinvariants. □ □

Denote by $H_{k,-}/H_k$ the kernel of the group homomorphism

$$s(\cdot)s^{-1}\colon N_0/H_k \to N_0/H_k\ .$$

It is a normal subgroup contained in the finite subgroup $H_0/H_k \leq N_0/H_k$ since $s(\cdot)s^{-1}$ is the multiplication by p map on $N_0/H_0 \cong \mathbb{Z}_p$ which is injective. If k is big enough so that H_k is contained in sH_0s^{-1} then we have $H_{k,-} = s^{-1}H_ks$, otherwise we always have $H_{k,-} = H_0\cap s^{-1}H_ks$. The ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h$$

factors through the quotient map $\Lambda(N_0/H_k)/\varpi^h \twoheadrightarrow \Lambda(N_0/H_{k,-})/\varpi^h$. We denote by $\tilde\varphi$ the induced ring homomorphism

$$\tilde\varphi\colon \Lambda(N_0/H_{k,-})/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h\ .$$

Note that $\tilde\varphi$ is injective and makes $\Lambda(N_0/H_k)/\varpi^h$ a free module of rank

$$\begin{aligned}\nu :=& \,|\operatorname{Coker}(s(\cdot)s^{-1}\colon N_0/H_k\to N_0/H_k)| =\\ =& \,p|\operatorname{Coker}(s(\cdot)s^{-1}\colon H_0/H_k\to H_0/H_k)| =\\ =& \,p|\operatorname{Ker}(s(\cdot)s^{-1}\colon H_0/H_k\to H_0/H_k)| = p|H_{k,-}/H_k|\end{aligned}$$

over $\Lambda(N_0/H_{k,-})/\varpi^h$ since the kernel and cokernel of an endomorphism of a finite group have the same cardinality.

Lemma 2.5. *We have a series of isomorphisms of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-mod-ules*

$$\begin{aligned}\mathrm{Tr}^{-1} = \mathrm{Tr}^{-1}_{H_{k,-}/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h\otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] \overset{(1)}{\to}\\ \overset{(1)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_k),\varphi}(\Lambda(N_0/H_k), M_k^\vee[1/X]) \overset{(2)}{\to}\\ \overset{(2)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_{k,-}),\tilde\varphi}(\Lambda(N_0/H_k), (M_k^\vee[1/X])^{H_{k,-}}) \overset{(3)}{\to}\\ \overset{(3)}{\to} \Lambda(N_0/H_k)\otimes_{\Lambda(N_0/H_{k,-}),\tilde\varphi} M_k^\vee[1/X]^{H_{k,-}} \overset{(4)}{\to}\\ \overset{(4)}{\to} \Lambda(N_0/H_k)\otimes_{\Lambda(N_0/H_{k,-}),\tilde\varphi} (M_k^\vee[1/X])_{H_{k,-}} \overset{(5)}{\to}\\ \overset{(5)}{\to} \Lambda(N_0/H_k)/\varpi^h\otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi} M_k^\vee[1/X]\ .\end{aligned}$$

Proof. (1) follows from the adjoint property of $\otimes$ and Hom. The second isomorphism follows from noting that the action of the ring $\Lambda(N_0/H_k)$ over itself via φ factors through the quotient $\Lambda(N_0/H_{k,-})$ therefore $H_{k,-}$ acts trivially on $\Lambda(N_0/H_k)$ via this map. So any module-homomorphism $\Lambda(N_0/H_k) \to M_k^\vee[1/X]$ lands in the $H_{k,-}$-invariant part $M_k^\vee[1/X]^{H_{k,-}}$ of $M_k^\vee[1/X]$. The third isomorphism follows from the fact that $\Lambda(N_0/H_k)$ is a free module over $\Lambda(N_0/H_{k,-})$ via $\tilde{\varphi}$. The fourth isomorphism is given by (the inverse of) the trace map $\mathrm{Tr}_{H_{k,-}/H_k}\colon (M_k^\vee[1/X])_{H_{k,-}} \to M_k^\vee[1/X]^{H_{k,-}}$ which is an isomorphism by Lemma 2.4. The last isomorphism follows from the isomorphism $(M_k^\vee[1/X])_{H_{k,-}} \cong \Lambda(N_0/H_{k,-}) \otimes_{\Lambda(N_0/H_k)} M_k^\vee[1/X]$. □ □

Remark. Here φ always acted only on the ring $\Lambda(N_0/H_k)$, hence denoting φ_t the action $n \mapsto tnt^{-1}$ for a fixed $t \in T_+$ and choosing k large enough such that $tH_0t^{-1} \geq H_k$ we get analogously an isomorphism

$$\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi_t,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] \to \\ \to \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k^\vee[1/X]\ .$$

One of the key points of Lemma 2.4 is that the trace map on $M_k^\vee[1/X]$ induces a bijection between $M_k^\vee[1/X]_{H_{k,-}}$ and $M_k^\vee[1/X]^{H_{k,-}}$ as noted in the isomorphism (4) above. We shall use this fact later on.

We denote the composite of the five isomorphisms in Lemma 2.5 by Tr^{-1} emphasising that all but (4) are tautologies. Our main result in this section is the following generalization of Lemma 2.6 in [2].

Proposition 2.6. *The map*

$$\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X]\colon \\ M_k^\vee[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] \tag{3}$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$*-modules. Therefore the natural action of* Γ *and the operator*

$$\begin{aligned} \varphi\colon M_k^\vee[1/X] &\to M_k^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X])^{-1}(1 \otimes f) \end{aligned}$$

make $M_k^\vee[1/X]$ *into an étale* (φ, Γ)*-module over the ring* $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Proof. Since M_k is finitely generated over $\Lambda(N_0/H_k)/\varpi^h[F_k]$ by Lemma 2.2, the cokernel C of the map

$$1 \otimes F_k\colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k \to M_k \tag{4}$$

is finitely generated as a module over $\Lambda(N_0/H_k)/\varpi^h$. Further, it is admissible as a representation of N_0 (again by Lemma 2.2), therefore C is finitely generated over o. In particular, we have $C^\vee[1/X] = 0$ showing that (3) is injective.

For the surjectivity put $Y_k := \sum_{u \in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})} u\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$. This is an $o/\varpi^h[[X]]$-submodule of M_k. By Lemma 2.3 we have

$$\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}(Y_k^\vee[1/X]) = \\ = |N_{\alpha,0} : s^rN_{\alpha,0}s^{-r}|\mathrm{length}_{o/\varpi^h((X))}(Y_k^\vee[1/X]) = p^rl\ .$$

By Lemma 2.4 we obtain

$$\mathrm{length}_{o/\varpi^h(\!(\varphi^r(X))\!)} M_k^\vee[1/X] = \\ = |H_0 : H_k| \cdot \mathrm{length}_{o/\varpi^h(\!(\varphi^r(X))\!)} Y_k^\vee[1/X] = |H_0 : H_k| p^r l \ .$$

Consider the ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \ . \tag{5}$$

Its image is the subring $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ over which the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a free module of rank $\nu = |N_0 : sN_0s^{-1}H_k| = p|H_{k,-} : H_k|$. So we obtain

$$\begin{aligned} p\mathrm{length}_{o(\!(\varphi^r(X))\!)} \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] = \\ = \mathrm{length}_{o(\!(\varphi^{r+1}(X))\!)} \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] = \\ = \nu \mathrm{length}_{o(\!(\varphi^{r+1}(X))\!)} \Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)] \\ \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] \overset{(*)}{=} \\ = \nu \mathrm{length}_{o(\!(\varphi^r(X))\!)} M_k^\vee[1/X]_{H_{k,-}} = \\ = \nu \mathrm{length}_{o(\!(\varphi^r(X))\!)} (o/\varpi^h[H_0/H_{k,-}] \otimes_{o/\varpi^h} Y_k^\vee[1/X]) = \\ = \nu |H_0 : H_{k,-}| p^r l = p|H_0 : H_k| p^r l = p\mathrm{length}_{o/\varpi^h(\!(\varphi^r(X))\!)} M_k^\vee[1/X] \ . \end{aligned}$$

Here the equality $(*)$ follows from the fact that the map φ induces an isomorphism between $\Lambda(N_0/H_{k,-})/\varpi^h[1/X]$ and $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ sending the subring $o(\!(\varphi^r(X))\!)$ isomorphically onto $o(\!(\varphi^{r+1}(X))\!)$.

This shows that (3) is an isomorphism as it is injective and the two sides have equal length as modules over the artinian ring $o/\varpi^h(\!(X)\!)$. □ □

Remark. We also obtain in particular that the map (4) has finite kernel and cokernel. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{k,*}$ of M_k such that the kernel of $1 \otimes F_k$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by M_k^* the image of $1 \otimes F_k$.

Note that for $k = 0$ we have $M_0 = M$. Let now $0 \leq j \leq k$ be two integers. By Lemma 2.4 the space of H_j-invariants of M_k is equal to $\mathrm{Tr}_{H_j/H_k}(M_k)$ upto finitely generated modules over o. On the other hand, we compute

$$\begin{aligned} N_0 F_j^{k-j}(M_j) = N_0 \mathrm{Tr}_{H_j/s^{k-j}H_js^{j-k}} \circ (s^{k-j}\cdot) \circ \mathrm{Tr}_{H_j/s^jH_0s^{-j}}(s^jM) = \\ = N_0 \mathrm{Tr}_{H_j/s^kH_0s^{-k}}(s^kM) = N_0 \mathrm{Tr}_{H_j/H_k} \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) = \\ = \mathrm{Tr}_{H_j/H_k}(N_0 \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = \mathrm{Tr}_{H_j/H_k}(M_k) \end{aligned}$$

since both H_k and H_j are normal in N_0 whence we have $(u\cdot) \circ \mathrm{Tr}_{H_j/H_k} = \mathrm{Tr}_{H_j/H_k} \circ (u\cdot)$ for all $u \in N_0$. So taking H_j/H_k-coinvariants of $M_k^\vee[1/X]$, we have a natural identification

$$M_k^\vee[1/X]_{H_j/H_k} \cong (M_k^{H_j/H_k})^\vee[1/X] \cong \\ \cong (\mathrm{Tr}_{H_j/H_k}(M_k))^\vee[1/X] = (N_0F_j^{k-j}(M_j))^\vee[1/X] \cong M_j^\vee[1/X] \tag{6}$$

induced by the inclusion $N_0F_j^{k-j}(M_j) \subseteq M_k^{H_j} \subseteq M_k$. The last identification follows from the fact that $M_j/N_0F_j^{k-j}(M_j)$ is finitely generated over o as noted in the beginning of the proof of Proposition 2.6 applied to j instead of k.

Lemma 2.7. *We have* $\mathrm{Tr}_{H_j/H_k}\circ F_k = F_j\circ \mathrm{Tr}_{H_j/H_k}$.

Proof. We compute

$$\begin{aligned}\mathrm{Tr}_{H_j/H_k}\circ F_k = \mathrm{Tr}_{H_j/H_k}\circ \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot) = \\ \mathrm{Tr}_{H_j/sH_ks^{-1}}\circ(s\cdot) = \mathrm{Tr}_{H_j/sH_js^{-1}}\circ \mathrm{Tr}_{sH_js^{-1}/sH_ks^{-1}}(s\cdot) = \\ \mathrm{Tr}_{H_j/sH_js^{-1}}\circ(s\cdot)\,\mathrm{Tr}_{H_j/H_k} = F_j\circ \mathrm{Tr}_{H_j/H_k}\ . \quad \square\end{aligned}$$

$\square$

Proposition 2.8. *The identification* (6) *is* φ *and* Γ*-equivariant.*

Proof. For fixed j it suffices to treat the case when k is large enough so that we have $H_{k,-} = s^{-1}H_ks$. Indeed, for fixed j and k we may choose a larger integer $k' > k$ with $H_{k',-} = s^{-1}H_{k'}s$ and the φ- and Γ equivariance of the identifications $M_k^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_k}$ and $M_j^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_j}$ will imply that of

$$M_j^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_j} = (M_{k'}^\vee[1/X]_{H_{k'}/H_j})_{H_k/H_j}\cong M_k^\vee[1/X]_{H_k/H_j}\ .$$

So from now on we assume $H_k\leq sH_0s^{-1}\leq sN_0s^{-1}$. As Γ acts both on M_k and M_j by multiplication coming from the action of Γ on π, the map (6) is clearly Γ-equivariant. In order to avoid confusion we are going to denote the map φ on $M_k^\vee[1/X]$ (resp. on $M_j^\vee[1/X]$) temporarily by φ_k (resp. by φ_j). Let f be in $M_k^\vee$ such that its restriction to $M_{k,*}$ is zero (see the Remark after Prop. 2.6). We regard f as an element in $(M_k^*/M_{k,*})^\vee\leq (M_k^*)^\vee$. We are going to compute $\varphi_k(f)$ and $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$ explicitly and find that the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k^*)$ is equal to $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$. Note that we have an isomorphism $M_k^\vee[1/X]\cong M_k^{*\vee}[1/X]\cong (M_k^*/M_{k,*})^\vee[1/X]$ (resp. $M_j^\vee[1/X]\cong \mathrm{Tr}_{H_j/H_k}(M_k^*)^\vee[1/X]$) obtained from the Remark after Prop. 2.6.

Let $m\in M_k^*\leq M_k$ be in the form

$$m = \sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)$$

with elements $m_u\in M_k$ for $u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})$. By the remark after Proposition 2.6 M_k^* is a finite index submodule of M_k. Note that the elements m_u are unique upto $M_{k,*}+\mathrm{Ker}(F_k)$. Therefore $\varphi_k(f)\in (M_k^*)^\vee$ is well-defined by our assumption that $f_{|M_{k,*}}=0$ noting that the kernel of F_k equals the kernel of $\mathrm{Tr}_{H_{k,-}/H_k}$ since the multiplication by s is injective and we have $F_k = s\circ \mathrm{Tr}_{H_{k,-}/H_k}$. So we compute

$$\begin{aligned}\varphi_k(f)(m) = ((1\otimes F_k)^\vee)^{-1}(\mathrm{Tr}_{H_{k,-}/H_k}(1\otimes f))(m) = \\ = ((1\otimes F_k)^\vee)^{-1}(1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)) = \\ = ((1\otimes F_k)^\vee)^{-1}(1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_u 1\otimes F_k(u\otimes m_u)) = \\ = (1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})} (u\otimes m_u)) = \\ = \mathrm{Tr}_{H_{k,-}/H_k}(f)(F_k^{-1}(u_0F_k(m_{u_0}))) = f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}u_0s)m_{u_0}))\end{aligned} \tag{7}$$

where u_0 is the single element in $J(N_0/sN_0s^{-1})$ corresponding to the coset of 1. The other terms in the above sum vanish as $1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f)$ is supported on $1\otimes M_k$ by definition. In order to simplify notation put f_* for the restriction of f to $\mathrm{Tr}_{H_j/H_k}(M_k)$ and

$$U := J(N_0/sN_0s^{-1})\cap H_jsN_0s^{-1} \ .$$

Note that we have $0=\varphi_j(f_*)(uF_j(m'))$ for all $m'\in M_j$ and

$$u\in J(N_0/sN_0s^{-1})\setminus U \ .$$

Therefore using Lemma 2.7 we obtain

$$\begin{aligned}\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k}\sum_{u\in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\ &= \varphi_j(f_*)(\sum_{u\in J(N_0/sN_0s^{-1})} uF_j\circ \mathrm{Tr}_{H_j/H_k}(m_u)) = \\ &= \sum_{u\in U} f(\mathrm{Tr}_{H_{j,-}/H_j}(s^{-1}\overline{u}s\,\mathrm{Tr}_{H_j/H_k}(m_u))) = \\ &= \sum_{u\in U} f(s^{-1}\overline{u}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u)) \end{aligned} \tag{8}$$

where for each $u\in U$ we choose a fixed $\overline{u}$ in $sN_0s^{-1}\cap H_ju$. Note that $f(s^{-1}\overline{u}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u))$ does not depend on this choice: If $\overline{u_1}\in sN_0s^{-1}\cap H_ju$ is another choice then we have $(\overline{u_1})^{-1}\overline{u}\in sN_0s^{-1}\cap H_j$ whence $s^{-1}(\overline{u_1})^{-1}\overline{u}s$ lies in $H_{j,-}=N_0\cap s^{-1}H_js$ so we have

$$\begin{aligned} s^{-1}\overline{u}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u) &= s^{-1}\overline{u_1}ss^{-1}(\overline{u_1})^{-1}\overline{u}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u) = \\ &= s^{-1}\overline{u_1}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u) \ . \end{aligned}$$

Moreover, the equation (8) also shows that $\varphi_j(f_*)$ is a well-defined element in $(\mathrm{Tr}_{H_j/H_k}(M_k^*))^\vee$. On the other hand, for the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k)$ we compute

$$\begin{aligned}\varphi_k(f)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_k(f)(\sum_{w\in J(H_j/H_k)} w \sum_{u\in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\ &= \sum_{w\in J(H_j/H_k)}\sum_{u\in J(N_0/sN_0s^{-1})} \varphi_k(f)(wuF_k(m_u)) = \\ &= \sum_{\substack{u\in U\\ w\in J(H_j/H_k)\cap(sN_0s^{-1}u^{-1})}} f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}wus)m_u)) = \\ &= f(\sum_{v:=s^{-1}wu\overline{u}^{-1}s\in J(H_{j,-}/H_{k,-})} \mathrm{Tr}_{H_{k,-}/H_k}\sum_{u\in U} vs^{-1}\overline{u}sm_u) = \\ &= \sum_{u\in U} f(s^{-1}\overline{u}s\,\mathrm{Tr}_{H_{j,-}/H_k}(m_u))\end{aligned}$$

that equals $\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m)$ by (8). Finally, let now $f\in M_k^\vee$ be arbitrary. Since $M_{k,*}$ is finite, there exists an integer $r\geq 0$ such that X^rf vanishes on $M_{k,*}$. By the above discussion we have $\varphi_k(X^rf)(\mathrm{Tr}_{H_j/H_k} m) = \varphi_j(X^rf_*)(\mathrm{Tr}_{H_j/H_k} m)$. The statement follows noting that $\varphi(X^r)$ is invertible in the ring $\Lambda(N_0/H_j)/\varpi^h[1/X]$. □ □

So we may take the projective limit $M^\vee_\infty[1/X] := \varprojlim_k M^\vee_k[1/X]$ with respect to these quotient maps. The resulting object is an étale (φ,Γ)-module over the ring

$$\varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X] \cong \Lambda_\ell(N_0)/\varpi^h\ .$$

Moreover, by taking the projective limit of (6) with respect to k we obtain a φ- and Γ-equivariant isomorphism $(M^\vee_\infty[1/X])_{H_j} \cong M^\vee_j[1/X]$. So we just proved

Corollary 2.9. *For any object $M \in \mathcal{M}(\pi^{H_0})$ the (φ,Γ)-module $M^\vee[1/X]$ over $o/\varpi^h((X))$ corresponds to $M^\vee_\infty[1/X]$ via the equivalence of categories in Theorem 8.20 in [10].*

Note that whenever $M \subset M'$ are two objects in $\mathcal{M}(\pi^{H_0})$ then we have a natural surjective map $M'^\vee_\infty[1/X] \twoheadrightarrow M^\vee_\infty[1/X]$. So in view of the above corollary we define

$$D^\vee_{\xi,\ell,\infty}(\pi) := \varprojlim_{k\geq 0, M\in\mathcal{M}(\pi^{H_0})} M^\vee_k[1/X] = \varprojlim_{M\in\mathcal{M}(\pi^{H_0})} M^\vee_\infty[1/X]\ .$$

We call two elements $M, M' \in \mathcal{M}(\pi^{H_0})$ equivalent ($M \sim M'$) if the inclusions $M \subseteq M+M'$ and $M' \subseteq M+M'$ induce isomorphisms $M^\vee[1/X] \cong (M+M')^\vee[1/X] \cong M'^\vee[1/X]$. This is equivalent to the condition that M equals M' upto finitely generated o-modules. In particular, this is an equivalence relation on the set $\mathcal{M}(\pi^{H_0})$. Similarly, we say that $M_k, M'_k \in \mathcal{M}_k(\pi^{H_k})$ are equivalent if the inclusions $M_k \subseteq M_k + M'_k$ and $M'_k \subseteq M_k + M'_k$ induce isomorphisms $M^\vee_k[1/X] \cong (M_k+M'_k)^\vee[1/X] \cong {M'_k}^\vee[1/X]$.

Proposition 2.10. *The maps*

$$\begin{array}{rcl} M & \mapsto & N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \\ \operatorname{Tr}_{H_0/H_k}(M_k) & \leftarrow\!\shortmid & M_k \end{array}$$

induce a bijection between the sets $\mathcal{M}(\pi^{H_0})/\sim$ and $\mathcal{M}_k(\pi^{H_k})/\sim$. In particaular, we have

$$D^\vee_{\xi,\ell,\infty}(\pi) = \varprojlim_{k\geq 0} \varprojlim_{M_k\in\mathcal{M}_k(\pi^{H_k})} M^\vee_k[1/X]\ .$$

Proof. We have $\operatorname{Tr}_{H_0/H_k}(N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = N_0 \operatorname{Tr}_{H_0/s^kH_0s^{-k}}(s^kM) = N_0F^k(M)$ which is equivalent to M. Conversely,

$$N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^k \operatorname{Tr}_{H_0/H_k}(M_k)) = N_0 \operatorname{Tr}_{H_k/s^kH_ks^{-k}}(s^kM_k) = N_0F^k_k(M_k)$$

is equivalent to M_k as it is the image of the map

$$1\otimes F^k_k\colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi^k,\Lambda(N_0/H_k)/\varpi^h} \to M_k$$

having finite cokernel. □ □

We equip the pseudocompact $\Lambda_\ell(N_0)$-module $D^\vee_{\xi,\ell,\infty}(\pi)$ with the weak topology, ie. with the projective limit topology of the weak topologies of $M^\vee_\infty[1/X]$. (The weak topology on $\Lambda_\ell(N_0)$ is defined in section 8 of [9].) Recall that the sets

$$O(M,l,l') := f^{-1}_{M,l}(\Lambda(N_0/H_l)\otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++}) \tag{9}$$

for $l, l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. Here $f_{M,l}$ is the natural projection map $f_{M,l}\colon D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M^\vee_l[1/X]$ and $M^\vee[1/X]^{++}$ denotes the set of elements $d \in M^\vee[1/X]$ with $\varphi^n(d) \to 0$ in the weak topology of $M^\vee[1/X]$ as $n\to\infty$.

2.2 A natural transformation from D_{SV} to $D^\vee_{\xi,\ell,\infty}$

In order to avoid confusion we denote by $D_{SV}(\pi)$ the $\Lambda(N_0)$-module with an action of B_+^{-1} associated to the smooth o-torsion representation π defined as $D(\pi)$ in [9] (note that in [9] the notation V is used for the o-torsion representation that we denote by π). For a brief review of this functor see section 1.2.

Lemma 2.11. *Let W be in $\mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$. There exists a positive integer $k_0 > 0$ such that for all $k \geq k_0$ we have $s^k M \subseteq W$. In particular, both $M_k = N_0 \operatorname{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)$ and $N_0 F^k(M)$ are contained in W for all $k \geq k_0$.*

Proof. By the assumption that M is finitely generated over $\Lambda(N_0/H_0)/\varpi^h[F]$ and W is a B_+-subrepresentation it suffices to find an integer s^{k_0} such that we have $s^{k_0} m_i$ lies in W for all the generators $m_1, \dots, m_r$ of M. This, however, follows from Lemma 2.1 in [9] noting that the powers of s are cofinal in T_+. □ □

In particular, we have a homomorphism $W^\vee \twoheadrightarrow M_k^\vee$ of $\Lambda(N_0)$-modules induced by this inclusion. We compose this with the localisation map $M_k^\vee \to M_k^\vee[1/X]$ and take projective limits with respect to k in order to obtain a $\Lambda(N_0)$-homomorphism

$$\operatorname{pr}_{W,M} \colon W^\vee \to M_\infty^\vee[1/X] \ .$$

Lemma 2.12. *The map $\operatorname{pr}_{W,M}$ is ψ_s- and Γ-equivariant.*

Proof. The Γ-equivariance is clear as it is given by the multiplication by elements of Γ on both sides. For the ψ_s-equivariance let $k > 0$ be large enough so that H_k is contained in $sH_0s^{-1} \leq sN_0s^{-1}$ (ie. $H_{k,-} = s^{-1}H_k s$) and M_k is contained in W. Let f be in $W^\vee = \operatorname{Hom}_o(W, o/\varpi^h)$ such that $f_{|N_0 s M_{k,*}} = 0$. By definition we have $\psi_s(f)(w) = f(sw)$ for any $w \in W$. Denote the restriction of f to M_k by $f_{|M_k}$ and choose an element $m \in M_k^* \leq M_k$ written in the form

$$m = \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u) = \sum_{u \in J(N_0/sN_0s^{-1})} us \operatorname{Tr}_{H_{k,-}/H_k}(m_u) \ .$$

Then we compute

$$\begin{aligned}
f_{|M_k}(m) &= \sum_{u \in J(N_0/sN_0s^{-1})} f(us \operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} (u^{-1}f)(s \operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} \psi_s(u^{-1}f)(\operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
&\overset{(7)}{=} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi(\psi_s(u^{-1}f)_{|M_k})(F_k(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(uF_k(m_u)) = \\
&= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(m)
\end{aligned}$$

as for distinct $u, v \in J(N_0/sN_0s^{-1})$ we have $u\varphi(f_0)(vF_k(m_v)) = 0$ for any $f_0 \in (M_k^*)^\vee$. So by inverting X and taking projective limits with respect to k we obtain

$$\mathrm{pr}_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\mathrm{pr}_{W,M}(\psi_s(u^{-1}f)))$$

as we have $(M_k^*)^\vee[1/X] \cong M_k^\vee[1/X]$. However, since $M_\infty^\vee[1/X]$ is an étale (φ, Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$ we have a unique decomposition of $\mathrm{pr}_{W,M}(f)$ as

$$\mathrm{pr}_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi(u^{-1}\,\mathrm{pr}_{W,M}(f)))$$

so we must have $\psi(\mathrm{pr}_{W,M}(f)) = \mathrm{pr}_{W,M}(\psi_s(f))$. For general $f \in W^\vee$ note that $N_0sM_{k,*}$ is killed by $\varphi(X^r)$ for $r \geq 0$ big enough, so we have $X^r\psi(\mathrm{pr}_{W,M}(f)) = \psi(\mathrm{pr}_{W,M}(\varphi(X^r)f)) = \mathrm{pr}_{W,M}(\psi_s(\varphi(X^r)f)) = X^r\,\mathrm{pr}_{W,M}(\psi_s(f))$. The statement follows since X^r is invertible in $\Lambda_\ell(N_0)$. □ □

By taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the injective limit with respect to $W \in \mathcal{B}_+(\pi)$ we obtain a ψ_s- and Γ-equivariant $\Lambda(N_0)$-homomorphism

$$\mathrm{pr} := \varinjlim_W \varprojlim_M \mathrm{pr}_{W,M} \colon D_{SV}(\pi) \to D_{\xi,\ell,\infty}^\vee(\pi)\ .$$

Remarks.

1. Taking Pontryagin dual of the inclusion $M_k \leq \pi$ for all $M \in \mathcal{M}(\pi^{H_k})$ and $k \geq 0$ we obtain a composite map $\pi^\vee \twoheadrightarrow M_k^\vee \to M_k^\vee[1/X]$. These are compatible with the projective limit construction therefore induce natural maps $\pi^\vee \to D_\xi^\vee(\pi)$ and $\pi^\vee \to D_{\xi,\ell,\infty}^\vee(\pi)$. Both of these maps factor through the map $\pi^\vee \twoheadrightarrow D_{SV}(\pi)$ by Lemma 2.11.

2. The natural topology on D_{SV} obtained as the quotient topology from the compact topology on $\pi^\vee$ via the surjective map $\pi^\vee \twoheadrightarrow D_{SV}(\pi)$ is compact, but may not be Hausdorff in general. However, if $\mathcal{B}_+(\pi)$ contains a minimal element (as in the case of the principal series [7]) then it is also Hausdorff. However, the map pr factors through the maximal Hausdorff quotient of $D_{SV}(\pi)$, namely $\overline{D}_{SV}(\pi) := (\bigcap_{W \in \mathcal{B}_+(\pi)} W)^\vee$. Indeed, pr is continuous and $D_{\xi,\ell,\infty}^\vee(\pi)$ is Hausdorff, so the kernel of pr is closed in $D_{SV}(\pi)$ (and contains 0).

3. Assume that $h = 1$, ie. π is a smooth representation in characteristic p. Then $D_{\xi,\ell,\infty}^\vee(\pi)$ has no nonzero $\Lambda(N_0)/\varpi$-torsion. Hence the $\Lambda(N_0)/\varpi$-torsion part of $D_{SV}(\pi)$ is contained in the kernel of pr.

4. If $D_{SV}(\pi)$ has finite rank and its torsion free part is étale over $\Lambda(N_0)$ then $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi)$ is also étale and of finite rank r over $\Lambda_\ell(N_0)$. Moreover, the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)}$ pr : $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi)$ has dense image by Lemma 2.11. Thus $D_{\xi,\ell,\infty}^\vee(\pi)$ has rank at most r over $\Lambda_\ell(N_0)$. In particular, for π being the principal series $D_{SV}(\pi)$ has rank 1 and its torsion free part is étale over $\Lambda(N_0)$ ([7]), hence we obtained that $D_{\xi,\ell,\infty}^\vee(\pi)$ has rank 1 over $\Lambda_\ell(N_0)$ (cf. Example 7.6 of [2]).

One can show the above Remark 2 algebraically, too. Let $M \in \mathcal{M}(\pi^{H_0})$ be arbitrary. Then the map $1 \otimes \mathrm{id}_{M^\vee} \colon M^\vee \to M^\vee[1/X]$ has finite kernel, so the image $(1 \otimes \mathrm{id}_{M^\vee})(M^\vee)$ is isomorphic to $M_0^\vee$ for some finite index submodule $M_0 \leq M$. Moreover, $M_0^\vee$ is a ψ- and Γ-invariant treillis in $D := M^\vee[1/X] = M_0^\vee[1/X]$. Therefore the map $(1 \otimes F)^\vee$ is injective on $M_0^\vee$ since it is injective after inverting X and $M_0^\vee$ has no X-torsion. This means that $1 \otimes F \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi} M_0 \to M_0$ is surjective, ie. we have $M_0 = N_0 F^k(M_0)$ for all $k \geq 0$. However, for any $W \in \mathcal{B}_+(\pi)$ and k large enough (depending a priori on W) we have $N_0 F^k(M_0) \subseteq W$, so we deduce $M_0 \subset \cap_{W \in \mathcal{B}_+} W$.

Corollary 2.13. *If $\pi = \mathrm{Ind}_{B_0}^B \pi_0$ is a compactly induced representation of B for some smooth o/ϖ^h-representation π_0 of B_0 then we have $D_\xi^\vee(\pi) = 0$. In particular, $D_\xi^\vee$ is not exact on the category of smooth o/ϖ^h-representations of B. (However, it may still be exact on a smaller subcategory with additional finiteness conditions.)*

Proof. By the 2nd remark above the map $\pi^\vee \to D_\xi^\vee(\pi)$ factors through the maximal Hausdorff quotient $\overline{D}_{SV}(\pi)$ of $D_{SV}(\pi)$. By Lemma 3.2 in [9], we have $\overline{D}_{SV}(\pi) = (\bigcap_\sigma W_\sigma)^\vee$ where the B_+-subrepresentations W_σ are indexed by order-preserving maps $\sigma \colon T_+/T_0 \to \mathrm{Sub}(\pi_0)$ where $\mathrm{Sub}(\pi_0)$ is the partially order set of B_0-subrepresentations of π_0. The explicit description of the B_+-subrepresentations W_σ (there denoted by M_σ) before Lemma 3.2 in [9] shows that we have in fact $\bigcap_\sigma W_\sigma = \{0\}$ whence the natural map $\pi^\vee \to D_\xi^\vee(\pi)$ is zero. However, by the construction of this map this can only be zero if $D_\xi^\vee(\pi) = 0$.

Since the principal series arises as a quotient of a compactly induced representation, the exactness of $D_\xi^\vee$ would imply the vanishing of $D_\xi^\vee$ on the principal series, too—which is not the case by Ex. 7.6 in [2]. □ □

Proposition 2.14. *Let D be an étale (φ, Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$, and $f : D_{SV}(\pi) \to D$ be a continuous ψ_s and Γ-equivariant $\Lambda(N_0)$-homomorphism. Then f factors uniquely through pr, ie. there exists a unique ψ- and Γ-equi-variant $\Lambda(N_0)$-homomorphism $\hat{f} : D_{\xi,\ell,\infty}^\vee(\pi) \to D$ such that $f = \hat{f} \circ \mathrm{pr}$.*

Proof. For the uniqueness of $\hat{f}$ note that Lemma 2.11 implies the density of the image of $\Lambda_\ell(N_0) \otimes D_{SV}(\pi)$ in $D_{\xi,\ell,\infty}^\vee(\pi)$ as its composite with the projection onto $M_k^\vee[1/X]$ is surjective for k large enough and $M \in \mathcal{M}(\pi^{H_0})$ arbitrary. Therefore if $\hat{f}'$ is another lift then $\hat{f} - \hat{f}'$ vanishes on a dense subset whence it is zero by continuity.

At first we construct a homomorphism $\hat{f}_{H_0} : D_\xi^\vee = (D_{\xi,\ell,\infty}^\vee)_{H_0} \to D_{H_0}$ such that the following diagram commutes:

$$\begin{array}{ccccc} D_{SV}(\pi) & \xrightarrow{\mathrm{pr}} & D_{\xi,\ell,\infty}^\vee(\pi) & \xrightarrow{(\cdot)_{H_0}} & D_\xi^\vee(\pi) \\ & \searrow^{f} & & & \downarrow{\scriptstyle \hat{f}_{H_0}} \\ & & D & \xrightarrow[(\cdot)_{H_0}]{} & D_{H_0} \end{array}$$

Consider the composite map $f' : \pi^\vee \to D_{SV}(\pi) \xrightarrow{f} D \to D_{H_0}$. Note that f' is continuous and D_{H_0} is Hausdorff, so $\mathrm{Ker}(f')$ is closed in $\pi^\vee$. Therefore $M_0 = (\pi^\vee / \mathrm{Ker}(f'))^\vee$ is naturally a subspace in π. We claim that M_0 lies in $\mathcal{M}(\pi^{H_0})$. Indeed, $M_0^\vee$ is a quotient of $\pi_{H_0}^\vee$, hence $M_0 \leq \pi^{H_0}$ and it is Γ-invariant since f' is Γ-equivariant. M_0 is admissible because it is discrete, hence $M_0^\vee$ is compact, equivalently finitely generated over $o/\varpi^h[[X]]$, because $M_0^\vee$ can be identified with a $o/\varpi^h[[X]]$-submodule of D_{H_0} which is finitely generated over $o/\varpi^h((X))$. The last thing to verify is that M is finitely generated over $o/\varpi^h[[X]][F]$, which follows from the following

Lemma 2.15. *Let D be an étale (φ,Γ)-module over $o/\varpi^h((X))$ and $D_0 \subset D$ be a ψ and Γ-invariant compact (or, equivalently, finitely generated) $o/\varpi^h[[X]]$ submodule. Then $D_0^\vee$ is finitely generated as a module over $o/\varpi^h[[X]][F]$ where for any $m \in D_0^\vee = \mathrm{Hom}_o(D_0, o/\varpi^h)$ we put $F(m)(f) := m(\psi(f))$ (for all $f \in D_0$).*

Proof. As the extension of finitely generated modules over a ring is again finitely generated, we may assume without loss of generality that $h = 1$ and D is irreducible, ie. D has no nontrivial étale (φ,Γ)-submodule over $o/\varpi((X))$.

If $D_0 = \{0\}$ then there is nothing to prove. Otherwise D_0 contains the smallest ψ and Γ stable $o[[X]]$-submodule $D^\natural$ of D. So let $0 \neq m \in D_0^\vee$ be arbitrary such that the restriction of m to $D^\natural$ is nonzero and consider the $o/\varpi[[X]][F]$-submodule $M := o/\varpi[[X]][F]m$ of $D_0^\vee$ generated by m. We claim that M is not finitely generated over o. Suppose for contradiction that the elements $F^r m$ are not linearly independent over o/ϖ. Then we have a polynomial $P(x) = \sum_{i=0}^n a_i x^i \in o/\varpi[x]$ such that $0 = P(F)m(f) = m(\sum a_i \psi^i(f)) = m(P(\psi)f)$ for any $f \in D^\natural \subset D_0$. However, $P(\psi)\colon D^\natural \to D^\natural$ is surjective by Prop. II.5.15. in [3], so we obtain $m_{|D^\natural} = 0$ which is a contradiction. In particular, we obtain that $M^\vee[1/X] \neq 0$. However, note that $M^\vee[1/X]$ has the structure of an étale (φ,Γ)-module over $o/\varpi((X))$ by Lemma 2.6 in [2]. Indeed, M is admissible, Γ-invariant, and finitely generated over $o/\varpi[[X]][F]$ by construction. Moreover, we have a natural surjective homomorphism $D = D_0[1/X] = (D_0^\vee)^\vee[1/X] \to M^\vee[1/X]$ which is an isomorphism as D is assumed to be irreducible. Therefore we have $(D_0^\vee/M)^\vee[1/X] = 0$ showing that $D_0^\vee/M$ is finitely generated over o. In particular, both M and $D_0^\vee/M$ are finitely generated over $o/\varpi[[X]][F]$ therefore so is $D_0^\vee$. □ □

Now $D_0 = M_0^\vee$ is a ψ- and Γ-invariant $o/\varpi^h[[X]]$-submodule of D therefore we have an injection $f_0\colon M_0^\vee[1/X] \hookrightarrow D$ of étale (φ,Γ)-modules. The map $\hat{f}_{H_0}\colon D_\xi^\vee \to D_{H_0}$ is the composite map $D_\xi^\vee \twoheadrightarrow M_0^\vee[1/X] \hookrightarrow D$. It is well defined and makes the above diagram commutative, because the map

$$\pi^\vee \to D_{SV}(\pi) \xrightarrow{\mathrm{pr}} D_{\xi,\ell,\infty}^\vee(\pi) \xrightarrow{(\cdot)_{H_0}} D_\xi^\vee(\pi) \to M_0^\vee[1/X]$$

is the same as $\pi^\vee \to M_0^\vee \to M_0^\vee[1/X]$.

Finally, by Corollary 2.9 $M^\vee[1/X]$ (resp. D_{H_0}) corresponds to $M_\infty^\vee[1/X]$ (resp. to D) via the equivalence of categories in Theorem 8.20 in [10] therefore f_0 can uniquely be lifted to a φ- and Γ-equivariant $\Lambda_\ell(N_0)$-homomorphism $f_\infty\colon M_\infty^\vee[1/X] \hookrightarrow D$. The map $\hat{f}$ is defined as the composite $D_{\xi,\ell,\infty}^\vee \twoheadrightarrow M_\infty^\vee[1/X] \hookrightarrow D$. Now the image of $f - \hat{f}\circ\mathrm{pr}$ is a ψ_s-invariant $\Lambda(N_0)$-submodule in $(H_0 - 1)D$ therefore it is zero by Lemma 8.17 and the proof of Lemma 8.18 in [10]. Indeed, for any $x \in D_{SV}(\pi)$ and $k \geq 0$ we may write $(f - \hat{f}\circ\mathrm{pr})(x)$ in the form $\sum_{u\in J(N_0/s^k N_0 s^{-k})} u\varphi^k((f - \hat{f}\circ\mathrm{pr})(\psi^k(u^{-1}x)))$ that lies in $(H_k - 1)D$. □ □

2.3 Étale hull

In this section we construct the étale hull of $D_{SV}(\pi)$: an étale T_+-module $\widetilde{D_{SV}}(\pi)$ over $\Lambda(N_0)$ with an injection $\iota : D_{SV}(\pi) \to \widetilde{D_{SV}}(\pi)$ with the following universal property: For any étale (φ,Γ)-module D' over $\Lambda(N_0)$, and ψ_s and Γ-equivariant map $f: D_{SV}(\pi)\to D'$, f factors through $\widetilde{D_{SV}}(\pi)$, ie. there exists a unique ψ- and Γ-equivariant $\Lambda(N_0)$-homomorphism $\widetilde{f}:\widetilde{D_{SV}}(\pi)\to D'$ making the diagram

$$\begin{array}{ccc} D_{SV}(\pi) & \xrightarrow{\iota} & \widetilde{D_{SV}}(\pi) \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \widetilde{f}} & \\ D' & & \end{array}$$

commutative. Moreover, if we assume further that D' is an étale T_+-module over $\Lambda(N_0)$ and the map f is ψ_t-equivariant for all $t\in T_+$ then the map $\widetilde{f}$ is T_+-equivariant.

Definition 2.16. *Let D be a $\Lambda(N_0)$-module and $T_*\leq T_+$ be a submonoid. Assume moreover that the monoid T_* (or in the case of ψ-actions the inverse monoid T_*^{-1}) acts o-linearly on D, as well.*

We call the action of T_* a φ-action (relative to the $\Lambda(N_0)$-action) and denote the action of t by $d\mapsto\varphi_t(d)$, if for any $\lambda\in\Lambda(N_0)$, $t\in T_*$ and $d\in D$ we have $\varphi_t(\lambda d)=\varphi_t(\lambda)\varphi_t(d)$. Moreover, we say that the φ-action is injective *if for all $t\in T_*$ the map φ_t is injective. The φ-action of T_* is* nondegenerate *if for all $t\in T_*$ we have*

$$D=\sum_{u\in J(N_0/tN_0t^{-1})}\mathrm{Im}(u\circ\varphi_t)=\sum_{u\in J(N_0/tN_0t^{-1})}u(\varphi_t(D))\ .$$

We call the action of T_*^{-1} a ψ-action of T_* (relative to the $\Lambda(N_0)$-action) and denote the action of $t^{-1}\in T_*^{-1}$ by $d\mapsto\psi_t(d)$, if for any $\lambda\in\Lambda(N_0)$, $t\in T_*$ and $d\in D$ we have $\psi_t(\varphi_t(\lambda)d)=\lambda\psi_t(d)$. Moreover, we say that the ψ-action of T_* is surjective *if for all $t\in T_*$ the map ψ_t is surjective. The ψ-action of T_* is* nondegenerate *if for all $t\in T_*$ we have*

$$\{0\}=\bigcap_{u\in J(N_0/tN_0t^{-1})}\mathrm{Ker}(\psi_t\circ u^{-1})\ .$$

The nondegeneracy is equivalent to the condition that for any $t\in T_*$ $\mathrm{Ker}(\psi_t)$ does not contain any nonzero $\Lambda(N_0)$-submodule of D.

We say that a φ- and a ψ-action of T_* are compatible *on D, if*

$(\varphi\psi)$ for any $t\in T_*$, $\lambda\in\Lambda(N_0)$, and $d\in D$ we have $\psi_t(\lambda\varphi_t(d))=\psi_t(\lambda)d$.

Note that with $\lambda=1$ we also have $\psi_t\circ\varphi_t=\mathrm{id}_D$ for any $t\in T_*$ assuming $(\varphi\psi)$.

We also consider φ- and ψ-actions of the monoid $\mathbb{Z}_p\setminus\{0\}$ on $\Lambda(N_0)$-modules via the embedding $\xi\colon\mathbb{Z}_p\setminus\{0\}\to T_+$. Modules with a φ-action (resp. ψ-action) of $\mathbb{Z}_p\setminus\{0\}$ are called (φ,Γ)-modules (resp. (ψ,Γ)-modules).

For example, the natural φ- and ψ-actions of T_+ on $\Lambda(N_0)$ are compatible.

Remarks. 1. Note that the ψ-action of the monoid T_* is in fact an action of the inverse monoid T_*^{-1}. However, we assume T_+ to be commutative so it may also be viewed as an action of T_*.

2. Pontryagin duality provides an equivalence of categories between compact $\Lambda(N_0)$-modules with a continuous ψ-action of T_* and discrete $\Lambda(N_0)$-modules with a continuous φ-action of T_*. The surjectivity of the ψ-action corresponds to the injectivity of φ-action. Moreover, the ψ-action is nondegenerate if and only if so is the corresponding φ-action on the Pontryagin dual.

If D is a $\Lambda(N_0)$-module with a φ-action of T_* then there exists a homomorphism

$$\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \to D, \lambda \otimes d \mapsto \lambda\varphi_t(d) \tag{10}$$

of $\Lambda(N_0)$-modules. We say that the T_*-action on D is *étale* if the above map is an isomorphism. The φ-action of T_* on D is étale if and only if it is injective and for any $t \in T_*$ we have

$$D = \bigoplus_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(D) \ . \tag{11}$$

Similarly, we call a $\Lambda(N_0)$-module together with a φ-action of the monoid $\mathbb{Z}_p \setminus \{0\}$ an étale (φ, Γ)-module over $\Lambda(N_0)$ if the action of $\varphi = \varphi_s$ is étale.

If D is an étale T_*-module over $\Lambda(N_0)$ then there exists a ψ-action of T_* compatible with the étale φ-action (see [9] Section 6).

Dually, if D is a $\Lambda(N_0)$-module with a ψ-action of T_* then there exists a map

$$\begin{aligned} \iota_t \colon D &\to \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \\ d &\mapsto \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}d) \ . \end{aligned}$$

Lemma 2.17. *Fix $t \in T_*$. For any $\lambda \in \Lambda(N_0)$ and $u, v \in N_0$ we put $\lambda_{u,v} := \psi_t(u^{-1}\lambda v)$. For any fixed $v \in N_0$ we have*

$$\lambda v = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v})$$

and for any fixed $u \in N_0$ we have

$$u^{-1}\lambda = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\lambda_{u,v})v^{-1} \ .$$

Proof. The above formulae follow from the usual identities

$$\sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\mu)) = \mu = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(\mu v))v^{-1}$$

for $\mu \in \Lambda(N_0)$ as the inverses of elements of $J(N_0/tN_0t^{-1})$ form a set of representatives of the right cosets of tN_0t^{-1}. $\square$ $\square$

Lemma 2.18. *For any $t \in T_*$ the map ι_t is a homomorphism of $\Lambda(N_0)$-modules. It is injective for all $t \in T_*$ if and only if the ψ-action of T_* on D is nondegenerate.*

Proof. Using Lemma 2.17 we compute

$$\begin{aligned}\iota_t(\lambda x) = \sum_{u\in J(N_0/tN_0t^{-1})} u\otimes\psi_t(u^{-1}\lambda x) = \\ = \sum_{u,v\in J(N_0/tN_0t^{-1})} u\otimes\psi_t(\varphi_t(\lambda_{u,v})v^{-1}x) = \\ = \sum_{u,v\in J(N_0/tN_0t^{-1})} u\otimes\lambda_{u,v}\psi_t(v^{-1}x) = \\ = \sum_{u,v\in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v})\otimes\psi_t(v^{-1}x) = \\ = \sum_{v\in J(N_0/tN_0t^{-1})} \lambda v\otimes\psi_t(v^{-1}x) = \lambda\iota_t(x)\ .\end{aligned}$$

The second statement follows from noting that $\Lambda(N_0)$ is a free right module over itself via the map φ_t with free generators $u\in J(N_0/tN_0t^{-1})$. □ □

Lemma 2.19. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_* and $t\in T_*$. Then there exists a ψ-action of T_* on $\varphi_t^*D := \Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_t} D$ making the homomorphism ι_t ψ-equivariant. Moreover, if we assume in addition that the ψ-action on D is nondegenerate then so is the ψ-action on φ_t^*D.*

Proof. Let $t'\in T_*$ be arbitrary and define the action of $\psi_{t'}$ on φ_t^*D by putting

$$\psi_{t'}(\lambda\otimes d) := \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d) \text{ for } \lambda\in\Lambda(N_0), d\in D\ ,$$

and extending $\psi_{t'}$ to φ_t^*D o-linearly. Note that we have

$$\begin{aligned}\psi_{t'}(\varphi_{t'}(\mu)\lambda\otimes d) = \\ = \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\varphi_{t'}(\mu)\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d) = \mu\psi_{t'}(\lambda\otimes d)\ .\end{aligned}$$

Moreover, the map $\psi_{t'}$ is well-defined since we have

$$
\begin{aligned}
\psi_{t'}(\lambda\varphi_t(\mu)\otimes d) &= \sum_{v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu)\varphi_t(v'))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu v'))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(\mu_{u',v'})))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\varphi_t(\mu_{u',v'})\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\mu_{u',v'}\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(\varphi_{t'}(\mu_{u',v'})v'^{-1}d) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}\mu d) = \psi_{t'}(\lambda\otimes\mu d)\ ,
\end{aligned}
$$

using Lemma 2.17 where $\mu_{u',v'} = \psi_{t'}(u'^{-1}\mu v')$. Introducing the notation $J' := J(N_0/t'N_0t'^{-1})$ and $J'' := J(N_0/t''N_0t''^{-1})$ we further compute

$$
\begin{aligned}
\psi_{t''}(\psi_{t'}(\lambda\otimes d)) &= \psi_{t''}\Big(\sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d)\Big) = \\
&= \sum_{u''\in J''}\sum_{u'\in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'))\varphi_t(u''))\otimes\psi_{t''}(u''^{-1}\psi_{t'}(u'^{-1}d)) = \\
&= \sum_{u''\in J''}\sum_{u'\in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(u''))))\otimes\psi_{t''}(\psi_{t'}(\varphi_{t'}(u'')^{-1}u'^{-1}d)) = \\
&= \psi_{t''t'}(\lambda\otimes d)
\end{aligned}
$$

showing that it is indeed a ψ-action of the monoid T_*.

For the second statement of the Lemma we compute

$$
\begin{aligned}
&\psi_{t'}(\iota_t(x)) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})}\sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}\psi_t(u^{-1}x)) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})}\sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u'))\otimes\psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x))\ .
\end{aligned}
$$

Note that in the above sum $u\varphi_t(u')$ runs through a set of representatives for the cosets $N_0/tt'N_0t'^{-1}t^{-1}$. Moreover, $v := \psi_{t'}(u\varphi_t(u'))$ is nonzero if and only if $u\varphi_t(u')$ lies in $t'N_0t'^{-1}$ and the nonzero values of v run through a set $J'(N_0/tN_0t^{-1})$ of representatives of the cosets N_0/tN_0t^{-1}. In case $v\neq 0$ we have $\varphi_{t'}(v)^{-1} = (u\varphi_t(u'))^{-1} = \varphi_t(u')^{-1}u^{-1}$. So we continue computing by replacing $\psi_{t'}(u\varphi_t(u'))$ by v and omitting the terms with $v=0$

$$\psi_{t'}(\iota_t(x)) =$$

$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x)) =$$

$$= \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(\psi_{t'}(\varphi_{t'}(v^{-1})x)) =$$

$$= \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(v^{-1}\psi_{t'}(x)) = \iota_t(\psi_{t'}(x)) \ .$$

Assume now that the ψ-action of T_* on D is nondegenerate. Any element in $x\in\varphi_t^*D$ can be uniquely written in the form $\sum_{u\in J(N_0/tN_0t^{-1})} u\otimes x_u$. Assume that for a fixed $t'\in T_*$ we have $\psi_{t'}(u_0'^{-1}x)=0$ for all $u_0'\in N_0$. Then we compute

$$0 = \psi_{t'}(u_0'^{-1}x) =$$

$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u_0'^{-1}u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}x_u) \ .$$

Put $y=u_0'^{-1}u\varphi_t(u')$. For any fixed u_0' the set $\{y \mid u\in J(N_0/tN_0t^{-1}), u'\in J(N_0/t'N_0t'^{-1})\}$ forms a set of representatives of $N_0/tt'N_0(tt')^{-1}$, and we have $\psi_{t'}(y)\neq 0$ if and only if y lies in $t'N_0t'^{-1}$ in which case we have $\psi_{t'}(y)=t'^{-1}yt'$. So the nonzero values of $\psi_{t'}(y)$ run through a set of representatives of N_0/tN_0t^{-1}. Since we have the direct sum decomposition $\varphi_t^*D = \bigoplus_{v\in J(N_0/tN_0t^{-1})} v\otimes D$ we obtain $\psi_{t'}(u'^{-1}x_u)=0$ for all $u'\in J(N_0/t'N_0t'^{-1})$ and $u\in J(N_0/tN_0t^{-1})$ such that $y=u_0'^{-1}u\varphi_t(u')$ is in $t'N_0t'^{-1}$. However, for any choice of u' and u there exists such a u_0', so we deduce $x=0$. □ □

Proposition 2.20. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_*. The following are equivalent:*

1. *There exists a unique φ-action on D, which is compatible with ψ and which makes D an étale T_*-module.*

2. *The ψ-action is surjective and for any $t\in T_*$ we have*

$$D = \bigoplus_{u_0\in J(N_0/tN_0t^{-1})} \bigcap_{\substack{u\in J(N_0/tN_0t^{-1})\\ u\neq u_0}} \mathrm{Ker}(\psi_t\circ u^{-1}) \ . \tag{12}$$

   In particular, the action of ψ is nondegenerate.

3. *The map ι_t is bijective for all $t\in T_*$.*

Proof. $1\Longrightarrow 3$ In this case the map ι_t is the inverse of the isomorphism (10) so it is bijective by the étale property.

$3\Longrightarrow 2$: The injectivity of ι_t shows the nondegeneracy of the ψ-action. Further if $1\otimes d=\iota_t(x)$ then we have $\psi_t(x)=d$ so the ψ-action is surjective. Moreover, $\iota_t^{-1}(u_0\otimes D)$ equals $\bigcap_{u_0\neq u\in J(N_0/tN_0t^{-1})}\mathrm{Ker}(\psi_t\circ u^{-1})$ therefore D can be written as a direct sum (12).

$2 \Longrightarrow 1$: In order to define the φ-action of T_* on D we fix $t \in T_*$. For any $d \in D$ we have to choose $\varphi_t(d)$ such that $\psi_t(\varphi_t(d)) = d$. By the surjectivity of ψ_t we can choose $x \in D$ such that $\psi_t(x) = d$. Using the assumption we can write $x = \sum_{u_0 \in J(N_0/tN_0t^{-1})} x_{u_0}$, with

$$x_{u_0} \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

By the compatibility $(\varphi\psi)$ we should have

$$\varphi_t(d) \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq 1}} \mathrm{Ker}(\psi_t \circ u^{-1})$$

as we have $\psi_t(u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$.

A convenient choice is $\varphi_t(d) = x_1$, and there exists exactly one such element in D: if x' would be an other, then

$$x_1 - x' \in \bigcap_{u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) = \{0\} \ .$$

This shows the uniqueness of the φ-action. Further, $x_1 = \varphi_t(d) = 0$ would mean that x lies in $\mathrm{Ker}(\psi_t)$ whence $d = \psi_t(x) = 0$—therefore the injectivity. Similarly, by definition we also have $x_{u_0} = u_0\varphi_t \circ \psi_t(u_0^{-1}x)$ for all $u_0 \in J(N_0/sN_0s^{-1})$. By the surjectivity of the ψ-action any element in D can be written of the form $\psi_t(u_0^{-1}x)$ for any fixed $u_0 \in J(N_0/tN_0t^{-1})$ so we obtain

$$u_0\varphi_t(D) = \bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

The étale property (11) follows from this using our assumption 2. Moreover, this also shows $\psi_t(u\varphi_t(d)) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ which implies $(\varphi\psi)$ using that $\psi_t \circ \varphi_t = \mathrm{id}_D$ by construction. Finally, $\varphi_t(\lambda)\varphi_t(d) - \varphi_t(\lambda d)$ lies in the kernel of $\psi_t \circ u_0^{-1}$ for any $u_0 \in J(N_0/tN_0t^{-1})$, $\lambda \in \Lambda(N_0)$ and $d \in D$, so it is zero. □ □

From now on if we have an étale T_*-module over $\Lambda(N_0)$ we a priori equip it with the compatible ψ-action, and if we have a $\Lambda(N_0)$-module with a ψ-action, which satisfies the above property 2, we equip it with the compatible φ-action, which makes it étale. The construction of the étale hull and its universal property is given in the following

Proposition 2.21. *For any $\Lambda(N_0)$-module D, with a ψ-action of T_* there exists an étale T_*-module $\widetilde{D}$ over $\Lambda(N_0)$ and a ψ-equivariant $\Lambda(N_0)$-homomor-phism $\iota \colon D \to \widetilde{D}$ with the following universal property: For any ψ-equivariant $\Lambda(N_0)$-homomorphism $f : D \to D'$ into an étale T_*-module D' we have a unique morphism $\widetilde{f} : \widetilde{D} \to D'$ of étale T_*-modules over $\Lambda(N_0)$ making the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\iota} & \widetilde{D} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \widetilde{f}} & \\ D' & & \end{array}$$

commutative. $\widetilde{D}$ is unique upto a unique isomorphism. If we assume the ψ-action on D to be nondegenerate then ι is injective.

Proof. We will construct $\widetilde{D}$ as the injective limit of φ_t^*D for $t \in T_*$. Consider the following partial order on the set T_*: we put $t_1 \leq t_2$ whenever we have $t_2t_1^{-1} \in T_*$. Note that by Lemma 2.19 we obtain a ψ-equivariant isomorphism $\varphi^*_{t_2t_1^{-1}}\varphi^*_{t_1}D \cong \varphi^*_{t_2}D$ for any pair $t_1 \leq t_2$ in T_*. In particular, we obtain a ψ-equivariant map $\iota_{t_1,t_2} \colon \varphi^*_{t_1}D \to \varphi^*_{t_2}D$. Applying this observation to $\varphi^*_{t_1}D$ for a sequence $t_1 \leq t_2 \leq t_3$ we see that the $\Lambda(N_0)$-modules φ_t^*D $(t \in T_*)$ with the ψ-action of T_* form a direct system with respect to the connecting maps ι_{t_1,t_2}. We put

$$\widetilde{D} := \varinjlim_{t \in T_*} \varphi_t^* D$$

as a $\Lambda(N_0)$-module with a ψ-action of T_*. For any fixed $t' \in T_*$ we have

$$\varphi^*_{t'}\widetilde{D} = \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t'}} \varinjlim_{t \in T_*} \varphi_t^* D \cong$$
$$\cong \varinjlim_{t \in T_*} \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t'}} \varphi_t^* D \cong \varinjlim_{t't \in T_*} \varphi^*_{t't} D \cong \widetilde{D}$$

showing that there exists a unique φ-action of T_* on $\widetilde{D}$ making $\widetilde{D}$ an étale T_*-module over $\Lambda(N_0)$ by Proposition 2.20.

For the universal property, let $f : D \to D'$ be an ψ-equivariant map into an étale T_*-module D' over $\Lambda(N_0)$. By construction of the map φ_t on $\widetilde{D}$ $(t \in T_*)$ we have $\varphi_t(\iota(x)) = (1 \otimes x)_t$ where $(1 \otimes x)_t$ denotes the image of $1 \otimes x \in \varphi_t^* D$ in $\widetilde{D}$. So we put

$$\widetilde{f}((\lambda \otimes x)_t) := \lambda\varphi_t(f(x)) \in D'$$

and extend it o-linearly to $\widetilde{D}$. Note right away that $\widetilde{f}$ is unique as it is φ_t-equivariant. The map $\widetilde{f} \colon \widetilde{D} \to D'$ is well-defined as we have

$$\widetilde{f}(\iota_{t,tt'}(1 \otimes_t x)) = \widetilde{f}(\sum_{u' \in N_0/t'N_0t'^{-1}} u' \otimes_{t'} \psi_{t'}(u'^{-1} \otimes_t x)) =$$
$$= \sum_{u',v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(u' \otimes_{t'} \psi_{t'}(u'^{-1}\varphi_t(v')) \otimes_t \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{u',v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(u'\varphi_{t'} \circ \psi_{t'}(u'^{-1}\varphi_t(v')) \otimes_{tt'} \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(\varphi_t(v') \otimes_{tt'} \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v')\varphi_{tt'}(f(\psi_{t'}(v'^{-1}x))) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v'\varphi_{t'} \circ \psi_{t'}(v'^{-1}f(x))) = \varphi_t(f(x)) = \widetilde{f}(1 \otimes_t x)$$

noting that $\iota_{t,tt'}$ is a $\Lambda(N_0)$-homomorphism. Here the notation $\otimes_t$ indicates that the tensor product is via the map φ_t. By construction $\widetilde{f}$ is a homomorphism of étale T_*-modules over $\Lambda(N_0)$ satisfying $\widetilde{f} \circ \iota = f$.

The injectivity of ι in case the ψ-action on D is nondegenerate follows from Lemmata 2.18 and 2.19. □ □

Example 2.22. *If D itself is étale then we have $\widetilde{D} = D$.*

Corollary 2.23. *The functor $D \mapsto \widetilde{D}$ from the category of $\Lambda(N_0)$-modules with a ψ-action of T_* to the category of étale T_*-modules over $\Lambda(N_0)$ is exact.*

Proof. $\Lambda(N_0)$ is a free $\varphi_t(\Lambda(N_0))$-module, so $\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} -$ is exact, and so is the direct limit functor. □ □

Corollary 2.24. *Assume that D is a $\Lambda(N_0)$-module with a nondegenerate ψ-action of T_* and $f\colon D \to D'$ is an injective ψ-equivariant $\Lambda(N_0)$-homomor-phism into the étale T_*-module D' over $\Lambda(N_0)$. Then $\widetilde{f}$ is also injective.*

Proof. Since D is nondegenerate we may identify $\varphi_t^* D$ with a $\Lambda(N_0)$-submodule of $\widetilde{D}$. Assume that $x = \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes_t x_u \in \varphi_t^* D$ lies in the kernel of $\widetilde{f}$. Then $x_u = \psi_t(u^{-1}x) \in D \subseteq \varphi_t^* D \subseteq \widetilde{D}$ ($u \in J(N_0/tN_0t^{-1})$) also lies in the kernel of $\widetilde{f}$. However, we have $\widetilde{f}(x_u) = f(x_u)$ showing that $x_u = 0$ for all $u \in J(N_0/tN_0t^{-1})$ as f is injective. □ □

Example 2.25. *Let D be a (classical) irreducible étale (φ, Γ)-module over $k((X))$ and $D_0 \subset D$ a ψ- and Γ-invariant treillis in D. Then we have $\widetilde{D_0} \cong D$ unless D is 1-dimensional and $D_0 = D^\natural$ in which case we have $\widetilde{D_0} = D_0$.*

Proof. If D is 1-dimensional then $D^\natural = D^+$ is an étale (φ, Γ)-module over $k[[X]]$ (Prop. II.5.14 in [3]) therefore it is equal to its étale hull. If $\dim D > 1$ then we have $D^\natural = D^\# \subseteq D_0$ by Cor. II.5.12 and II.5.21 in [3]. By Corollary 2.24 $\widetilde{D^\#} \subseteq \widetilde{D_0}$ injects into D and it is φ- and ψ-invariant. Since $D^\#$ is not φ-invariant (Prop. II.5.14 in [3]) and it is the maximal compact $o[[X]]$-submodule of D on which ψ acts surjectively (Prop. II.4.2 in [3]) we obtain that $\widetilde{D_0}$ is not compact. In particular, its X-divisible part is nonzero therefore equals D as the X-divisible part of $\widetilde{D_0}$ is an étale (φ, Γ)-submodule of the irreducible D. □ □

Proposition 2.26. *The T_+^{-1} action on $D_{SV}(\pi)$ is a surjective nondegenerate ψ-action of T_+.*

Proof. Let $d \in D_{SV}(\pi)$ and $t \in T_+$. Since the action of both t and $\Lambda(N_0)$ on $D_{SV}(\pi)$ comes from that on $\pi^\vee$ we have $t^{-1}\varphi_t(\lambda)d = t^{-1}t\lambda t^{-1}d = \lambda t^{-1}d$, so this is indeed a ψ-action. The surjectivity of each ψ_t follows from the injectivity of the multiplication by t on each $W \in \mathcal{B}_+(\pi)$ and the exactness of $\varinjlim$ and $(\cdot)^\vee$. Finally, if W is in $\mathcal{B}_+(\pi)$ then so is $t^*W := \sum_{u \in J(N_0/tN_0t^{-1})} utW$ for any $t \in T_+$. Take an element $d \in D_{SV}(\pi)$ lying in the kernel of $\psi_t(u^{-1}\cdot)$ for all $u \in J(N_0/tN_0t^{-1})$. Now $D_{SV}(\pi)$ is by definition the direct limit of $W^\vee$ for all $W \in \mathcal{B}_+(\pi)$, so $\psi_t(u^{-1}d) = 0$ means that $t^{-1}u^{-1}d$ vanishes on some $W \in \mathcal{B}_+(\pi)$ (depending a priori on u). Since the set $J(N_0/tN_0t^{-1})$ is finite, we may even choose a common W for all u (taking the intersection and using Lemma 2.2 in [9]). Then the restriction of d to t^*W is zero showing that d is zero in $D_{SV}(\pi)$ therefore the nondegeneracy. Alternatively, the nondegeneracy of the ψ-action also follows from the existence of a ψ-equivariant injective map $D_{SV}(\pi) \hookrightarrow D_{SV}^0(\pi)$ into an étale T_+-module $D_{SV}^0(\pi)$ ([9] Proposition 3.5 and Remark 6.1). □ □

Question 1. *Let $D_{SV}^{(0)}(\pi)$ as in [9]. We have that $D_{SV}^{(0)}(\pi)$ is an étale T_*-module over $\Lambda(N_0)$ ([9] Proposition 3.5) and $f : D_{SV}(\pi) \hookrightarrow D_{SV}^{(0)}(\pi)$ is a ψ-equivariant map ([9] Remark 6.1). By the universal property of the étale hull and Corollary 2.24 $\widetilde{D_{SV}}(\pi)$ also injects into $D_{SV}^{(0)}(\pi)$. Whether or not this injection is always an isomorphism is an open question. In case of the Steinberg representation this is true by Proposition 11 in [12].*

We call the submonoid $T'_* \leq T_* \leq T_+$ cofinal in T_* if for any $t \in T_*$ there exists a $t' \in T'_*$ such that $t \leq t'$. For example $\xi(\mathbb{Z}_p \setminus \{0\})$ is cofinal in T_+.

Corollary 2.27. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_* and denote by $\widetilde{D}$ (resp. by $\widetilde{D}'$) the étale hull of D for the ψ-action of T_* (resp. of T'_*). Then we have a natural isomorphism $\widetilde{D}' \xrightarrow{\sim} \widetilde{D}$ of étale T'_*-modules over $\Lambda(N_0)$. More precisely, if $f\colon D \to D_1$ is a ψ-equivariant $\Lambda(N_0)$-homomorphism into an étale T'_*-module D_1 then f factors uniquely through $\iota\colon D \to \widetilde{D}$.*

Proof. Since $T'_* \leq T_*$ is cofinal in T_* we have $\varinjlim_{t' \in T'_*} \varphi^*_{t'} D \cong \varinjlim_{t \in T_*} \varphi^*_t D = \widetilde{D}$. □ □

By Corollary 2.27 there exists a homomorphism $\widetilde{\mathrm{pr}} : \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ of étale (φ,Γ)-modules over $\Lambda(N_0)$ such that $\mathrm{pr} = \widetilde{\mathrm{pr}} \circ \iota$. Our main result in this section is the following

Theorem 2.28. *$D^\vee_{\xi,\ell,\infty}(\pi)$ is the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ in the category of étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$, ie. we have*

$$D^\vee_{\xi,\ell,\infty}(\pi) \cong \varprojlim_{D} D$$

where D runs through the finitely generated étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule. This holds in any *topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ making both the maps $1 \otimes \iota\colon D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \widetilde{\mathrm{pr}}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ continuous.*

Remark. Since the map $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous, there exists such a topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$. For instance we could take either the final topology of the map $D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ or the initial topology of the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$.

Proof. The homomorphism $\widetilde{\mathrm{pr}}$ factors through the map $1 \otimes \mathrm{id}\colon \widetilde{D_{SV}}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ since $D^\vee_{\xi,\ell,\infty}(\pi)$ is a module over $\Lambda_\ell(N_0)$, so we obtain a homomorphism

$$1 \otimes \widetilde{\mathrm{pr}}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

of étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$. At first we claim that $1 \otimes \widetilde{\mathrm{pr}}$ has dense image. Let $M \in \mathcal{M}(\pi^{H_0})$ and $W \in \mathcal{B}_+(\pi)$ be arbitrary. Then by Lemma 2.11 the map $\mathrm{pr}_{W,M,k}\colon W^\vee \to M_k^\vee$ is surjective for $k \geq 0$ large enough. This shows that the natural map

$$1 \otimes \mathrm{pr}_{W,M,k}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} W^\vee \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} M_k^\vee \cong M_k^\vee[1/X]$$

is surjective. However, $1 \otimes \mathrm{pr}_{W,M,k}$ factors through $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi)$ by the Remarks after Lemma 2.12. In particular, the natural map

$$1 \otimes \mathrm{pr}_{M,k}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to M_k^\vee[1/X]$$

is surjective for all $M \in \mathcal{M}(\pi^{H_0})$ and $k \geq 0$ large enough (whence in fact for all $k \geq 0$). This shows that the image of the map

$$1 \otimes \mathrm{pr}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

is dense whence so is the image of $1 \otimes \widetilde{\mathrm{pr}}$. By the assumption that $1 \otimes \widetilde{\mathrm{pr}}$ is continuous we obtain a surjective homomorphism

$$\widehat{1 \otimes \widetilde{\mathrm{pr}}} \colon \varprojlim_D D \to D^\vee_{\xi,\ell,\infty}(\pi)$$

of pseudocompact (φ, Γ)-modules over $\Lambda_\ell(N_0)$ where D runs through the finitely generated étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$.

Let $0 \neq (x_D)_D$ be in the kernel of $\widehat{1 \otimes \widetilde{\mathrm{pr}}}$. Then there exists a finitely generated étale (φ, Γ)-module D over $\Lambda_\ell(N_0)$ with a surjective continuous homomorphism $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \twoheadrightarrow D$ such that $x_D \neq 0$. By Proposition 2.14 this map factors through $D^\vee_{\xi,\ell,\infty}(\pi)$ contradicting to the assumption $\widehat{1 \otimes \widetilde{\mathrm{pr}}}((x_D)_D) = 0$. □ □

Remark. Breuil's functor $D^\vee_\xi$ can therefore be computed from D_{SV} the following way: For a smooth o/ϖ^h-representation π we have $D^\vee_\xi(\pi) \cong (\varprojlim_D D)_{H_0} \cong \varprojlim_D D_{H_0}$ where D runs through the finitely generated étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule.

3 Nongeneric ℓ

Assume from now on that $\ell = \ell_\alpha$ is a nongeneric Whittaker functional defined by the projection of N_0 onto $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root $\alpha \in \Delta$.

Remark. In [2] the Whittaker functional ℓ is assumed to be generic. However, even if ℓ is not generic, the functor $D^\vee_\xi$ (hence also $D^\vee_{\xi,\ell,\infty}$) is right exact even though the restriction of $D^\vee_\xi$ to the category SP_{o/ϖ^h} may not be exact in general.

3.1 Compatibility with parabolic induction

Let $P = L_P N_P$ be a parabolic subgroup of G containing B with Levi component L_P and unipotent radical N_P and let π_P be a smooth o/ϖ^h-representation of L_P that we view as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$ where $P^- = L_P N_{P^-}$ is the parabolic subgroup opposite to P. Since T is contained in L_P, we may consider the same cocharacter $\xi \colon \mathbb{Q}_p^\times \to T$ for the group L_P instead of G. Further, we put $N_{L_P} := N \cap L_P$ and $N_{L_P,0} := N_0 \cap L_P$.

As in [2] denote by $W := N_G(T)/T$ (resp. by $W_P := (N_G(T) \cap L_P)/T$) the Weyl group of G (resp. of L_P) and by $w_0 \in W$ the element of maximal length. We have a canonical system

$$K_P := \{w \in W \mid w^{-1}(\Phi_P^+) \subseteq \Phi^+\}$$

of representatives (the Kostant representatives) of the right cosets $W_P \backslash W$ where Φ_P^+ denotes the set of positivie roots of L_P with respect to the Borel subgroup $L_P \cap B$. We have a generalized Bruhat decomposition

$$G = \coprod_{w \in K_P} P^- w B = \coprod_{w \in K_P} P^- w N \ .$$

Now let π_P be a smooth representation of L_P over A. We regard π_P as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$. Then the parabolically induced representation $\mathrm{Ind}_{P^-}^G \pi_P$ admits [11] (see also [6] §4.3) a filtration by B-subrepresentations whose graded pieces are contained in

$$\mathcal{C}_w(\pi_P) := c - \mathrm{Ind}_{P^-}^{P^-wN} \pi_P$$

for $w \in K_P$ where $c - \mathrm{Ind}_{P^-}^*$ stands for the space of locally constant functions on $* \supseteq P^-$ with compact support modulo P^-. B acts on $\mathcal{C}_w(\pi_P)$ by right translations. Moreover, the first graded piece equals $\mathcal{C}_1(\pi_P)$.

Lemma 3.1. *Let $\pi' \leq \mathcal{C}_w(\pi_P)$ be any B-subrepresentation for some $w \in K_P \setminus \{1\}$. Then we have $D_\xi^\vee(\pi') = 0$.*

Proof. By the right exactness of $D_\xi^\vee$ (Prop. 2.7(*ii*) in [2]) it suffices to treat the case $\pi' = \mathcal{C}_w(\pi_P)$. For this the same argument works as in Prop. 6.2 [2] with the following modification:

The particular shape of ℓ is only used in Lemma 6.5 in [2] (note that the subgroup $H_0 = \mathrm{Ker}(\ell\colon N_0 \to \mathbb{Z}_p)$ is denoted by N_1 therein). For an element $w \neq 1$ in the Weyl group we have $(w^{-1}N_{P^-}w \cap N_0)\backslash N_0/H_0 = \{1\}$ if and only if H_0 does not contain $w^{-1}N_{P^-}w \cap N_0$. Whenever $w^{-1}N_{P^-}w \cap N_0 \not\subseteq H_0$, the statement of Lemma 6.5 in [2] is true and there is nothing to prove.

In case we have $\{1\} \neq w^{-1}N_{P^-}w \cap N_0 \subseteq H_0$, the statement of Lemma 6.5 is not true for $\ell = \ell_\alpha$. However, the argument using it in the proof of Prop. 6.2 can be replaced by the following: the operator F acts on the space $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$ nilpotently. Indeed, the trace map $\mathrm{Tr}_{H_0/sH_0s^{-1}}$

$$\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}} \to \mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$$

is zero as each double coset $(w^{-1}N_{P^-}w \cap H_0)\backslash H_0/sH_0s^{-1}$ has size divisible by p and any function in $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}}$ is constant on these double cosets. The statement follows from Prop. 2.7(*iii*) in [2]. □ □

In order to extend Thm. 6.1 in [2] (the compatibility with parabolic induction) to our situation ($\ell = \ell_\alpha$) we need to distinguish two cases: whether the root subgroup N_α is contained in L_P or in N_P. Similarly to [6] we define the $s^{\mathbb{Z}}N_{L_P}$-ordinary part $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ of a smooth representation π_P of L_P as follows. We equip $\pi_P^{N_{L_P,0}}$ with the Hecke action $F_P := \mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s\cdot)$ of s making $\pi_P^{N_{L_P,0}}$ a module over the polynomial ring $o/\varpi^h[F_P]$ and put

$$\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P) := \mathrm{Hom}_{o/\varpi^h[F_p]}(o/\varpi^h[F_P, F_P^{-1}], \pi_P^{N_{L_P,0}})_{F_P-fin}$$

where $F_P - fin$ stands for those elements in the Hom-space whose orbit under the action of F_P is finite. By Lemmata 3.1.5 and 3.1.6 in [6] we may identify $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ with an $o/\varpi^h[F_P]$-submodule in $\pi_P^{N_{L_P,0}}$ by sending a map $f \in \mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ to its value $f(1) \in \pi_P^{N_{L_P,0}}$ at $1 \in o/\varpi^h[F_P, F_P^{-1}]$.

Proposition 3.2. *Let π_P be a smooth locally admissible representation of L_P over A which we view by inflation as a representation of P^-. We have an isomorphism*

$$D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G \pi_P\right) \cong \begin{cases} D_\xi^\vee(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases}$$

as étale (φ,Γ)-modules. In particular, for $P=B$ we have $D^\vee_\xi(\mathrm{Ind}_{B^-}\pi_B)\cong o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\pi_B^\vee$, ie. the value of $D^\vee_\xi$ at the principal series is the same (φ,Γ)-module of rank 1 regardless of the choice of ℓ (generic or not).

Proof. By Lemma 3.1 and the right exactness of $D^\vee_\xi$ (Prop. 2.7(ii) in [2]) it suffices to show that $D^\vee_\xi(\mathcal{C}_1(\pi_P))\cong D^\vee_\xi(\pi_P)$. Moreover, the proof of Prop. 6.7 in [2] goes through without modification so we have an isomorphism $D^\vee_\xi(\mathcal{C}_1(\pi_P))\cong D^\vee((\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0})$. Hence we are reduced to computing $D^\vee((\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0})$ in terms of π_P. We further have an identification

$$\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P\cong\mathcal{C}(N_{P,0},\pi_P)\cong\mathcal{C}(N_{P,0},o/\varpi^h)\otimes_{o/\varpi^h}\pi_P$$

by equation (40) in [2]. We need to distinguish two cases.

Case 1: $N_\alpha\subseteq L_P$. In this case we have $N_{P,0}\subseteq H_0$. Hence we deduce $(\mathcal{C}(N_{P,0},o/\varpi^h)\otimes_{o/\varpi^h}\pi_P)^{H_0}=\pi_P^{H_0/N_{P,0}}=\pi_P^{H_{P,0}}$. So we have

$$D^\vee_\xi\left(\mathrm{Ind}_{P^-}^G\pi_P\right)\cong D^\vee(\mathrm{Ind}_{P^-\cap N_0}^{N_0}\pi_P)^{H_0})\cong D^\vee(\pi_P^{H_{P,0}})\cong D^\vee_\xi(\pi_P)$$

in this case as claimed.

Case 2: $N_\alpha\subseteq N_P$. In this case we have $N_{L_P,0}\subseteq H_0$ and $N_{P,0}/(N_{P,0}\cap H_0)\cong\mathbb{Z}_p$. So we have an identification

$$\mathcal{C}(N_{P,0},\pi_P)^{H_0}\cong\mathcal{C}(N_{P,0}/(N_{P,0}\cap H_0),\pi_P^{N_{L_P,0}})\cong\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})\ .$$

Here the Hecke action $F=F_G=\mathrm{Tr}_{H_0/sH_0s^{-1}}\circ(s\cdot)$ of s on the right hand side is given by the formula

$$F_G(f)(a)=\begin{cases}F_P(f(a/p)) & \text{if } a\in p\mathbb{Z}_p\\ 0 & \text{if } a\in\mathbb{Z}_p\setminus p\mathbb{Z}_p\end{cases},$$

where $F_P=\mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}}\circ(s\cdot)$ denotes the Hecke action of s on $\pi_P^{N_{L_P,0}}$.

Now let M be a finitely generated $o/\varpi^h[[X]][F]$ submodule of $\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})$ that is stable under the action of Γ and is admissible as a representation of $\mathbb{Z}_p$. By possibly passing to a finite index submodule of M we may assume without loss of generality that the natural map $M^\vee\to M^\vee[1/X]$ is injective whence the map $\mathrm{id}\otimes F\colon o/\varpi^h[[X]]\otimes_{o/\varpi^h[[X]],F}M\to M$ is surjective. Let $f\in M$ be arbitrary. By continuity of f there exists an integer $n\geq 0$ such that f is constant on the cosets of $p^n\mathbb{Z}_p$. Writing $f=\sum_{i=0}^{p^n-1}[i]\cdot F^n(f_i)$ (where $[i]\cdot$ denotes the multiplication by the group element $i\in\mathbb{Z}_p$) by the surjectivity of $\mathrm{id}\otimes F$ we find that each f_i is necessarily constant as a function on $\mathbb{Z}_p$ satisfying $F_P^n(f_0(0))=f(0)$. Put $M_*:=\{f(0)\mid f\in M\}\subseteq\pi_P^{N_{L_P,0}}$. By the previous discussion F_P acts surjectively on M_* and is generated by the values of elements in $M^{\mathbb{Z}_p}$ (ie. constant functions) as a module over $A[F_P]$. By the admissibility of M we deduce that $M^{\mathbb{Z}_p}$ hence M_* is finite (or, equivalently, finitely generated over o/ϖ^h). We deduce that in fact we have $M=\mathcal{C}(\mathbb{Z}_p,M_*)$, ie. $M^\vee\cong o/\varpi^h[[X]]\otimes_{o/\varpi^h}M_*^\vee$. Conversely, whenever we have a $o/\varpi^h[F_P]$-submodule $M'\leq\pi_P^{N_{L_P,0}}$ that is finitely generated over o/ϖ^h and on which F_P acts surjectively (hence bijectively as the cardinality of o/ϖ^h is finite) then for $M:=\mathcal{C}(\mathbb{Z}_p,M')$ we have $M'=M_*$, $M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}}))$, and $M^\vee\cong o/\varpi^h[[X]]\otimes_{o/\varpi^h}(M')^\vee$ is X-torsion free. In particular, we compute

$$
\begin{aligned}
D_\xi^\vee(\mathcal{C}_1(\pi_P)) \cong & \varprojlim_{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}}))} M^\vee[1/X] \cong \\
\cong & \varprojlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} o/\varpi^h((X))\otimes_{o/\varpi^h} M_*^\vee \cong \\
o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}(& \varinjlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} M_*)^\vee = \\
= & o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee
\end{aligned}
$$

as claimed. □ □

Corollary 3.3. *Assume $L_P \cong \mathrm{GL}_2(\mathbb{Q}_p)\times T'$ where T' is a torus and let $\pi_P\cong\pi_2\otimes_k\chi$ be the twist of a supercuspidal modulo p representation π_2 of $\mathrm{GL}_2(\mathbb{Q}_p)$ by a character χ of the torus. Then we have*

$$
\dim_{k((X))} D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G \pi_P\right) = \begin{cases} 0 & \text{if } N_\alpha \not\subseteq L_P \\ 2 & \text{if } N_\alpha\subseteq L_P \end{cases} .
$$

Proof. Let the superscript $^{(2)}$ denote the analogous construction of the subgroups B, T, N, T_0 and element s of G in case $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Note that the torus $T^{(2)}$ is generated by $s^{(2)}$ and $T_0^{(2)}$. So in this case we have an isomorphism $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)\cong(\mathrm{Ord}_{B^{(2)}}(\pi_2)\otimes\chi)_{|k[F_P]}=0$ by the adjunction formula of Emerton's ordinary parts (Thm. 4.4.6 in [6]). In the other case we apply Thm. 0.10 in [4]. □ □

3.2 The action of T_+

Our goal in this section is to define a φ-action of T_+ on $D_{\xi,\ell,\infty}^\vee(\pi)$ or, equivalently, on $D_\xi^\vee(\pi)$ extending the action of $\xi(\mathbb{Z}_p\setminus\{0\})\leq T_+$ and making $D_{\xi,\ell,\infty}^\vee(\pi)$ an étale T_+-module over $\Lambda_\ell(N_0)$. Let $t\in T_+$ be arbitrary. Note that by the choice of this ℓ we have $tH_0t^{-1}\subseteq H_0$. In particular, T_+ acts via conjugation on the ring $\Lambda(N_0/H_0)\cong o[[X]]$; we denote the action of $t\in T_+$ by φ_t. This action is via the character α mapping T_+ onto $\mathbb{Z}_p\setminus\{0\}$. In particular, $o[[X]]$ is a free module of finite rank over itself via φ_t. Moreover, we define the Hecke action of $t\in T_+$ on π^{H_0} by the formula $F_t(m):=\mathrm{Tr}_{H_0/tH_0t^{-1}}(tm)$ for any $m\in\pi^{H_0}$. For $t,t'\in T_+$ we have

$$
\begin{aligned}
F_{t'}\circ F_t = \mathrm{Tr}_{H_0/t'H_0t'^{-1}}\circ(t'\cdot)\circ\mathrm{Tr}_{H_0/tH_0t^{-1}}\circ(t\cdot) = \\
= \mathrm{Tr}_{H_0/t'H_0t'^{-1}}\circ\mathrm{Tr}_{t'H_0t'^{-1}/t'tH_0t^{-1}t'^{-1}}\circ(t't\cdot) = F_{t't}\ .
\end{aligned}
$$

For any $M\in\mathcal{M}(\pi^{H_0})$ we put $F_t^*M := N_0F_t(M)$.

Lemma 3.4. *For any $M\in\mathcal{M}(\pi^{H_0})$ we have $F_t^*M\in\mathcal{M}(\pi^{H_0})$.*

Proof. We have

$$
\begin{aligned}
F(F_t^*M) = F(N_0F_t(M)) \subset N_0FF_t(M) = \\
= N_0F_{st}(M) = N_0F_t(F(M)) \subseteq F_t^*M\ .
\end{aligned}
$$

So F_t^*M is a module over $\Lambda(N_0/H_0)/\varpi^h[F]$. Moreover, if $m_1, \dots m_r$ generates M, then the elements $F_t(m_i)$ $(1 \leq i \leq r)$ generate F_t^*M, so it is finitely generated. The admissibility is clear as $F_t^*M = \sum_{u \in J(N_0/tN_0t^{-1})} uF_t(M)$ is the sum of finitely many admissible submodules. Finally, F_t^*M is stable under the action of Γ as F_t commutes with the action of Γ. □ □

By the definition of F_t^*M we have a surjective $o/\varpi^h[[X]]$-homomorphism

$$1 \otimes F_t \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M \twoheadrightarrow F_t^*M$$

which gives rise to an injective $o/\varpi^h((X))$-homomorphism

$$(1 \otimes F_t)^\vee[1/X] \colon (F_t^*M)^\vee[1/X] \hookrightarrow o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X] \ . \tag{13}$$

Moreover, there is a structure of an $o/\varpi^h[[X]][F]$-module on

$$o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M$$

by putting $F(\lambda \otimes m) := \varphi_t(\lambda) \otimes F(m)$. Similarly, the group Γ also acts on $o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi_t} M$ semilinearly. The map $1 \otimes F_t$ is F and Γ-equivariant as F_t, F, and the action of Γ all commute. We deduce that $(1 \otimes F_t)^\vee[1/X]$ is a φ- and Γ-equivariant map of étlae (φ, Γ)-modules.

Note that for any $t \in T_+$ there exists a positive integer $k \geq 0$ such that $t \leq s^k$, ie. $t' := t^{-1}s^k$ lies in T_+. So we have $F_t^*(F_{t'}^*M) = F_{s^k}^*M = N_0F^k(M) \subseteq M$. So we obtain an isomorphism $M^\vee[1/X] \cong (F_{s^k}^*M)^\vee[1/X] = (F_t^*(F_{t'}^*M))^\vee[1/X]$ as $M/N_0F^k(M)$ is finitely generated over o.

Lemma 3.5. *The map* (13) *is an isomorphism of étale* (φ, Γ)*-modules for any* $M \in \mathcal{M}(\pi^{H_0})$ *and* $t \in T_+$.

Proof. The composite $(1 \otimes F_{t'})^\vee[1/X] \circ (1 \otimes F_t)^\vee[1/X] = (1 \otimes F^k)^\vee[1/X]$ is an isomorphism by Lemma 2.6 in [2]. So $(1 \otimes F_t)^\vee[1/X]$ is also an isomorphism as both $(1 \otimes F_t)^\vee[1/X]$ and $(1 \otimes F_{t'})^\vee[1/X]$ are injective. □ □

Now taking projective limits we obtain an isomorphism of pseudocompact étale (φ, Γ)-modules

$$\begin{aligned} (1 \otimes F_t)^\vee[1/X] \colon D_\xi^\vee(\pi) &\to \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X]) \\ (m)_{(F_t^*M)^\vee[1/X]} &\mapsto ((1 \otimes F_t)^\vee[1/X](m))_{M^\vee[1/X]} \ . \end{aligned}$$

Moreover, since $o((X))$ is finite free over itself via φ_t, we have an identification

$$\begin{aligned} \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X]) &\cong \\ &\cong o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} D_\xi^\vee(\pi) \ . \end{aligned}$$

Using the maps $(1 \otimes F_t)^\vee[1/X]$ we define a φ-action of T_+ on $D_\xi^\vee(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d)$ for $d \in D_\xi^\vee(\pi)$.

Proposition 3.6. *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D^\vee_\xi(\pi)$ into an étale T_+-module over $o/\varpi^h[[X]]$.*

Proof. By the definition of the T_+-action it is indeed an extension of the action of the monoid $\mathbb{Z}_p \setminus \{0\}$. For $t, t' \in T_+$ we compute

$$\begin{aligned}\varphi_{t'} \circ \varphi_t(d) &= ((1 \otimes F_{t'})^\vee[1/X])^{-1} \circ ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \\ &= ((1 \otimes F_t)^\vee[1/X] \circ (1 \otimes F_{t'})^\vee[1/X])^{-1}(1 \otimes d) = \\ &= ((1 \otimes F_{tt'})^\vee[1/X])^{-1}(1 \otimes d) = \varphi_{tt'}(d) = \varphi_{t't}(d) \ .\end{aligned}$$

Further, we have

$$\begin{aligned}\varphi_t(\lambda d) &= ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes \lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(\varphi_t(\lambda) \otimes d) = \\ &= \varphi_t(\lambda)((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \varphi_t(\lambda)\varphi_t(d)\end{aligned}$$

showing that this is indeed a φ-action of T_+. The étale property follows from the fact that $(1 \otimes F_t)^\vee[1/X]$ is an isomorphism for each $t \in T_+$. □ □

The inclusion $u_\alpha \colon \mathbb{Z}_p \to N_{\alpha,0} \leq N_0$ induces an injective ring homomor-phism—still denoted by u_α by a certain abuse of notation—$u_\alpha \colon \widehat{o((X))}^p \hookrightarrow \Lambda_\ell(N_0)$ where $\widehat{o((X))}^p$ denotes the p-adic completion of the Laurent-series ring $o((X))$. For each $t \in T_+$ this gives rise to a commutative diagram

$$\begin{array}{ccc} \widehat{o((X))}^p & \xrightarrow{u_\alpha} & \Lambda_\ell(N_0) \\ \varphi_t \downarrow & & \downarrow \varphi_t \\ \widehat{o((X))}^p & \xrightarrow{u_\alpha} & \Lambda_\ell(N_0) \end{array}$$

with injective ring homomorphisms. On the other hand, by the equivalence of categories in Thm. 8.20 in [10] we have a φ- and Γ-equivariant identification $M^\vee_\infty[1/X] \cong \Lambda_\ell(N_0) \otimes_{\widehat{o((X))}^p, u_\alpha} M^\vee[1/X]$. Therefore tensoring the isomorphism (13) with $\Lambda_\ell(N_0)$ via u_α we obtain an isomorphism

$$\begin{aligned}(1 \otimes F_t)^\vee_\infty[1/X] \colon (F_t^* M)^\vee_\infty[1/X] &\cong \Lambda_\ell(N_0) \otimes_{u_\alpha} (F_t^* M)^\vee[1/X] \to \\ &\to \Lambda_\ell(N_0) \otimes_{u_\alpha} o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X] \cong \\ \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} \Lambda_\ell(N_0) \otimes_{u_\alpha} M^\vee[1/X] &\cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} M^\vee_\infty[1/X] \ . \end{aligned} \tag{14}$$

Taking projective limits again we deduce an isomorphism

$$\begin{aligned}(1 \otimes F_t)^\vee_\infty[1/X] \colon D^\vee_{\xi,\ell,\infty}(\pi) &\to \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} D^\vee_{\xi,\ell,\infty}(\pi) \\ (m)_{(F_t^* M)^\vee_\infty[1/X]} &\mapsto ((1 \otimes F_t)^\vee_\infty[1/X](m))_{M^\vee_\infty[1/X]}\end{aligned}$$

for all $t \in T_+$ using the identification

$$\varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (\Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} M^\vee_\infty[1/X]) \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} D^\vee_{\xi,\ell,\infty}(\pi) \ .$$

Using the maps $(1 \otimes F_t)^\vee_\infty[1/X]$ we define a φ-action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)^\vee_\infty[1/X])^{-1}(1 \otimes d)$ for $d \in D^\vee_{\xi,\ell,\infty}(\pi)$.

Corollary 3.7. *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D^\vee_{\xi,\ell,\infty}(\pi)$ into an étale T_+-module over $\Lambda_\ell(N_0)$. The reduction map $D^\vee_{\xi,\ell,\infty}(\pi) \to D^\vee_\xi(\pi)$ is T_+-equivariant for the φ-action.*

We can view this φ-action of T_+ in a different way: Let us define $F_{t,k} := \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot)$. Then we have a map

$$1 \otimes F_{t,k} \colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k \to F^*_{t,k}M_k := N_0 F_{t,k}(M_k) \ , \tag{15}$$

where we have $F^*_{t,k}M \in \mathcal{M}_k(\pi^{H_k})$. Let k be large enough such that we have $tH_0t^{-1} \geq H_k$. After taking Pontryagin duals, inverting X, taking projective limit and using the remark after Lemma 2.5 we obtain a homomorphism of étale (φ, Γ)-modules

$$\varprojlim_k \mathrm{Tr}^{-1}_{t^{-1}H_kt} \circ (1 \otimes F_{t,k})^\vee[1/X] \colon (F^*_t M)^\vee_\infty[1/X] \to \Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X] \ . \tag{16}$$

This map is indeed Γ- and φ-equivariant because we compute

$$\begin{aligned} F_k \circ F_{t,k} = \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) \circ \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) = \\ = \mathrm{Tr}_{H_k/s^k tH_kt^{-1}s^{-k}} \circ (s^k t\cdot) = \\ = \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) \circ \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) = F_{t,k} \circ F_k \ . \end{aligned}$$

Now we have two maps (14) and (16) between $(F^*_t M)^\vee_\infty[1/X]$ and $\Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X]$ that agree after taking H_0-coinvariants by definition. Hence they are equal by the equivalence of categories in Thm. 8.20 in [10].

We obtain in particular that the map (15) has finite kernel and cokernel as it becomes an isomorphism after taking Pontryagin duals and inverting X. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{t,k,*}$ of M_k such that the kernel of $1 \otimes F_{t,k}$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{t,k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by $M^*_{t,k} \leq F^*_{t,k}M_k$ the image of $1 \otimes F_{t,k}$. We conclude that as in Proposition 2.6, we can describe the φ_t-action in the following way:

$$\begin{aligned} \varphi_t \colon M^\vee_k[1/X] &\to (F^*_{t,k}M_k)^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k} \circ (1 \otimes F_{t,k})^\vee[1/X])^{-1}(1 \otimes f) \end{aligned} \tag{17}$$

Being an étale T_+-module over $\Lambda_\ell(N_0)$ we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with the ψ-action of T_+: ψ_t is the canonical left inverse of φ_t for all $t \in T_+$.

Proposition 3.8. *The map* $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ *is ψ-equivariant for the ψ-actions of T_+ on both sides.*

Proof. We proceed as in the proofs of Proposition 2.8 and Lemma 2.12. We fix $t \in T_+$, $W \in \mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$ and show that $\mathrm{pr}_{W,M}$ is ψ_t-equivariant. Fix k such that $F^*_{t,k}M_k \leq W$ and $tH_0t^{-1} \geq H_k$.

At first we compute the formula analogous to (7). Let f be in $M^\vee_k$ such that its restriction to $M_{t,k,*}$ is zero and $m \in M^*_{t,k} \leq F^*_{t,k}M_k$ be in the form

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u)$$

with elements $m_u \in M_k$ for $u \in J(N_0/tN_0t^{-1})$. $M_{t,k}^*$ is a finite index submodule of $F_{t,k}^*M_k$. Note that the elements m_u are unique upto $M_{t,k,*} + \mathrm{Ker}(F_{t,k})$. Therefore $\varphi_t(f) \in (M_{t,k}^*)^\vee$ is well-defined by our assumption that $f_{|M_{t,k,*}} = 0$ noting that the kernel of $F_{t,k}$ equals the kernel of $\mathrm{Tr}_{t^{-1}H_kt/H_k}$ since the multiplication by t is injective and we have $F_{t,k} = t \circ \mathrm{Tr}_{t^{-1}H_kt/H_k}$. So we compute

$$\begin{aligned}\varphi_t(f)(m) = ((1\otimes F_{t,k})^\vee)^{-1}(\mathrm{Tr}_{t^{-1}H_kt/H_k}(1\otimes f))(m) = \\ = ((1\otimes F_{t,k})^\vee)^{-1}(1\otimes \mathrm{Tr}_{t^{-1}H_kt/H_k}(f))(\sum_{u\in J((N_0/H_k)/t(N_0/H_k)t^{-1})} uF_{t,k}(m_u)) = \\ = \mathrm{Tr}_{t^{-1}H_kt/H_k}(f)(F_{t,k}^{-1}(u_0F_{t,k}(m_{u_0}))) = f(\mathrm{Tr}_{t^{-1}H_kt/H_k}((t^{-1}u_0t)m_{u_0}))\end{aligned} \tag{18}$$

where u_0 is the single element in $J(N_0/tN_0t^{-1})$ corresponding to the coset of 1.

Now let f be in $W^\vee$ such that the restriction $f_{|N_0tM_{t,k,*}} = 0$. By definition we have $\psi_t(f)(w) = f(tw)$ for any $w \in W$. Choose an element $m \in M_{t,k}^* \in F_{t,k}^*M_k$ written in the form

$$m = \sum_{u\in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u) = \sum_{u\in J(N_0/tN_0t^{-1})} ut\,\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)\ .$$

Then we compute

$$\begin{aligned} f_{|F_{t,k}^*M_k}(m) &= \sum_{u\in J(N_0/tN_0t^{-1})} f(ut\,\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) = \\ &= \sum_{u\in J(N_0/tN_0t^{-1})} \psi_t(u^{-1}f)(\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) = \\ &\overset{(18)}{=} \sum_{u\in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(u^{-1}f)_{|F_{t,k}^*M_k})(F_{t,k}(m_u)) = \\ &= \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(uF_{t,k}(m_u)) = \\ &= \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(m)\end{aligned}$$

as for distinct $u, v \in J(N_0/tN_0t^{-1})$ we have $u\varphi_t(f_0)(vF_{t,k}(m_v)) = 0$ for any $f_0 \in (M_{t,k}^*)^\vee$. So by inverting X and taking projective limits with respect to k we obtain

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\mathrm{pr}_{W,M}(\psi_t(u^{-1}f)))$$

as we have $(M_{t,k}^*)^\vee[1/X] \cong (F_{t,k}^*M)^\vee[1/X]$. Since the map (14) is an isomorphism we may decompose $\mathrm{pr}_{W,F_t^*M}(f)$ uniquely as

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\,\mathrm{pr}_{W,F_t^*M}(f)))$$

so we must have $\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$. For general $f \in W^\vee$ note that $N_0sM_{t,k,*}$ is killed by $\varphi_t(X^r)$ for $r \geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \psi_t(\mathrm{pr}_{W,F_t^*M}(\varphi_t(X^r)f)) = \\ = \mathrm{pr}_{W,M}(\psi_t(\varphi_t(X^r)f)) = X^r\,\mathrm{pr}_{W,M}(\psi_t(f))\ .\end{aligned}$$

Since X^r is invertible in $\Lambda_\ell(N_0)$, we obtain

$$\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$$

for any $f \in W^\vee$. The statement follows taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the inductive limit with respect to $W \in \mathcal{B}_+(\pi)$. □ □

We end this section by proving a Lemma that will be needed several times later on.

Lemma 3.9. *For any $M \in \mathcal{M}(\pi^{H_0})$ there exists an open subgroup $T' = T'(M) \leq T$ such that M is T'-stable.*

Proof. Choose $m_1, \dots, m_a \in M$ $(a \geq 1)$ generating M as a module over $o/\varpi^h[[X]][F]$. Since π is smooth, there exists an open subgroup $T' \leq T_0$ stabilizing all $m_1, \dots, m_a$. Now T' normalizes N_0 and all the elements $t \in T'$ commute with F we deduce that T' acts on M. □ □

4 Compatibility with a reverse functor

Assume $\ell = \ell_\alpha$ for some simple root $\alpha \in \Delta$ so we may apply the results of section 3.

4.1 A G-equivariant sheaf $\mathfrak{Y}$ on G/B attached to $D^\vee_{\xi,\ell,\infty}(\pi)$

Let D be an étale (φ, Γ)-module over the ring $\Lambda_\ell(N_0)/\varpi^h$. Recall that the $\Lambda(N_0)$-submodule D^{bd} of bounded elements in D is defined [10] as

$$D^{bd} = \{x \in D \mid \{\ell_D(\psi_s^k(u^{-1}x)) \mid k \geq 0, u \in N_0\} \subseteq D_{H_0} \text{ is bounded}\} \ .$$

where ℓ_D denotes the natural map $D \to D_{H_0}$. Note that D_{H_0} is an étale (φ, Γ)-module over $o/\varpi^h((X))$, so the bounded subsets of D_{H_0} are exactly those contained in a compact $o/\varpi^h[[X]]$-submodule of D_{H_0}.

Lemma 4.1. *Assume that D is a finitely generated étale (φ, Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$. Then $d \in D$ lies in D^{bd} if and only if d is contained in a compact ψ_s-invariant $\Lambda(N_0)$-submodule of D.*

Proof. If d is in D^{bd} then it is contained in

$$D^{bd}(D_0) = \{x \in D \mid \ell_D(\psi_s^k(u^{-1}x)) \subseteq D_0\}$$

for some treillis $D_0 \subset D_{H_0}$ where $D^{bd}(D_0)$ is a compact ψ_s-stable $\Lambda(N_0)$-submodule of D by Prop. 9.10 in [10]. On the other hand if $x \in D_1$ for some compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_1 \subset D$ then we have

$$\{\ell_D(\psi_s^k(u^{-1}x)) \mid k \geq 0, u \in N_0\} \subseteq \ell_D(D_1)$$

where $\ell_D(D_1)$ is bounded as D_1 is compact and ℓ_D is continuous. □ □

We call a pseudocompact $\Lambda_\ell(N_0)$-module together with a φ-action of the monoid T_+ (resp. $\mathbb{Z}_p \setminus \{0\}$) a pseudocompact étale T_+-module (resp. (φ, Γ)-module) over $\Lambda_\ell(N_0)$ if it is a topologically étale $o[B_+]$-module in the sense of section 4.1 in [10]. Recall that a pseudocompact module over the pseudocompact ring $\Lambda_\ell(N_0)$ is the projective limit of finitely generated $\Lambda_\ell(N_0)$-modules. As for $D = D^\vee_{\xi,\ell,\infty}(\pi)$ in section 2.1 we equip the pseudocompact $\Lambda_\ell(N_0)$-modules D with the weak topology, ie. with the projective limit topology of the weak topologies of these finitely generated quotients of D. Recall from section 4.1 in [10] that the condition for D to be topologically étale means in this case that the map

$$\begin{aligned} B_+ \times D &\to D \\ (b, x) &\mapsto \varphi_b(x) \end{aligned} \tag{19}$$

is continuous and $\psi = \psi_s \colon D \to D$ is continuous (Lemma 4.1 in [10]).

Lemma 4.2. *$D^\vee_{\xi,\ell,\infty}(\pi)$ is a pseudocompact étale T_+-module over $\Lambda_\ell(N_0)$.*

Proof. At first we show that the map (19) is continuous in the weak topology of $D = D^\vee_{\xi,\ell,\infty}(\pi)$. Let $b = ut \in B_+$ ($u \in N_0$, $t \in T_+$), $x, y \in D^\vee_{\xi,\ell,\infty}(\pi)$ be such that $u\varphi_t(y) = x$ and let $M \in \mathcal{M}(\pi^{H_0})$, $l, l' \geq 0$ be arbitrary. Recall from (9) that the sets

$$O(M, l, l') := f^{-1}_{M,l}(\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'} M^\vee[1/X]^{++})$$

form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. We need to verify that the preimage of $x + O(M, l, l')$ under (19) contains a neighbourhood of (b, y). By Lemma 3.9 there exists an open subgroup $T' \leq T_0 \leq T$ acting on M therefore also on $M_l^\vee[1/X]$ as T_0 normalizes H_l for all $l \geq 0$ by the assumption $\ell = \ell_\alpha$. Moreover, this action is continuous in the weak topology of $M_l^\vee[1/X]$, so there exists an open subgroup $T_1 \leq T'$ such that we have $(T_1 - 1)x \subset O(M, l, l')$. Moreover, since we have $D^\vee_{\xi,\ell,\infty}(\pi)/O(M, l, l') \cong M_l^\vee[1/X]/(\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'} M^\vee[1/X]^{++})$ is a smooth representation of N_0, we have an open subgroup $N_1 \leq N_0$ with $(N_1 - 1)x \subset O(M, l, l')$. Moreover, we may assume that T_1 normalizes N_1 so that $B_1 := N_1 T_1$ is an open subgroup in $B_0 \leq B_+$ for which we have $(B_1 - 1)x \subset O(M, l, l')$ as $O(M, l, l')$ is N_0-invariant. Choose an elemet $t' \in T_+$ such that $tt' = s^r$ for some $r \geq 0$. Note that the composite map $D^\vee_{\xi,\ell,\infty}(\pi) \overset{\varphi_t}{\to} D^\vee_{\xi,\ell,\infty} \twoheadrightarrow M^\vee[1/X]$ factors through the φ_s-equivariant map

$$((1 \otimes F_t)^\vee[1/X])^{-1} \colon (F^*_{t'} M)^\vee[1/X] \to M^\vee[1/X]$$

mapping $X^{l'}(F^*_{t'} M)^\vee[1/X]^{++}$ into $X^{l'} M^\vee[1/X]^{++}$. Since $X^{l'} M^\vee[1/X]^{++}$ is B_1-invariant (as each φ_{t_1} for $t_1 \in T_1$ commutes with φ_s), so is $O(M, l, l')$. We deduce that

$$B_1 b \times (y + O(F^*_{t'} M, l, l')) \subset B_+ \times D^\vee_{\xi,\ell,\infty}(\pi)$$

maps into $x + O(M, l, l')$ via (19).

The continuity of ψ_s follows from Proposition 8.22 in [10] since ψ_s on $D^\vee_{\xi,\ell,\infty}(\pi)$ is the projective limit of the maps $\psi_s \colon M^\vee_\infty[1/X] \to M^\vee_\infty[1/X]$ for $M \in \mathcal{M}(\pi^{H_0})$. $\square$ $\square$

In view of the above Lemmata we define D^{bd} for a pseudocompact étale (φ, Γ)-module D over $\Lambda_\ell(N_0)$ as

$$D^{bd} = \bigcup_{D_c \in \mathfrak{C}_0(D)} D_c$$

where we denote the set of ψ_s-invariant compact $\Lambda(N_0)$-submodules $D_c \subset D$ by $\mathfrak{C}_0 = \mathfrak{C}_0(D)$.

The following is a generalization of Prop. 9.5 in [10].

Proposition 4.3. *Let D be a pseudocompact étale (φ,Γ)-module over $\Lambda_\ell(N_0)$. Then D^{bd} is an étale (φ,Γ)-module over $\Lambda(N_0)$. If we assume in addition that D is an étale T_+-module over $\Lambda_\ell(N_0)$ (for a φ-action of the monoid T_+ extending that of $\xi(\mathbb{Z}_p\setminus\{0\})$) then D^{bd} is an étale T_+-module over $\Lambda(N_0)$ (with respect to the action of T_+ restricted from D).*

Proof. We prove the second statement assuming that D is an étale T_+-module. The first statement follows easily the same way.

At first note that D^{bd} is ψ_t-invariant for all $t\in T_+$ as for $D_c\in\mathfrak{C}_0$ we also have $\psi_t(D_c)\in\mathfrak{C}_0$. So it suffices to show that it is also stable under the φ-action of T_+ since these two actions are clearly compatible (as they are compatible on D). At first we show that we have $\varphi_s(D^{bd})\subset D^{bd}$. Let $D_c\in\mathfrak{C}_0$ be arbitrary. Then the ψ-action of the monoid $p^{\mathbb{Z}}$ (ie. the action of ψ_s) is nondegenerate on D_c as D_c is a ψ_s-invariant submodule of an étale module D. So by the remark after Proposition 2.21 and by Corollary 2.24 we obtain an injective ψ_s and φ_s-equivariant homomorphism $i\colon \widetilde{D}_c\hookrightarrow D$. However, each $\varphi^*_{s^k}D_c\subseteq\widetilde{D}_c$ is compact and ψ-equivariant therefore the image of $\widetilde{D}_c$ is contained in D^{bd} showing that $\varphi_s(D_c)\subset N_0\varphi_s(D_c)=i(\varphi_s^*D_c)\subseteq D^{bd}$. However, for each $t\in T_+$ there exists a $t'\in T_+$ with $tt'=s^k$ for some $k\geq 0$, so $\varphi_t(D_c)=\psi_{t'}(\varphi_{s^k}(D_c))\subseteq D^{bd}$ showing that D^{bd} is φ_t-invariant for all $t\in T_+$. □ □

Corollary 4.4. *The image of the map $\widetilde{\mathrm{pr}}\colon\widetilde{D_{SV}}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$.*

Proof. By Propositions 2.21 and 4.3 it suffices to show that the image of $\mathrm{pr}\colon D_{SV}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ lies in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$. However, this is clear since $\mathrm{pr}(D_{SV}(\pi))$ is a ψ_s-invariant compact $\Lambda(N_0)$-submodule of $D^\vee_{\xi,\ell,\infty}(\pi)$. □ □

Let $\mathfrak{C}$ be the set of all compact subsets C of $D^\vee_{\xi,\ell,\infty}(\pi)$ contained in one of the compact subsets $D_c\in\mathfrak{C}_0=\mathfrak{C}_0(D^\vee_{\xi,\ell,\infty}(\pi))$. Recall from Definition 6.1 in [10] that the family $\mathfrak{C}$ is said to be special if it satisfies the following axioms:

$\mathfrak{C}(1)$ Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.

$\mathfrak{C}(2)$ If $C_1,C_2,\dots,C_n\in\mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.

$\mathfrak{C}(3)$ For all $C\in\mathfrak{C}$ we have $N_0C\in\mathfrak{C}$.

$\mathfrak{C}(4)$ $D(\mathfrak{C}):=\bigcup_{C\in\mathfrak{C}}C$ is an étale T_+-submodule of D.

Lemma 4.5. *The set $\mathfrak{C}$ is a special family of compact sets in $D^\vee_{\xi,\ell,\infty}(\pi)$ in the sense of Definition 6.1 in [10].*

Proof. $\mathfrak{C}(1)$ is satisfied by construction. So is $\mathfrak{C}(3)$ by noting that any $C\in\mathfrak{C}$ is contained in a $D_c\in\mathfrak{C}_0$ which is N_0-stable. For $\mathfrak{C}(2)$ note that for any $D_{c,1},\dots,D_{c,r}\in\mathfrak{C}_0$ we have $\sum_{i=1}^r D_{c,i}\in\mathfrak{C}_0$. Finally, $\mathfrak{C}(4)$ is just Proposition 4.3. □ □

Our next goal is to construct a G-equivariant sheaf $\mathfrak{Y}=\mathfrak{Y}_{\alpha,\pi}$ on G/B in [10] with sections $\mathfrak{Y}(\mathcal{C}_0)$ on $\mathcal{C}_0:=N_0w_0B/B$ isomorphic to $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ as a B_+-module. Here $w_0\in N_G(T)$ is a representative of an element in the Weyl group $N_G(T)/C_G(T)$ of *maximal length*. For this we identify $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ with the global sections of a B_+-equivariant sheaf on N_0 as in [10]. The restriction maps $\mathrm{res}^{N_0}_{us^kN_0s^{-k}}$ are defined as $u\circ\varphi_s^k\circ\psi_s^k\circ u^{-1}$. The open sets $us^kN_0s^{-k}$ form a basis of the topology on N_0, so it suffices to give these restriction maps. Indeed, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^k N_0 s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\mathrm{res}_{\mathcal{U}} = \mathrm{res}_{\mathcal{U}}^{N_0} := \sum_{u \in J(N_0/s^k N_0 s^{-k}) \cap \mathcal{U}} u\varphi_{s^k} \circ \psi_s^k \circ (u^{-1} \cdot) \ .$$

This is independent of the choice of $k \geq k'(\mathcal{U})$ by Prop. 3.16 in [10]. Note that the map

$$u \mapsto x_u := uw_0 B/B \in \mathcal{C}_0$$

is a B_+-equivariant homeomorphism from N_0 to $\mathcal{C}_0$ therefore we may view $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ as the global sections of a sheaf on $\mathcal{C}_0$. For an open subset $U \subseteq N_0$ we denote the image of U by $x_U \subseteq \mathcal{C}_0$ under the above map $u \mapsto x_u$. Moreover, we regard res as an $\mathrm{End}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi))$-valued measure on $\mathcal{C}_0$, ie. a ring homomorphism res : $C^\infty(\mathcal{C}_0, o) \to \mathrm{End}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi))$. We restrict res to a map res: $C^\infty(\mathcal{C}_0, o) \to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$. Put $\mathcal{C} := Nw_0B/B \supset \mathcal{C}_0$. By the discussion in section 5 of [10] in order to construct a G-equivariant sheaf on G/B with the required properties we need to integrate the map

$$\begin{aligned} \alpha_g \colon \mathcal{C}_0 &\to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi)) \\ x_u &\mapsto \alpha(g,u) \circ \mathrm{res}(1_{\alpha(g,u)^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) \end{aligned}$$

with respect to the measure res where for $x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 \subset g^{-1}\mathcal{C} \cap \mathcal{C}$ we take $\alpha(g,u)$ to be the unique element in B with the property

$$guw_0N = \alpha(g,u)uw_0N \ .$$

Note that since x_u lies in $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ we also have $x_u \in \alpha(g,u)^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ so the latter set is nonempty and open in G/B. Recall from section 6.1 in [10] that a map $F \colon \mathcal{C}_0 \to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ is called integrable with respect to (s, res, $\mathfrak{C}$) if the limit

$$\int_{\mathcal{C}_0} F d\,\mathrm{res} := \lim_{k\to\infty} \sum_{u \in J(N_0/s^k N s^{-k})} F(x_u) \circ \mathrm{res}(1_{x_{us^kNs^{-k}}})$$

exists in $\mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ and does not depend on the choice of the sets of representatives $J(N_0/s^k N s^{-k})$.

Proposition 4.6. *The map α_g is $(s, \mathrm{res}, \mathfrak{C})$-integrable for any $g \in G$.*

Proof. By Proposition 6.8 in [10] it suffices to show that $\mathfrak{C}$ satisfies:

$\mathfrak{C}(5)$ For any $C \in \mathfrak{C}$ the compact subset $\psi_s(C) \subseteq D^\vee_{\xi,\ell,\infty}(\pi)$ also lies in $\mathfrak{C}$.

$\mathfrak{T}(1)$ For any $C \in \mathfrak{C}$ such that $C = N_0C$, any open $o[N_0]$-submodule $\mathcal{D}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$, and any compact subset $C_+ \subseteq T_+$ there exists a compact open subgroup $B_1 = B_1(C, \mathcal{D}, C_+) \subseteq B_0$ and an integer $k(C, \mathcal{D}, C_+) \geq 0$ such that

$$\varphi_s^k \circ (1 - B_1)C_+\psi_s^k(C) \subseteq \mathcal{D} \qquad \text{for any } k \geq k(C, \mathcal{D}, C_+) \ .$$

Here the multiplication by C_+ is via the φ-action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$.

The condition $\mathfrak{C}(5)$ is clearly satisfied as for any $D_c \in \mathfrak{C}_0$ we have $\psi_s(D_c) \in \mathfrak{C}_0$, as well. For the condition $\mathfrak{T}(1)$ choose a $C \in \mathfrak{C}$ with $C = N_0 C$, a compact subset $C_+ \subset T_+$, and an open $o[N_0]$-submodule $\mathcal{D} \subseteq D^\vee_{\xi,\ell,\infty}(\pi)$. As $D^\vee_{\xi,\ell,\infty}(\pi)$ is the topological projective limit $\varprojlim_{M\in\mathcal{M}(\pi^{H_0}),n\geq 0} M_n^\vee[1/X]$ we may assume without loss of generality that $\mathcal{D}$ is the preimage of a compact $\Lambda(N_0)$-submodule $D_n \leq M_n^\vee[1/X]$ with $D_n[1/X] = M_n^\vee[1/X]$ under the natural surjective map $f_{M,n}\colon D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M_n^\vee[1/X]$ for some $M \in \mathcal{M}(\pi^{H_0})$ and $n \geq 0$. Moreover, since $B_0 = T_0 N_0$ is compact and normailizes H_0, the T_0-orbit of any element $m \in M \leq \pi^{H_0}$ is finite and contained in π^{H_0}. Therefore we also have $B_0 M = T_0 M \in \mathcal{M}(\pi^{H_0})$. So we may assume without loss of generality that M is B_0-invariant whence we have an action of B_0 on $M_n^\vee[1/X]$. Choose a $D_c \in \mathfrak{C}_0$ with $C \subseteq D_c$. Since D_c is ψ_s-invariant, we have $C_+\psi_s^k(C) \subseteq C_+\psi_s^k(D_c) \subseteq C_+ D_c$. Moreover, $C_+ D_c$ is compact as both C_+ and D_c are compact, so $f_{M,n}(C_+\psi_s^k(C)) \subset M_n^\vee[1/X]$ is bounded. In particular, we have a compact $\Lambda(N_0)$-submodule D' of $M_n^\vee[1/X]$ containing $f_{M,n}(C_+\psi_s^k(C))$. So by the continuity of the action of B_0 on $M_n^\vee[1/X]$ there exists an open subgroup $B_1 \leq B_0$ such that we have

$$(1 - B_1) f_{M,n}(C_+\psi_s^k(C)) \subset \Lambda(N_0/H_n) \otimes_{\Lambda(N_{\alpha,0})} (M^\vee[1/X]^{++}) \leq \\ \leq \Lambda(N_0/H_n) \otimes_{\Lambda(N_{\alpha,0})} M^\vee[1/X] \cong M_n^\vee[1/X]$$

for any $k \geq 0$. Here $M^\vee[1/X]^{++}$ denotes the treillis in $M^\vee[1/X]$ consisting of those elements $d \in M^\vee[1/X]$ such that $\varphi_s^n(d) \to 0$ in $M^\vee[1/X]$ as $n \to \infty$ (cf. section I.3.2 in [4]). Finally, since D_n is open and $M^\vee[1/X]^{++}$ is finitely generated over $\Lambda(N_{\alpha,0}) \cong o[[X]]$ there exists an integer $k_1 \geq 0$ such that $\varphi_s^k(\Lambda(N_0/H_n) \otimes_{\Lambda(N_{\alpha,0})} (M^\vee[1/X]^{++}))$ is contained in D_n for all $k \geq k_1$. In particular, we have

$$f_{M,n}(\varphi_s^k \circ (1 - B_1) C_+\psi_s^k(C)) = \varphi_s^k \circ (1 - B_1)(f_{M,n}(C_+\psi_s^k(C))) \subseteq \\ \subseteq \varphi_s^k \circ (1 - B_1)(M^\vee[1/X]^{++}) \subseteq D_n$$

showing that $\varphi_s^k \circ (1 - B_1) C_+\psi_s^k(C)$ is contained in $\mathcal{D}$. □ □

For all $g \in G$ we denote by $\mathcal{H}_g \in \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ the integral

$$\mathcal{H}_g := \int_{\mathcal{C}_0} \alpha_g d\,\mathrm{res} = \lim_{k\to\infty} \sum_{u \in J(N_0/s^k N s^{-k})} \alpha_g(x_u) u \circ \varphi_s^k \circ \psi_s^k \circ u^{-1}$$

we have just proven to converge. We denote the kth term of the above sequence by

$$\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^k N_0 s^{-k})} := \sum_{u \in J(N_0/s^k N s^{-k})} \alpha_g(x_u) u \circ \varphi_s^k \circ \psi_s^k \circ u^{-1} \ . \qquad (20)$$

Our main result in this section is the following

Proposition 4.7. *The image of the map $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$. There exists a G-equivariant sheaf $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$ on G/B with sections $\mathfrak{Y}(\mathcal{C}_0)$ on $\mathcal{C}_0$ isomorphic B_+-equivariantly to $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ such that we have $\mathcal{H}_g = \mathrm{res}^{G/B}_{\mathcal{C}_0} \circ (g\cdot) \circ \mathrm{res}^{G/B}_{\mathcal{C}_0}$ as maps on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathfrak{Y}(\mathcal{C}_0)$.*

Proof. By Prop. 5.14 and 6.9 in [10] it suffices to check the following conditions:

$\mathfrak{C}(6)$ For any $C \in \mathfrak{C}$ the compact subset $\varphi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.

$\mathfrak{T}(2)$ Given a set $J(N_0/s^k N_0 s^{-k}) \subset N_0$ of representatives for all $k \geq 1$, for any $x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ and $g \in G$ there exists a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_{x,g} \in \mathfrak{C}$ and a positive integer $k_{x,g}$ such that $\mathcal{H}_g^{(k)}(x) \subseteq D_{x,g}$ for any $k \geq k_{x,g}$.

The condition $\mathfrak{C}(6)$ follows from (the proof of) Prop. 4.3 as for $C \subseteq D_c \in \mathfrak{C}_0$ we have $\varphi_s(C) \subseteq \varphi_s(D_c) \subseteq i(\varphi_s^* D_c) \in \mathfrak{C}_0$.

The proof of $\mathfrak{T}(2)$ is very similar to the proof of Corollary 9.15 in [10]. However, it is not a direct consequence of that as $D^\vee_{\xi,\ell,\infty}(\pi)$ is not necessarily finitely generated over $\Lambda_\ell(N_0)$, so we recall the details. For any x in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$, the element $\mathcal{H}_g^{(k)}(x)$ also lies in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ for any fixed k since the set of bounded elements form an étale T_+-submodule (by axiom $\mathfrak{C}(4)$) whence they are closed under the operations (φ-, ψ-, and N_0-actions) defining the map $\mathcal{H}_g^{(k)}$. So by axiom $\mathfrak{C}(2)$ we only need to show that for k large enough the difference

$$s_g^{(k)}(x) := \mathcal{H}_g^{(k)}(x) - \mathcal{H}_g^{(k+1)}(x)$$

lies in a compact submodule $D_{x,g} \leq D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ in $\mathfrak{C}_0$ independent of k. In order to do so we proceed in four steps. In steps 1, 2, and 3 the goal is to show that for a fixed choice $M \in \mathcal{M}(\pi^{H_0})$ the image of $s_g^{(k)}(x)$ lies in a compact ψ-invariant $\Lambda(N_0)$-submodule of $M^\vee_\infty[1/X]$ under the projection map $D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M^\vee_\infty[1/X]$ for k large enough *not depending* on M. This compact submodule of $M^\vee_\infty[1/X]$ will be of the form

$$\{m \in M^\vee_\infty[1/X] \mid \ell_M(\psi_s^r(u^{-1}m)) \text{ is in } D_0 \text{ for all } r \geq 0, u \in N_0\}$$

for some treillis $D_0 \subset M^\vee[1/X]$ where $\ell_M \colon M^\vee_\infty[1/X] \to M^\vee[1/X]$ is the natural projection map. Step 1 is devoted to showing this for smaller r (compared to k) with some choice of a treillis and in Step 2 we take care of all larger r (using a different treillis in $M^\vee[1/X]$). In both of these steps $k \geq k(M)$ is large enough *depending* on M. In Step 3 we eliminate this dependence on M of the lower bound for k by choosing a third treillis so that the sum D_0 of these three different choices of a treillis will do. In Step 4 we take the projective limit of these compact sets for all possible choices of M to obtain a compact subset of $D^\vee_{\xi,\ell,\infty}(\pi)$.

Step 1. Equation (43) in [10] shows that for any compact open subgroup $B_1 \leq B_0$ there exist integers $0 \leq k_g^{(1)} \leq k_g^{(2)}(B_1)$ and a compact subset $\Lambda_g \subset T_+$ such that for $k \geq k_g^{(2)}(B_1)$ we have

$$s_g^{(k)} \in \langle N_0 s^{k-k_g^{(1)}}(1-B_1)\Lambda_g s \psi_s^{k+1} N_0\rangle_o \ , \tag{21}$$

where we denote by $\langle\cdot\rangle_o$ the generated o-submodule. Here $k_g^{(1)}$ is chosen so that $\{\alpha(g,u)us^{k_g^{(1)}} \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$ is contained in $B_+ = N_0 T_+$. There exists such an integer $k_g^{(1)}$ since $\{\alpha(g,u)u \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$ is a compact subset in $N_0 T$. Choose a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_c \in \mathfrak{C}_0$ containing the element $x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ and pick an M in $\mathcal{M}(\pi^{H_0})$. Applying $\mathfrak{T}(1)$ in the situation $C = D_c$, $C_+ = \Lambda_g s$, and $\mathcal{D} = f_{M,0}^{-1}(M^\vee[1/X]^{++})$ we find an integer $k_1 \geq 0$ and a compact open subgroup $B_1 \leq B_0$ such that $\varphi_s^k \circ (1-B_1)\Lambda_g s D_c \subseteq \mathcal{D}$ for all $k \geq k_1$. Noting that D_c is ψ_s-stable and $\mathcal{D}$ is a $\Lambda(N_0)$-submodule we obtain $s_g^{(k)}(D_c) \subseteq N_0\varphi_s^r(\mathcal{D})$ for $k \geq r + k_1 + k_g^{(2)}(B_1)$. Applying ψ_s^r to this using (21) and putting $k_g(M) := k_1 + k_g^{(2)}(B_1)$ we deduce

$$\psi_s^r(\Lambda(N_0)s_g^{(k)}(D_c)) \subseteq \mathcal{D} \qquad \text{for all } k \geq k_g(M) \text{ and } r \leq k - k_g(M) \ . \tag{22}$$

Note that the subgroup B_1 depends on M therefore so do $k_g^{(2)}(B_1)$ and $k_g(M)$, but not $k_g^{(1)}$.

Step 2. We are going to find another treillis $D_1 \leq M^\vee[1/X]$ such that for all $k \geq k_g(M)$ and $r \geq k - k_g(M)$ we have

$$\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D}_1 := f_{M,0}^{-1}(D_1)\ . \tag{23}$$

For $x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ write $\alpha(g,u)u$ in the form $\alpha(g,u)u = n(g,u)t(g,u)$ with $n(g,u) \in N_0$ and $t(g,u) \in T$. Since $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ is compact, $t(g,\cdot)$ is continuous, and $k_g(M) \geq k_g^{(1)}$ the set $C'_+ := \{t(g,u)s^{k_g(M)} \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\} \subset T$ is compact and contained in T_+. So we compute

$$\begin{aligned} &\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) = \\ &= \psi_s^r(\Lambda(N_0)\sum_{u\in J(N_0/s^kN_0s^{-k})} n(g,u)\varphi_{t(g,u)s^k}\circ\psi_s^k(u^{-1}D_c)) \subseteq \\ &\subseteq \psi_s^r(\Lambda(N_0)\varphi_s^{k-k_g(M)}\circ\varphi_{t(g,u)s^{k_g(M)}}(D_c)) \subseteq \psi_s^{r-k+k_g(M)}(\Lambda(N_0)C'_+(D_c))\ . \end{aligned}$$

Since $C'_+ \subset T_+$ is compact , there exists an integer $k(C'_+)$ such that s^kt^{-1} lies in T_+ for all $t \in C'_+$. So we have $C'_+(D_c) \subseteq i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd}) \in \mathfrak{C}_0$ showing that

$$D_1 := f_{M,0}(i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd}))$$

is a good choice as $i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$ is a ψ_s-stable $\Lambda(N_0)$ submodule.

Step 3. For each fixed $k \geq k_g^{(1)}$ there exists a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_{c,k} \in \mathfrak{C}_0$ containing $\mathcal{H}_g^{(k)}(D_c)$. In particular, we may choose a treillis $D_2 \leq M^\vee[1/X]$ containing

$$f_{M,0}(\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)))$$

for all $k_g^{(1)} \leq k \leq k_g(M)$ and $r \geq 0$. Putting $\mathcal{D}_2 := f_{M,0}^{-1}(D_2)$ and combining this with (22) and (23) we obtain

$$\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D} + \mathcal{D}_1 + \mathcal{D}_2 \tag{24}$$

for all $k \geq k_{x,g} := k_g^{(1)}$ and $r \geq 0$. Denote by $f_{M,\infty}$ the natural surjective map $f_{M,\infty}\colon D^\vee_{\xi,\ell,\infty} \to M^\vee_\infty[1/X]$. Note that $f_{M,0}$ factors through $f_{M,\infty}$. The equation (24) implies (in fact, is equivalent to) that

$$f_{M,\infty}\left(\bigcup_{k\geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c)\right) \subseteq M^\vee_\infty[1/X]^{bd}(D_0)$$

where

$$\begin{aligned} M^\vee_\infty[1/X]^{bd}(D_0) = \{m \in M^\vee_\infty[1/X] \mid \ell_M(\psi_s^r(u^{-1}m)) \text{ is in} \\ D_0 := M^\vee[1/X]^{++} + D_1 + D_2 \text{ for all } r \geq 0, u \in N_0\} \end{aligned}$$

is a compact ψ_s-invariant $\Lambda(N_0)$-submodule in $M^\vee_\infty[1/X]$ (Prop. 9.10 in [10]).

Step 4. We put $D_{x,g}(M) := \bigcap \mathfrak{D}$ where $\mathfrak{D}$ runs through all the ψ_s-invariant compact $\Lambda(N_0)$-submodules of $M^\vee_\infty[1/X]$ containing $f_{M,\infty}(\bigcup_{k\geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c))$. Therefore

$$D_{x,g} := \varprojlim_{M\in\mathcal{M}(\pi^{H_0})} D_{x,g}(M)$$

is a ψ_s-invariant compact $\Lambda(N_0)$-submodule of $D^\vee_{\xi,\ell,\infty}(\pi)$ (ie. we have $D_{x,g} \in \mathfrak{C}_0$) containing $\bigcup_{k\geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c)$. □ □

We end this section by putting a natural topology (called the weak topology) on the global sections $\mathfrak{Y}(G/B)$ that will be needed in the next section. At first we equip $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ with the inductive limit topology of the compact topologies of each $D_c \in \mathfrak{C}_0$. This makes sense as the inclusion maps $D_c \hookrightarrow D'_c$ for $D_c \subseteq D'_c \in \mathfrak{C}_0$ are continuous as these compact topologies are obtained as the subspace topologies in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. We call this topology the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$.

Lemma 4.8. *The operators $\mathcal{H}_g$ and $\mathrm{res}_\mathcal{U}$ on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ are continuous in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ for all $g \in G$ and $\mathcal{U} \subseteq N_0$ compact open. In particular, $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is the topological direct sum of $\mathrm{res}_\mathcal{U}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$ and $\mathrm{res}_{N_0\setminus\mathcal{U}}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$.*

Proof. By the property $\mathfrak{T}(2)$ the restriction of $\mathcal{H}_g^{(k)}$ to a compact subset D_c in $\mathfrak{C}_0$ has image in a compact set $D_{c,g} \in \mathfrak{C}_0$ for all large enough k. Moreover, each $\mathcal{H}_g^{(k)}$ is continuous by Lemma 4.2. On the other hand, the limit $\mathcal{H}_g = \lim_{k\to\infty} \mathcal{H}_g^{(k)}$ is uniform on each compact subset $D_c \in \mathfrak{C}_0$ by Proposition 6.3 in [10], so the limit $\mathcal{H}_g \colon D_c \to D_{c,g}$ is also continuous. Taking the inductive limit on both sides we deduce that $\mathcal{H}_g \colon D^\vee_{\xi,\ell,\infty}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is also continuous. The continuity of $\mathrm{res}_\mathcal{U}$ follows in a similar but easier way. □ □

So far we have put a topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathfrak{Y}(\mathcal{C}_0)$. The multiplication by an element $g \in G$ gives an o-linear bijection $g\colon \mathfrak{Y}(\mathcal{C}_0) \to \mathfrak{Y}(g\mathcal{C}_0)$. We define the weak topology on $\mathfrak{Y}(g\mathcal{C}_0)$ so that this is a homeomorphism. Now we equip $\mathfrak{Y}(G/B)$ with the coarsest topology such that the restriction maps $\mathrm{res}^{G/B}_{g\mathcal{C}_0} \colon \mathfrak{Y}(G/B) \to \mathfrak{Y}(g\mathcal{C}_0)$ are continuous for all $g \in G$. We call this the weak topology on $\mathfrak{Y}(G/B)$ making $\mathfrak{Y}(G/B)$ a linear-topological o-module.

Lemma 4.9. *a) The multiplication by g on $\mathfrak{Y}(G/B)$ is continuous (in fact a homeomorphism) for each $g \in G$.*

b) The weak topology on $\mathfrak{Y}(G/B)$ is Hausdorff.

Proof. For a) we need to check that the composite of the function

$$(g\cdot)_{G/B} \colon \mathfrak{Y}(G/B) \to \mathfrak{Y}(G/B)$$

with the projections $\mathrm{res}^{G/B}_{h\mathcal{C}_0}$ is continuous for all $h \in G$. However, $\mathrm{res}^{G/B}_{h\mathcal{C}_0} \circ (g\cdot)_{G/B} = (g\cdot)_{g^{-1}h\mathcal{C}_0} \circ \mathrm{res}^{G/B}_{g^{-1}h\mathcal{C}_0}$ is the composite of two continuous maps hence also continuous.

For b) note that the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is finer than the subspace topology inherited from $D^\vee_{\xi,\ell,\infty}(\pi)$ therefore it is Hausdorff. To see this we need to show that the inclusion $D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \hookrightarrow D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous. As the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as a direct limit, it suffices to check this on the defining compact sets $D_c \in \mathfrak{C}_0$. However, on these compact sets the inclusion map is even a homeomorphism by definition.

So the topology on $\mathfrak{Y}(G/B)$ is also Hausdorff as for any two different global sections $x \neq y \in \mathfrak{Y}(G/B)$ there exists an element $g \in G$ such that $\mathrm{res}^{G/B}_{g\mathcal{C}_0}(x) \neq \mathrm{res}^{G/B}_{g\mathcal{C}_0}(y)$. □ □

4.2 A G-equivariant map $\pi^\vee \to \mathfrak{Y}(G/B)$

Here we generalize Thm. IV.4.7 in [4] to $\mathbb{Q}_p$-split reductive groups G over $\mathbb{Q}_p$ with connected centre. Assume in this section that π is an *admissible* smooth o/ϖ^h-representation of G of *finite length.*

By Corollary 4.4 we have the composite maps

$$\beta_{g\mathcal{C}_0}\colon \pi^\vee \xrightarrow{g^{-1}\cdot} \pi^\vee \xrightarrow{\mathrm{pr}_{SV}} D_{SV}(\pi) \xrightarrow{\mathrm{pr}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \xrightarrow{\sim} \mathfrak{Y}(\mathcal{C}_0) \xrightarrow{g\cdot} \mathfrak{Y}(g\mathcal{C}_0)$$

for each $g \in G$. By definition we have $\beta_{g\mathcal{C}_0}(\mu) = g\beta_{\mathcal{C}_0}(g^{-1}\mu)$ for all $\mu \in \pi^\vee$ and $g \in G$. Our goal is to show that these maps glue together to a G-equivariant map $\beta_{G/B}\colon \pi^\vee \to \mathfrak{Y}(G/B)$.

Let $n_0 = n_0(G) \in \mathbb{N}$ be the maximum of the degrees of the algebraic characters $\beta\circ\xi\colon \mathbb{G}_m \to \mathbb{G}_m$ for all β in Φ^+ and put $U^{(k)} := \mathrm{Ker}(G_0 \twoheadrightarrow G(\mathbb{Z}_p/p^k\mathbb{Z}_p))$ where $G_0 = \mathbf{G}(\mathbb{Z}_p)$.

Lemma 4.10. *For any fixed $r_0 \geq 1$ we have $t^{-1}U^{(k)}t \leq U^{(k-n_0r_0)}$ for all $t \leq s^{r_0}$ in T_+ and $k \geq r_0n_0$.*

Proof. The condition $t \leq s^{r_0}$ implies that $v_p(\beta(t)) \leq v_p(\beta(s^{r_0})) = v_p(\beta\circ\xi(p^{r_0})) \leq r_0n_0$ for all $\beta \in \Phi^+$ by the maximality of n_0. On the other hand, by the Iwahori factorization we have $U^{(k)} = (U^{(k)} \cap \overline{N})(U^{(k)} \cap T)(U^{(k)} \cap N)$. Since t is in T_+ we deduce

$$\begin{aligned} t^{-1}(U^{(k)} \cap \overline{N})t \leq (U^{(k)} \cap \overline{N}) &\leq (U^{(k-r_0n_0)} \cap \overline{N}) \\ t^{-1}(U^{(k)} \cap T)t = (U^{(k)} \cap T) &\leq (U^{(k-r_0n_0)} \cap T) \\ t^{-1}(U^{(k)} \cap N)t = & \\ \prod_{\beta\in\Phi^+} t^{-1}(U^{(k)} \cap N_\beta)t &\leq \prod_{\beta\in\Phi^+} (U^{(k-r_0n_0)} \cap N_\beta) \\ &= (U^{(k-r_0n_0)} \cap N)\ . \quad \square \end{aligned}$$

$\square$

Lemma 4.11. *Assume that π is an admissible representation of G of finite length. Then there exists a finitely generated o-submodule $W_0 \leq \pi$ such that $\pi = BW_0$.*

Proof. Since π has finite length, by induction we may assume it is irreducible (hence killed by ϖ). In this case we may take $W_0 = \pi^{U^{(1)}}$ which is G_0-stable as $U^{(1)}$ is normal in G_0. It is nonzero since π is smooth, and finitely generated over o as π is admissible. By the Iwasawa decomposition we have $\pi = GW_0 = BG_0W_0 = BW_0$. $\square$ $\square$

Let W_0 be as in Lemma 4.11 and put $W := B_+W_0$, $W_r := \bigcup_{t\leq s^r} N_0tW_0$ so we have

$$W = \varinjlim_r W_r = \bigcup_{r\geq 0} W_r \tag{25}$$

where W_r is finitely generated over o for all $r \geq 0$. By construction W is a generating B_+-subrepresentation of π. So the map pr_{SV} factors through the natural projection map $\mathrm{pr}_W\colon \pi^\vee \to W^\vee$. Here the Pontryagin dual $W^\vee$ is a compact $\Lambda(N_0)$-module with a ψ-action of T_+ coming from the multiplication by T_+ on W. By Proposition 2.21 we may form the étale hull $\widetilde{W^\vee}$ of $W^\vee$ which is an étale T_+-module over $\Lambda(N_0)$. Since $D^\vee_{\xi,\ell,\infty}(\pi)$ is an étale

T_+-module over $\Lambda(N_0)$ and the composite map $W^\vee \to D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is ψ-equivariant, it factors through $\widetilde{W^\vee}$. All in all we have factored the map $\mathrm{pr} \circ \mathrm{pr}_{SV}$ as

$$\mathrm{pr} \circ \mathrm{pr}_{SV} \colon \pi^\vee \xrightarrow{\widetilde{\mathrm{pr}_W}} \widetilde{W^\vee} \xrightarrow{\mathrm{pr}_D^{\widetilde{W^\vee}}} D^\vee_{\xi,\ell,\infty}(\pi) \ .$$

The advantage of considering $\widetilde{W^\vee}$ is that the operators $\mathcal{H}_g^{(k)}$ make sense as maps $\widetilde{W^\vee} \to \widetilde{W^\vee}$ and the map $\widetilde{W^\vee} \to D^\vee_{\xi,\ell,\infty}(\pi)$ is $\mathcal{H}_g^{(k)}$-equivariant as it is a morphism of étale T_+-modules over $\Lambda(N_0)$. More precisely, let g be in G and put $\mathcal{U}_g := \{u \in N_0 \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$, $\mathcal{U}_g^{(k)} := J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}_g$. For any $u \in \mathcal{U}_g$ we write gu in the form $gu = n(g,u)t(g,u)\overline{n}(g,u)$ for some unique $n(g,u) \in N_0$, $t(g,u) \in T$, $\overline{n}(g,u) \in \overline{N}$.

Lemma 4.12. *There exists an integer $k_0 = k_0(g)$ such that for all $k \geq k_0$ and $u \in \mathcal{U}_g$ we have $us^kN_0s^{-k} \subseteq \mathcal{U}_g$, $s^kt(g,u) \in T_+$, and $s^{-k}\overline{n}(g,u)s^k \in \overline{N}_0 = G_0 \cap \overline{N}$. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ we have $\mathcal{U}_g = \bigcup_{u \in \mathcal{U}_g^{(k)}} us^kN_0s^{-k}$.*

Proof. Since $\mathcal{U}_g$ is compact and open in N_0, it is a union of finitely many cosets of the form $us^kN_0s^{-k}$ for k large enough. Moreover, the maps $t(g,\cdot)$ and $\overline{n}(g,\cdot)$ are continuous in the p-adic topology. So the image of $t(g,\cdot)$ is contained in finitely many cosets of T/T_0 as T_0 is open. For the statement regarding $\overline{n}(g,u)$ note that we have $\overline{N} = \bigcup_{k\geq 0} s^k\overline{N}_0s^{-k}$. □ □

For $k \geq k_0 = k_0(g)$ let $J(N_0/s^kN_0s^{-k}) \subset N_0$ be an arbitrary set of representatives of $N_0/s^kN_0s^{-k}$. Recall from the proof of Prop. 4.7 Step 2 (see also [10]) that for fixed $g \in G$ and all $u \in N_0$ we may write $\alpha_g(x_u)u$ in the form $n(g,u)t(g,U)$ for some $n(g,u) \in N_0$ and $t(g,U) \in s^{-k_0}T_+$. In particular the equation (20) defining $\mathcal{H}_g^{(k)}$ reads

$$\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^kN_0s^{-k})} := \sum_{u \in \mathcal{U}_g^{(k)}} n(g,u)\varphi_{t(g,u)s^k} \circ \psi_s^k \circ (u^{-1}\cdot)$$

where $t(g,u)s^k$ lies in T_+. Further, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^kN_0s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\mathrm{res}_{\mathcal{U}} := \sum_{u \in J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}} u\varphi_{s^k} \circ \psi_s^k \circ (u^{-1}\cdot) \ .$$

The operators $\mathcal{H}_g^{(k)}$ and $\mathrm{res}_{\mathcal{U}}$ make sense in any étale T_+-module over $\Lambda(N_0)$, in particular also in $\widetilde{W^\vee}$ and $D^\vee_{\xi,\ell,\infty}(\pi)$. Moreover, $\mathrm{res}_{\mathcal{U}}$ is independent of the choice of $k \geq k'(\mathcal{U})$. Further, any morphism between étale T_+-modules over $\Lambda(N_0)$ is $\mathcal{H}_g^{(k)}$- and $\mathrm{res}_{\mathcal{U}}$-equivariant.

Lemma 4.13. *Let g be in G, u be in $\mathcal{U}_g$, and $k \geq k_0 + 1$ be an integer. Then the map*

$$n(g,\cdot)\colon us^kN_0s^{-k} \to n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1} \tag{26}$$

is a bijection. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ the set $\mathcal{U}_{g^{-1}}$ is the disjoint union of the cosets $n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1}$ for $u \in \mathcal{U}_g^{(k)}$.

Proof. By our assumption $k \geq k_0+1$, $s^{-k}\overline{n}(g,u)s^k$ lies in $s^{-1}\overline{N}_0 s \subseteq U^{(1)}$. So for any $v \in N_0$ we have $s^{-k}\overline{n}(g,u)s^k v = vv_1$ for some v_1 in $v^{-1}U^{(1)}v = U^{(1)}$. Further, by the Iwahori factorization we have $U^{(1)} = (N \cap U^{(1)})(T \cap U^{(1)})(\overline{N} \cap U^{(1)})$. So we obtain that $s^{-k}\overline{n}(g,u)s^k v w_0 B \subset \mathcal{C}_0$ for all $v \in N_0$, whence we deduce $s^{-k}\overline{n}(g,u)s^k\mathcal{C}_0 \subseteq \mathcal{C}_0$. Similarly we have $s^{-k}\overline{n}(g,u)^{-1}s^k\mathcal{C}_0 \subseteq \mathcal{C}_0$ showing that in fact $s^{-k}\overline{n}(g,u)s^k\mathcal{C}_0 = \mathcal{C}_0$. We compute

$$\begin{aligned} g(us^kN_0s^{-k})w_0B = gus^kN_0w_0B = n(g,u)t(g,u)s^k(s^{-k}\overline{n}(g,u)s^k)\mathcal{C}_0 = \\ = n(g,u)t(g,u)s^k\mathcal{C}_0 = n(g,u)(t(g,u)s^kN_0s^{-k}t(g,u)^{-1})w_0B \ . \end{aligned}$$

Since the map $n(g,\cdot)$ is induced by the multiplication by g on $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ (identified with $\mathcal{U}_g$), we deduce that the map (26) is a bijection. The second statement follows as $n(g,\cdot)\colon \mathcal{U}_g \to \mathcal{U}_{g^{-1}}$ is a bijection and we have a partition of $\mathcal{U}_g$ into cosets $us^kN_0s^{-k}$ for $u \in \mathcal{U}_g^{(k)}$ by Lemma 4.12. □ □

Lemma 4.14. *Let M be arbitrary in $\mathcal{M}(\pi^{H_0})$ and $l, l' \geq 0$ be integers. There exists an integer $k_1 = k_1(M, W_0, l, l') \geq 0$ such that for all $r \geq k_1$ the image of the natural composite map*

$$(W/W_r)^\vee \hookrightarrow W^\vee \to D^\vee_{\xi,\ell,\infty}(\pi) \overset{f_{M,l}}{\twoheadrightarrow} M_l^\vee[1/X]$$

lies in $\Lambda(N_0/H_l)\otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++} \subset \Lambda(N_0/H_l)\otimes_{u_\alpha} M^\vee[1/X] \cong M_l^\vee[1/X]$. Here $M^\vee[1/X]^{++}$ denotes the $o/\varpi^h[[X]]$-submodule of $M^\vee[1/X]$ consisting of elements $d \in M^\vee[1/X]$ with $\varphi_s^n(d) \to 0$ as $n \to \infty$.

Proof. By (25) the $\Lambda(N_0)$-submodules $(W/W_r)^\vee$ form a system of neighbourhoods of 0 in $W^\vee$. On the other hand, $X^{l'}M^\vee[1/X]^{++}$ being a treillis in $M^\vee[1/X]$ (Prop. II.2.2 in [3]), $\Lambda(N_0/H_l)\otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++}$ is open in the weak topology of $M_l^\vee[1/X]$. Therefore its preimage in $W^\vee$ contains $(W/W_r)^\vee$ for r large enough. □ □

Since $t(g,\cdot)$ is continuous and $\mathcal{U}_g$ is compact, there exists an integer $c \geq 0$ such that for all $u \in \mathcal{U}_g$ there is an element $t'(g,u) \in T_+$ such that $t(g,u)s^{k_0}t'(g,u) = s^c$.

Lemma 4.15. *For any fixed $M \in \mathcal{M}(\pi^{H_0})$ there are finitely many different values of $F^*_{t'(g,u)}M$ where $g \in G$ is fixed and u runs on $\mathcal{U}_g$.*

Proof. By Lemma 3.9 there exists an open subgroup $T' \leq T$ acting on M. In particular, $F^*_{t'(g,u)}M$ only depends on the coset $t'(g,u)T'$. Now $t'(g,\cdot) = s^{c-k_0}t(g,\cdot)^{-1}$ is continuous and $\mathcal{U}_g$ is compact therefore there are only finitely many cosets of the form $t'(g,u)T'$. □ □

Our key proposition is the following:

Proposition 4.16. *For all $g \in G$ we have $\mathrm{res}^{\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0} \circ \beta_{\mathcal{C}_0} = \mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0} \circ \beta_{g\mathcal{C}_0}$.*

Proof. Note that since G/B is totally disconnected in the p-adic topology, in particular $g\mathcal{C}_0\cap\mathcal{C}_0$ is both open and closed in $\mathcal{C}_0$, we have $\mathfrak{Y}(\mathcal{C}_0) = \mathfrak{Y}(g\mathcal{C}_0 \cap \mathcal{C}_0) \oplus \mathfrak{Y}(\mathcal{C}_0 \setminus g\mathcal{C}_0)$. By Prop. 4.7 $\mathcal{H}_g$ is the composite map

$$D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathfrak{Y}(\mathcal{C}_0) \overset{g\cdot}{\to} \mathfrak{Y}(g\mathcal{C}_0) \overset{\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}}{\twoheadrightarrow} \mathfrak{Y}(g\mathcal{C}_0 \cap \mathcal{C}_0) \hookrightarrow \mathfrak{Y}(\mathcal{C}_0) = D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \ ,$$

ie. we obtain $\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}\circ(g\cdot)=\mathcal{H}_g$ as maps on $\mathfrak{Y}(\mathcal{C}_0)$ once we identify $\mathfrak{Y}(g\mathcal{C}_0\cap\mathcal{C}_0)$ with a subspace in $\mathfrak{Y}(\mathcal{C}_0)$ via the above direct sum decomposition. On the other hand, by definition $\beta_{\mathcal{C}_0}=\mathrm{pr}\circ\mathrm{pr}_{SV}\colon\pi^\vee\to D^\vee_{\xi,\ell,\infty}(\pi)^{bd}\subset D^\vee_{\xi,\ell,\infty}(\pi)$ is the natural map and we have $\beta_{g\mathcal{C}_0}(\mu)=g\beta_{\mathcal{C}_0}(g^{-1}\mu)$ for any $g\in G$ and $\mu\in\pi^\vee$. Further, as maps on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}=\mathfrak{Y}(\mathcal{C}_0)$ we have $\mathrm{res}_{\mathcal{U}_{g^{-1}}}=\mathrm{res}^{\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}$. Putting these together our equation to show reads

$$\mathrm{res}_{\mathcal{U}_{g^{-1}}}\circ\mathrm{pr}\circ\mathrm{pr}_{SV}(\mu)=\mathcal{H}_g(\mathrm{pr}\circ\mathrm{pr}_{SV}(g^{-1}\mu))\ .$$

We want to write $\mathcal{H}_g$ as the limit of the maps $\mathcal{H}_g^{(k)}$, so we set $\mathcal{U}_g^{(k)}:=\{u\in J(N_0/s^kN_0s^{-k})\mid x_u\in g^{-1}\mathcal{C}_0\cap\mathcal{C}_0\}$ and compute

$$\begin{aligned}
&\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\psi_s^k(u^{-1}\widetilde{\mathrm{pr}_W}(g^{-1}\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\widetilde{\mathrm{pr}_W}(s^{-k}u^{-1}g^{-1}\mu)=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W(s^{-k}u^{-1}g^{-1}\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W(s^{-k}\overline{n}(g,u)^{-1}t(g,u)^{-1}n(g,u)^{-1}\mu))\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W((s^{-k}\overline{n}(g,u)^{-1}s^k)t(g,u)^{-1}s^{-k}n(g,u)^{-1}\mu))
\end{aligned}\tag{27}$$

where $\iota_{t(g,u)s^k,\infty}\colon\varphi^*_{t(g,u)s^k}W^\vee\to\varinjlim_t\varphi_t^*W^\vee=\widetilde{W^\vee}$ is the natural map. By Lemma 4.12 we have

$$s^{-k}\overline{n}(g,u)^{-1}s^k\in s^{-k+k_0}(G_0\cap\overline{N})s^{k-k_0}\leq U^{(k-k_0)}\ .$$

As π is a smooth representation of G and W_0 is finite, there exists an integer $k_2=k_2(W_0)$ such that for all $k'\geq k_2$ the subgroup $U^{(k')}$ acts trivially on W_0. By Lemma 4.10 we deduce

$$\mathrm{pr}_W(s^{-k}\overline{n}(g,u)^{-1}t(g,u)^{-1}n(g,u)^{-1}\mu)\mid_{W_r}=\mathrm{pr}_W(s^{-k}t(g,u)^{-1}n(g,u)^{-1}\mu)\mid_{W_r}$$

for all $r\leq\frac{k-k_2-k_0}{n_0}$ since N_0 normalizes $U^{(k-k_0)}$. Therefore by Lemma 4.13 and (27) we obtain

$$\begin{aligned}
&\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)-\mathrm{res}_{\mathcal{U}_{g^{-1}}}\circ\widetilde{\mathrm{pr}_W}(\mu)=\\
&=\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)-\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\psi_{t(g,u)s^k}(n(g,u)^{-1}\widetilde{\mathrm{pr}_W}(\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota(n(g,u)\otimes\mathrm{pr}_W((s^{-k}\overline{n}(g,u)^{-1}s^k-1)s^{-k}t(g,u)^{-1}n(g,u)^{-1}\mu))\\
&\in\sum_{u\in\mathcal{U}_g^{(k)}}\iota(\Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_{t(g,u)s^k}}(W/W_r)^\vee)
\end{aligned}$$

where $\iota = \iota_{t(g,u)s^k,\infty}$.

Finally, the sets $O(M,l,l') \subset D^\vee_{\xi,\ell,\infty}(\pi)$ in (9) form a system of open neighbourhoods of 0 in $D^\vee_{\xi,\ell,\infty}(\pi)$. Moreover, for any fixed choice $l,l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ there exists an integer $k_1 \geq 0$ such that for all $r \geq k_1$ and $u \in \mathcal{U}_g$ we have

$$\mathrm{pr}_{W,F^*_{t'(g,u)}M_l}((W/W_r)^\vee) \subseteq \Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'}(F^*_{t'(g,u)}M)^\vee[1/X]^{++}$$

(see Lemmata 4.14 and 4.15). Note that the composite map $D^\vee_{\xi,\ell,\infty}(\pi) \overset{\varphi_{t(g,u)s^k}}{\to} D^\vee_{\xi,\ell,\infty}(\pi) \overset{f_{M,0}}{\twoheadrightarrow} M^\vee[1/X]$ factors through the φ_s-equivariant map

$$((1 \otimes F_{t(g,u)s^k})^\vee[1/X])^{-1} \colon (F^*_{t'(g,u)}M)^\vee[1/X] \to M^\vee[1/X]$$

mapping $X^{l'}(F^*_{t'(g,u)}M)^\vee[1/X]^{++}$ into $X^{l'}M^\vee[1/X]^{++}$. So we deduce that

$$\mathcal{H}_g^{(k)} \circ \mathrm{pr} \circ \mathrm{pr}_{SV}(g^{-1}\mu) - \mathrm{res}_{\mathcal{U}_{g^{-1}}} \circ \mathrm{pr} \circ \mathrm{pr}_{SV}(\mu)$$

lies in $O(M,l,l')$ for all $k \geq k_0 + k_2 + n_0 k_1$ and any choice of $J(N_0/s^k N_0 s^{-k})$. The result follows by taking the limit $\mathcal{H}_g = \lim_{k\to\infty} \mathcal{H}_g^{(k)}$. □ □

Now for any fixed $\mu \in \pi^\vee$ consider the the elements $\beta_{g\mathcal{C}_0}(\mu) \in \mathfrak{Y}(g\mathcal{C}_0)$ for $g \in G$. By Proposition 4.16 we also deduce

$$\begin{aligned}\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0} \circ \beta_{g\mathcal{C}_0}(\mu) = \mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(g\beta_{\mathcal{C}_0}(g^{-1}\mu)) =\\ = g\,\mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0\cap g^{-1}h\mathcal{C}_0} \circ \beta_{\mathcal{C}_0}(g^{-1}\mu) \overset{4.16}{=} g\,\mathrm{res}^{g^{-1}h\mathcal{C}_0}_{\mathcal{C}_0\cap g^{-1}h\mathcal{C}_0} \circ \beta_{g^{-1}h\mathcal{C}_0}(g^{-1}\mu) =\\ = \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(g(g^{-1}h)\beta_{\mathcal{C}_0}((g^{-1}h)^{-1}g^{-1}\mu)) = \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(h\beta_{\mathcal{C}_0}(h^{-1}\mu)) =\\ = \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0} \circ \beta_{h\mathcal{C}_0}(\mu)\end{aligned}$$

for all $g,h \in G$. Since $\mathfrak{Y}$ is a sheaf and we have $\bigcup_{g\in G} g\mathcal{C}_0 = G/B$, there exists a unique element $\beta_{G/B}(\mu)$ in the global sections $\mathfrak{Y}(G/B)$ with

$$\mathrm{res}^{G/B}_{g\mathcal{C}_0}(\beta_{G/B}(\mu)) = \beta_{g\mathcal{C}_0}(\mu)$$

for all $g \in G_0$. So we obtained a map $\beta_{G/B} \colon \pi^\vee \to \mathfrak{Y}(G/B)$. Our main result in this section is the following

Theorem 4.17. *The family of morphisms $\beta_{G/B,\pi}$ for smooth, admissible o-torsion representations π of G of finite length form a natural transformation between the functors $(\cdot)^\vee$ and $\mathfrak{Y}_{\alpha,\cdot}(G/B)$. Whenever $D^\vee_{\xi,\ell}(\pi)$ is nonzero, the map $\beta_{G/B,\pi}$ is nonzero either. In particular, if we further assume that π is irreducible then $\beta_{G/B}$ is injective.*

Proof. At first we need to check that $\beta_{G/B,\pi} \colon \pi^\vee \to \mathfrak{Y}_{\alpha,\pi}(G/B)$ is G-equivari-ant and continuous for all π. For $g,h \in G$ and $\mu \in \pi^\vee$ we compute

$$\begin{aligned}\mathrm{res}^{G/B}_{g\mathcal{C}_0}(\beta_{G/B}(h\mu)) = \beta_{g\mathcal{C}_0}(h\mu) = g\beta_{\mathcal{C}_0}(g^{-1}h\mu) =\\ = h\beta_{h^{-1}g\mathcal{C}_0}(\mu) = h\,\mathrm{res}^{G/B}_{h^{-1}g\mathcal{C}_0} \circ \beta_{G/B}(\mu) = \mathrm{res}^{G/B}_{g\mathcal{C}_0}(h\beta_{G/B}(\mu))\end{aligned}$$

showing that $\beta_{G/B}(h\mu)$ and $h\beta_{G/B}(\mu)$ are equal locally everywhere, so they are equal globally, too. The continuity follows from the fact that $\beta_{g\mathcal{C}_0}$ is continuous for each $g \in G$.

By Thm. 9.24 in [10] the assignment $\pi \mapsto \mathfrak{Y}_{\alpha,\pi}$ is functorial. Moreover, by definition we have $\beta_{g\mathcal{C}_0,\pi} = (g\cdot)\circ\beta_{\mathcal{C}_0,\pi}\circ(g^{-1}\cdot)$ so we are reduced to showing the naturality of $\beta_{\mathcal{C}_0,\cdot}$. This follows from the fact that for any morphism $f \colon \pi \to \pi'$ of smooth, admissible o-torsion representations of G of finite length and $M_k \in \mathcal{M}_k(\pi^{H_k})$ for any $k \geq 0$ we have $f(M_k) \in \mathcal{M}_k(\pi'^{H_k})$. □ □

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References

[1] Breuil Ch., The emerging p-adic Langlands programme, in: *Proceedings of the International Congress of Mathematicians* Volume II, Hindustan Book Agency, New Delhi (2010), 203–230.

[2] Breuil Ch., Induction parabolique et (φ, Γ)-modules, *Algebra & Number Theory* **9**(10) (2015), 2241–2291.

[3] Colmez P., (φ, Γ)-modules et représentations du mirabolique de $GL_2(\mathbb{Q}_p)$, *Astérisque* **330** (2010), 61–153.

[4] Colmez P., Représentations de $GL_2(\mathbb{Q}_p)$ et (φ, Γ)-modules, *Astérisque* **330** (2010), 281–509.

[5] Emerton M., On a class of coherent rings with applications to the smooth representation theory of $GL_2(\mathbb{Q}_p)$ in characteristic p, preprint (2008)

[6] Emerton M., Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties, *Astérisque* **331** (2010), 355–402.

[7] Erdélyi M., The Schneider–Vigneras functor for principal series, *J. of Number Theory* **162** (2016), 68–85.

[8] Fontaine J.-M., Représentations p-adiques des corps locaux, in "The Grothendieck Festschrift", vol 2, Prog. in Math. 87, 249–309, Birkhäuser 1991.

[9] Schneider P., Vigneras M.-F., A functor from smooth o-torsion representations to (φ, Γ)-modules, Volume in honour of F. Shahidi, Clay Mathematics Proceedings Volume **13** (2011), 525–601.

[10] Schneider P., Vigneras M.-F., Zábrádi G., From étale P_+-representations to G-equivariant sheaves on G/P, in: *Automorphic forms and Galois representations* (Volume 2), LMS Lecture Note Series **415** (2014), Cambridge Univ. Press, 248–366.

[11] Vigneras M.-F., Série principale modulo p de groupes réductifs p-adiques, *Geom. and Funct. Analysis* **17** (2008), 2090–2112.

[12] Zábrádi G., Exactness of the reduction on étale modules, *Journal of Algebra* **331** (2011), 400–415.

[13] Zábrádi G., (φ, Γ)-modules over noncommutative overconvergent and Robba rings, *Algebra & Number Theory* **8**(1) (2014), 191–242.