Ramsey number of a connected triangle matching

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Abstract

We determine the 2-color Ramsey number of a *connected* triangle matching $c(nK_3)$ which is any connected graph containing n vertex disjoint triangles. We obtain that $R(c(nK_3), c(nK_3)) = 7n - 2$, somewhat larger than in the classical result of Burr, Erdős and Spencer for a triangle matching, $R(nK_3, nK_3) = 5n$. The motivation is to determine the Ramsey number $R(C_n^2, C_n^2)$ of the square of a cycle C_n^2. We apply our Ramsey result for connected triangle matchings to show that the Ramsey number of an "almost" square of a cycle $C_n^{2,c}$ (a cycle of length n in which all but at most a constant number c of short diagonals are present) is asymptotic to $7n/3$.

1 Introduction

Denote by $\delta(G)$ the minimum degree in a graph G. K_n is the complete graph on n vertices and $K_{n,n}$ is the complete bipartite graph between two sets of n vertices each. If $G_1, G_2, \ldots, G_r$ are graphs, then the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the

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smallest positive integer n such that in any edge-coloring with colors $1, 2, \ldots, r$, for some i the edges of color i contain a subgraph isomorphic to G_i. In this paper we will deal with 2- and 3-color Ramsey numbers (so $r = 2$ or 3) and we will think of color 1 as red, color 2 as blue and color 3 (if it exists) as white.

Among well-known early results in generalized Ramsey theory is the exact value of $R(nK_2, nK_2) = 3n - 1$ determined by Cockayne and Lorimer [8], and $R(nK_3, nK_3) = 5n$ determined by Burr, Erdős and Spencer [3]. Here nG denotes n vertex disjoint copies of the graph G. It turned out in many applications that it is important to study the case when nK_2, the matching, is replaced by a *connected matching*, $c(nK_2)$, defined as any connected graph containing nK_2 (see for example applications [2, 9, 11, 12, 13, 14, 15, 16, 22]). The 2-color Ramsey number of connected matchings is the same as the Ramsey number of matchings (in fact, [10] proves more) and one of the key arguments of [13] was that this remains true for 3 colors as well. However, for more than 3 colors the Ramsey numbers of matchings and connected matchings are different. For example, $R(nK_2, nK_2, nK_2, nK_2) = 5n - 3$ [8], but $R(c(nK_2), c(nK_2), c(nK_2), c(nK_2)) > 6n - 3$ when $2n - 1$ is divisible by three. This can be seen by the 4-coloring obtained from the parallel classes of an affine plane of order 3 by replacing each point with a point set of size $\frac{2n-1}{3}$.

In this paper we look at the connected version of the "matching of triangles". Let $c(nK_3)$ denote any connected graph containing n vertex disjoint triangles. We shall prove that here already the 2-color Ramsey number of $c(nK_3)$ is different from its counterpart nK_3.

Theorem 1. *For $n \geq 2$, $R(c(nK_3), c(nK_3)) = 7n - 2$.*

(While we have $R(nK_3, nK_3) = 5n$.) To see that $R(c(nK_3), c(nK_3)) > 7n-3$, consider pairwise disjoint sets A, B, C such that $V(K_{7n-3}) = A \cup B \cup C$, $|A| = |B| = 3n - 1$ and $|C| = n - 1$. Edges inside A and inside B are red, all other edges are blue. In this coloring there is no monochromatic $c(nK_3)$ (in fact there is not even an nK_3 in blue).

To prove that $R(c(nK_3), c(nK_3)) \leq 7n - 2$, we need the Ramsey number of connected triangle matchings versus ordinary matchings that might be interesting on its own.

Lemma 2. *For $1 \leq m \leq n$, $R(c(nK_3), mK_2) = 3n + m - 1$.*

Notice that $R(c(nK_3), mK_2) > 3n+m-2$ is shown by the disjoint sets A, X such that $V(K_{3n+m-2}) = A \cup X$, $|A| = 3n - 1, |X| = m - 1$ and edges inside A are colored red, other edges are colored with blue. In this coloring there is no red nK_3 or blue mK_2.

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The motivation of Theorem 1 comes from the effort to determine or estimate the 2-color Ramsey number $R(C_n^2, C_n^2)$ where of C_n^2 is the square of the cycle on n vertices, i.e. the cycle C_n with all short diagonals (diagonals between vertices at a distance 2 on the cycle). A very recent paper of Allen, Brightwell and Skokan [1] gives lower bound $3n - 4$ and upper bound $\frac{20n}{3} + o(n)$ for that Ramsey number.

Density questions for the square of a cycle have also received a lot of attention (for example the well-known Pósa-Seymour problem, see [4], [17], [21]). Note also that there has been a lot of research on the Ramsey numbers of constant maximum degree graphs (such as C_n^2, where the maximum degree is 4). It is known that for a graph G on n vertices with maximum degree Δ the Ramsey number is linear, it is at most $c(\Delta)n$ [5]. The current best bound $c(\Delta) \leq 2^{c\Delta \log \Delta}$ is due to Conlon, Fox and Sudakov [6].

To determine $R(C_n^2, C_n^2)$ exactly for every n is hopeless since for $n = 5, 6$, $C_5^2 = K_5$, $C_6^2 = K_6 - 3K_2$ and these Ramsey numbers are both unknown. Also, the constant $c(4)$ in the linear bounds $c(4)n$ is very large. However, combining Theorem 1 with the Regularity Lemma, we shall make a step forward and prove the following. Let $C_n^{2,c}$ denote an "almost" square of the cycle C_n, a cycle of length n in which all but at most a constant number c of short diagonals are present.

Theorem 3. *For every fixed $\eta > 0$ there is a $c = c(\eta)$ so that for any $n \geq c$ we have $R(C_n^{2,c}, C_n^{2,c}) \leq (1 + \eta)7n/3$.*

It is worth noting that Theorem 3, although asymptotically sharp (shown by a similar construction as in Theorem 1), does not give the right asymptotics for $R(C_n^2, C_n^2)$, where we insist on *all* short diagonals. Indeed, $R(C_n^2, C_n^2) \geq 3n - 4$ is proved in [1]. Thus perhaps surprisingly removing these constant number of diagonals makes a big difference in the Ramsey number.

The following easy lemma from [15] will be used. It extends (when $\delta(G) = |V(G)| - 1$) the well-known remark of Erdős and Rado that in a 2-colored complete graph there is a monochromatic spanning tree.

Lemma 4. *(Lemma 1.5 in [15]) Suppose that the edges of a graph G with $\delta(G) \geq \frac{3|V(G)|}{4}$ are 2-colored. Then there is a monochromatic connected subgraph with order larger than $\delta(G)$. This estimate is sharp.*

Theorem 1 and Lemma 2 are proved in Section 2. Their perturbed versions are worked out in Section 3. Section 4 outlines the (rather standard) argument how to obtain Theorem 3 from the Regularity and Blow-up lemmas.

2 The proof of Lemma 2 and Theorem 1

Proof of Lemma 2. We prove by induction on m. Since for $m = 1$ the statement is trivially true for every n, suppose we have a blue matching $M = (m-1)K_2$ in a 2-coloring of a K_{3n+m-1} with vertex set V. If there is no blue mK_2 then every edge $e_i \in M$ has a vertex p_i adjacent in red to all but at most one vertex of $X = V - V(M)$ (otherwise e_i could be replaced by two independent blue edges). Also, X induces a red complete graph. Since

$$|X| = 3n + m - 1 - 2(m-1) = 3n - m + 1 \geq 2m + 1,$$

we can select greedily $m - 1$ pairwise disjoint red triangles with one vertex as p_i and two vertices from X. Then we find red triangles greedily in the remainder of X. We are guaranteed to find n red triangles this way. These triangles can be certainly included into a connected red subgraph so we have the required $c(nK_3)$. $\square$

Proof of Theorem 1. Consider a 2-coloring of the edges of $K = K_{7n-2}$ with red and blue, assume w.l.o.g. that the blue color class has only one connected component (since one color class is connected by using Lemma 4 with $\delta(G) = |V(G)| - 1$). Since $R(nK_3, nK_3) = 5n$ for $n \geq 2$, we have a monochromatic nK_3. If it is blue, we are done, therefore it is red and the red color class must define at least two connected components. Suppose that the red components have vertex sets $X_1, X_2, \ldots, X_s$, where $s \geq 2$ and $|X_1| \geq |X_2| \geq \ldots \geq |X_s| \geq 1$. We may suppose that $|X_1| \leq 5n$, otherwise we have the required monochromatic $c(nK_3)$ from $R(nK_3, nK_3) = 5n$.
Case 1. $|X_1| \geq |X_2| \geq 3n$, $|X_1| = 3n + k_1$, $|X_2| = 3n + k_2$. Since by Lemma 2 we have $R(c(nK_3), (k_i + 1)K_2) = 3n + k_i$ for $i = 1, 2$, X_i contains either a red $c(nK_3)$ or a blue $(k_i + 1)K_2$ and we are done if the first possibility appears. Thus we have blue matchings M_i of size $k_i + 1$ in X_i for $i = 1, 2$. We can take $k_1 + 1$ vertices in $X_2 \setminus M_2$ and $k_2 + 1$ vertices in $X_1 \setminus M_1$ to form a blue $T = (k_1 + k_2 + 2)K_3$ using the blue edges between X_1 and X_2 since from $k_1 + k_2 \leq n - 2$ it follows that

$$3n + k_1 - 2(k_1 + 1) = 3n - k_1 - 2 \geq k_2 + 1,$$

$$3n + k_2 - 2(k_2 + 1) = 3n - k_2 - 2 \geq k_1 + 1.$$

If $k_1 + k_2 + 2 = n$ (i.e. $s = 2$) we have the required blue $c(nK_3)$. Otherwise we have $l = n - (k_1 + k_2) - 2$ (> 0) vertices in $A = V - (X_1 \cup X_2)$ and we can form l vertex disjoint blue triangles taking one vertex from each of the sets $A, X_1 - T, X_2 - T$. We have enough vertices for that, because

$$|X_1 - T| = 3n + k_1 - 2(k_1 + 1) - (k_2 + 1) = 3n - (k_1 + k_2) - 3 \geq n - (k_1 + k_2) - 2$$

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and the same is true for $|X_2 - T|$. Thus we have a connected blue triangle matching of size at least $k_1 + k_2 + 2 + n - (k_1 + k_2) - 2 = n$, as desired finishing Case 1.

Case 2. $|X_1| \geq 3n$, $2n \leq |X_2| < 3n$, $|X_1| = 3n + k_1$, $|X_2| = 3n - k_2$, $1 \leq k_2 \leq n$. Again since by Lemma 2 we have $R(c(nK_3), (k_1 + 1)K_2) = 3n + k_1$, we may suppose that we have a blue $M_1 = (k_1 + 1)K_2$ in X_1. We transform M_1 to a blue triangle matching $T = (k_1 + 1)K_3$ using $k_1 + 1$ vertices from X_2 and then extend T using $q = n - k_1 + k_2 - 2$ vertices in $A = V - (X_1 \cup X_2)$ and q vertices from both X_1, X_2. We have enough room for that because

$$3n + k_1 - 2(k_1 + 1) \geq n - k_1 + k_2 - 2, \ 3n - k_2 - (k_1 + 1) \geq n - k_1 + k_2 - 2$$

are both true since $k_2 \leq n$. Thus we have a connected blue triangle matching of size at least $(k_1 + 1) + (n - k_1 + k_2 - 2) = n + k_2 - 1 \geq n$.

Case 3. $|X_1| \geq 3n$, $n \leq |X_2| < 2n$, $|X_1| = 3n + k_1$, $|X_2| = n + k_2$, $0 \leq k_2 < n$. Again since by Lemma 2 we have $R(c(nK_3), (k_1 + 1)K_2) = 3n + k_1$, we may suppose that we have a blue $M_1 = (k_1 + 1)K_2$ in X_1. Furthermore, we may suppose that $k_1 < n$ otherwise M_1 can be transformed to a blue triangle matching $T = (k_1 + 1)K_3$ using vertices from X_2. Since $k_2 < n$ also holds, $|V - (X_1 \cup X_2)| = 3n - (k_1 + k_2) - 2 \geq n$. Thus we have at least n vertices in all of the three sets $X_1, X_2, V - (X_1 \cup X_2)$ implying that we have a connected blue $c(nK_3)$.

Case 4. $|X_1| \geq 3n$, $|X_2| < n$ or $|X_1| < 3n$.

From Lemma 2 we may assume $|X_1| \leq 4n - 2$. Indeed, otherwise since by Lemma 2 we have $R(c(nK_3), nK_2) = 4n - 1$, we may suppose that we have a blue $M_1 = nK_2$ in X_1. This blue M_1 can be transformed into a blue $c(nK_3)$ using n vertices in $V - X_1$ ($|X_1| < 5n$ ensures that there are n vertices).

Define the set S_1 so that $|S_1| = n$ and, starting with X_1, all vertices of X_i are selected before taking vertices from X_{i+1}. Then, starting from the next X_i, define S_2 in the same way. Now set

$$A = \{\cup X_i : (S_1 \cup S_2) \cap X_i = \emptyset\}.$$

Then we have the following claim.

$$|A| \geq n \text{ (or equivalently } |V \setminus A| \leq 6n - 2).$$

Indeed, this is true either because $|X_1| \leq 4n - 2$ and $|X_2| < n$ so the last X_i which intersects S_2 satisfies $|X_i| < n$ or because $3n > |X_1| \geq |X_2|$. But then we can select $S_3 \subset A$ with $|S_3| = n$, and the blue complete tripartite graph $[S_1, S_2, S_3]$ defines the required blue $c(nK_3)$. $\square$

3 Perturbed version of Theorem 1

As in many applications of the Regularity Lemma, one has to handle a few irregular pairs and the corresponding edges will not be present in the reduced graph. We say that the graph G on n vertices is ε-*perturbed* if it is almost complete, at most $\varepsilon\binom{n}{2}$ edges are missing. We cannot apply Theorem 1 in the reduced graph because in Theorem 1 we have a 2-colored complete graph, yet the reduced graph will be a 2-colored ε-perturbed graph. Thus we need perturbed versions of Theorem 1 and first Lemma 2. It will be convenient to think of the missing edges as edges in a third color class (white or "invisible"), so we move up from 2-color Ramsey numbers to 3-color Ramsey numbers. $K_{1,t}$ denotes the star with t leaves.

Lemma 5. *For $1 \le m \le n$, $0 \le t \le n$, $R(c(nK_3), mK_2, K_{1,t}) \le 3n + m - 1 + 2t$.*

Proof.   We prove by induction on m as in the non-perturbed case. The starting case, $m = 1$ follows easily from a well-known result of Corradi and Hajnal [7] (or it could also be proved directly by an easy induction on n). Indeed, if there is no blue edge, we have a red graph on $N = 3n + 2t$ vertices with minimum degree at least $3n + t > \frac{2N}{3}$ and it contains at least $\lfloor N/3 \rfloor \ge n$ vertex disjoint red triangles. Since our red graph is automatically connected from the minimum degree condition, we have the required red $c(nK_3)$. Thus, we may select a blue matching $M = (m-1)K_2$ in a 2-coloring of a $K_{3n+m-1+2t}$ with vertex set V. We may assume that from every vertex fewer than t edges are missing (or white edges). If there is no blue mK_2 then every edge $e_i \in M$ again has a vertex p_i that is adjacent in blue to at most one vertex in $X = V - V(M)$. However, now p_i is not necessarily adjacent in red to all other vertices in X since some edges might be missing. But all the edges that are actually *present* are indeed red to the other vertices. Furthermore, in X all edges that are present are red as well. Since

$$|X| = 3n + m - 1 + 2t - 2(m-1) = 3n - m + 1 + 2t \ge 2m + 1 + 2t,$$

again we can select greedily $m - 1$ pairwise disjoint red triangles with one vertex as p_i and two vertices from X. Indeed, p_i is still adjacent in red to more than $(2t + 3) - t = t + 3 > t$ vertices in X but then there is a (red) edge among these neighbors, giving a red triangle as desired. Then we find red triangles greedily in the remainder of X similarly. Finally, we find the n-th red triangle in the remainder of X as follows. Select an arbitrary remaining vertex of X. Since it has more than t neighbors left in X, there is an edge among these neighbors and all edges are red in X. The red graph spanned by X is connected because $|X|/2 > t$, thus the n red triangles form a $c(nK_3)$. $\square$

We will also need a perturbed version of the classical result of Burr, Erdős and Spencer, $R(nK_3, nK_3) = 5n$.

Lemma 6. *For $n \geq 2$, $1 \leq t \leq n$, $R(nK_3, nK_3, K_{1,t}) \leq 6n - 2 + 5t$.*

Proof.   Consider the largest blue triangle matching, remove it, then consider the largest red triangle matching in the remainder and remove it. We have at least $5t + 4$ vertices left and there are no more monochromatic triangles. However, consider an arbitrary vertex, it is still adjacent to at least $4t + 4$ vertices in the leftover. Then in one of the colors (say blue) it is adjacent to at least $2t + 2$ vertices. These neighbors will induce a triangle which must be red (otherwise we get a blue triangle), a contradiction. Indeed, consider again an arbitrary vertex from these at least $2t + 2$ vertices, it is still adjacent to at least $t + 2$ vertices from these at least $2t + 2$ vertices. But then there must be an edge within these at least $t + 2$ vertices, giving a triangle.

We note that more is true ($5n + ct$) but for our purposes this weaker statement is sufficient. □

Next we are ready to give the perturbed version of Theorem 1.

Theorem 7. *For $n \geq 2$, $0 \leq t \leq 2n/3$, $R(c(nK_3), c(nK_3), K_{1,t}) \leq 7n - 2 + 7t$.*

Proof.   Again suppose we have a 2-coloring of a $K_{7n-2+7t}$ with vertex set V. We may assume that from every vertex fewer than t edges are missing (edges in the third color). Applying Lemma 4, we get a monochromatic (say blue) connected component X of size at least $(7n - 2 + 7t) - t = 7n - 2 + 6t$. By Lemma 6, since $7n - 2 + 6t \geq 6n - 2 + 5t$, we have a monochromatic nK_3 in X. If it is blue, we are done, therefore it is red and thus the red color class must define at least two connected components within X. Suppose that the red components of V have vertex sets $X_1, X_2, \ldots, X_s$, where $s \geq 2$ and $|X_1| \geq |X_2| \geq \ldots \geq |X_s| \geq 1$. We may suppose that $|X_1| \leq 6n - 2 + 5t$ otherwise we have the required monochromatic $c(nK_3)$ from Lemma 6.
Case 1. $|X_1| \geq |X_2| \geq 3n + 2t$, $|X_1| = 3n + 2t + k_1$, $|X_2| = 3n + 2t + k_2$. Here we apply Lemma 5 to the two subgraphs induced by X_1 and X_2. We find either a red $c(nK_3)$ or a blue $(k_i + 1)K_2$ in them and we are done if the first possibility appears. Thus we have blue matchings M_1, M_2 of size $k_1 + 1, k_2 + 1$, respectively.

We extend M_1 to a blue $(k_1 + 1)K_3$ by taking $k_1 + 1$ vertices in $X_2 - M_2$. This can be done if $|X_2 - M_2| \geq k_1 + 1 + 2t - 2$, extending the edges of M_1 one by one to blue triangles, at each step we have at most $2t - 2$ vertices in $X_2 - M_2$ not adjacent (in blue) to at least one of the ends of the edge to be extended. Indeed,

$$|X_2 - M_2| = 3n + 2t + k_2 - 2(k_2 + 1) = 3n + 2t - k_2 - 2 \geq k_1 + 1 + 2t - 2$$

i.e $3n \geq k_1 + k_2 + 1$ which is true since $k_1 + k_2 + 1 \leq n + 3t - 1 \leq n + 2n - 1$ from the assumption $t \leq 2n/3$. The same argument allows to extend M_2 to a blue $(k_2 + 1)K_3$ with $k_2 + 1$ vertices of $X - M_1$. Thus we have a blue $T = (k_1 + k_2 + 2)K_3$ and noticing

that the blue graph spanned by $X_1 \cup X_2$ is connected (by $|X_1|, |X_2| > 2t$, any two vertices of X_1 and of X_2 has a common blue neighbor) we are done if $k_1 + k_2 + 2 \geq n$.

Otherwise we have $l = n + 3t - (k_1 + k_2) - 2 \ (> 0)$ vertices in $A = V - (X_1 \cup X_2)$ and we plan to extend T to nK_3 with $n - (k_1 + k_2 + 2)$ vertex disjoint blue triangles taking one vertex from each of the sets $A, X_1 - T, X_2 - T$. Since T is already connected, the extension will be automatically connected as well. We have enough vertices for that if all the three sets have size at least $n - (k_1 + k_2 + 2) + 2t$ (in fact two of them can be only at least $n - (k_1 + k_2 + 2) + t$). In our case the condition holds for A with t to save and for $X_i - T$ it holds with about $2n$ to save:

$$|X_1 - T| = 3n + 2t + k_1 - 2(k_1 + 1) - (k_2 + 1) =$$

$$= 3n + 2t - (k_1 + k_2) - 3 \geq n - (k_1 + k_2 + 2) + 2t$$

and the same is true for $|X_2 - T|$. Thus we have a blue $c(nK_3)$.

Case 2. $|X_1| \geq 3n+2t$, $2n+2t \leq |X_2| < 3n+2t$, $|X_1| = 3n+2t+k_1$, $|X_2| = 3n+2t-k_2$, $1 \leq k_2 \leq n$.
Here we apply Lemma 5 to the subgraph induced by X_1. We may suppose we have a blue $M_1 = (k_1 + 1)K_2$ in X_1, we transform M_1 to a blue triangle matching $T = (k_1 + 1)K_3$ using $k_1 + 1$ vertices from X_2 and then extend T using $q = n - k_1 + k_2 - 2$ vertices in $A = V - (X_1 \cup X_2)$ and q vertices from both X_1, X_2. We have enough room for that because $|A| = n + 3t - k_1 + k_2 - 2 \geq q + 2t$ and

$$3n + 2t + k_1 - 2(k_1 + 1) \geq q + 2t, \ 3n + 2t - k_2 - (k_1 + 1) \geq q + 2t$$

are both true since $k_2 \leq n$. Thus we have a connected blue triangle matching of size at least $(k_1 + 1) + q = n + k_2 - 1 \geq n$.

Case 3. $|X_1| \geq 3n+2t$, $n+2t \leq |X_2| < 2n+2t$, $|X_1| = 3n+2t+k_1$, $|X_2| = n+2t+k_2$, $0 \leq k_2 < n$.
Again we apply Lemma 5 to the subgraph induced by X_1 and select the blue $M_1 = (k_1 + 1)K_2$ in X_1. We may suppose that $k_1 < n$ otherwise M_1 can be transformed to a blue triangle matching $T = (k_1 + 1)K_3$ using vertices from X_2. Since $k_2 < n$ also holds, $|V - (X_1 \cup X_2)| = 3n + 3t - (k_1 + k_2) - 2 \geq n + 2t$. Thus we have at least $n + 2t$ vertices in all of the three sets $X_1, X_2, V - (X_1 \cup X_2)$ implying that we have a connected blue $c(nK_3)$.

Case 4. $|X_1| \geq 3n + 2t$, $|X_2| < n + 2t$ or $|X_1| < 3n + 2t$.
We may assume $|X_1| \leq 4n - 2 + 2t$, otherwise we can apply Lemma 5 with $m = n$ to X_1 to find a blue nK_2 and, since from Lemma 7 $|X_1| \leq 6n - 2 + 5t$ we have at least $7n - 2 + 7t - (6n - 2 + 5t) = n + 2t$ vertices in $V - X_1$, the blue nK_2 can be transformed into a blue $c(nK_3)$ using n vertices of $V - X_1$.

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If $|X_1| \geq n+2t$, take an $(n+2t)$-vertex subset $S_1 \subset X_1$ then take an $(n+2t)$-vertex set S_2 from $\cup_{i>1}X_i$ so that in S_2 we use all vertices of X_i before taking vertices from X_{i+1}. Define

$$A = \{\cup_{i>1}X_i : S_2 \cap X_i \neq \emptyset\}$$

Then $|X_1 \cup A| \leq 6n-2+6t$ either because $|X_1| \leq 4n-2+2t$ and $|X_2| < n+2t$ so the last X_i which intersects S_2 satisfies $|X_i| < n+2t$ or because $3n+2t > |X_1| \geq |X_2|$. Thus we can select $S_3 \subset V - A$ with $|S_3| \geq n+t$, and the blue tripartite graph $[S_1, S_2, S_3]$ has lower bounds $n+2t, n+2t, n+t$ for its vertex classes which allows to pick the vertices of the required blue $c(nK_3)$.

If $|X_1| < n+2t$, define S_1 so that $|S_1| = n+t$ and all vertices of X_i are selected before taking vertices from X_{i+1}. Then, starting from the next X_i, define S_2 in the same way. Now set

$$B = \{\cup X_i : (S_1 \cup S_2) \cap X_i \neq \emptyset\}$$

and observe that $|B| \leq 4n+6t$ thus we can select $S_3 \subset V - B$ with $|S_3| \geq 7n-2+7t-(4n+6t) = 3n-2+t \geq n+2t$, and the blue complete tripartite graph $[S_1, S_2, S_3]$ has lower bounds $n+t, n+t, n+2t$ for its vertex classes which allows to pick the vertices of the required blue $c(nK_3)$. $\square$

4 Proof of Theorem 3; applying the Regularity Lemma

Next we show how to prove Theorem 3 from Theorem 7, the Regularity Lemma [23] and the Blow-up Lemma. The material of this section is fairly standard by now (see [2, 11, 12, 13, 14, 15, 16] for similar techniques) so we omit some of the details. In particular in [13] Section 2 follows a similar outline.

Let $e(X, Y)$ denote the number of edges between X and Y in a graph G. For disjoint X, Y, we define the density

$$d(X, Y) = \frac{e(X, Y)}{|X| \cdot |Y|}.$$

For two disjoint subsets A, B of $V(G)$, the bipartite graph with vertex set $A \cup B$ which has all the edges of G with one endpoint in A and the other in B is called the pair (A, B).

A pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

A pair (A, B) is (ε, δ)-super-regular if it is ε-regular and furthermore,

$$deg(a) \geq \delta|B| \quad \text{for all} \quad a \in A,$$

$$\text{and} \quad deg(b) \geq \delta|A| \quad \text{for all} \quad b \in B.$$

We need a 2-edge-colored version of the Szemerédi Regularity Lemma.[1]

Lemma 8. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that*

- *$m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \ldots = |V_\ell|$, $|V_0| < \varepsilon n$,*

- *apart from at most $\varepsilon \binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are ε-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 2$.*

Our other main tool is the Blow-up Lemma (see [18, 19]). It basically says that super-regular pairs behave like complete bipartite graphs from the point of view of bounded degree subgraphs.

Lemma 9. *Given a graph R of order r and positive parameters δ, Δ, there exists an $\varepsilon > 0$ such that the following holds. Let m be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint m-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The graph $R(m)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{m,m}$, and a sparser graph G is constructed by replacing the edges of R with some (ε, δ)-super-regular pairs. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $R(m)$ then it is already embeddable into G.*

Actually we will need the following consequence of the Blow-up Lemma (where R is a triangle).

Lemma 10. *For every $\delta > 0$ there exist an $\varepsilon > 0$ and m_0 such that the following holds. Let G be a tripartite graph with tripartition $V(G) = V_1 \cup V_2 \cup V_3$ such that $|V_1| = |V_2| = |V_3| = m \geq m_0$, and let all the 3 pairs (V_1, V_2), (V_1, V_3), (V_2, V_3) be (ε, δ)-super-regular. Then for every pair of vertices $v_1 \in V_1, v_2 \in V_3$ and for every integer p, $4 \leq p \leq 3m$, G contains an "almost" P_p^2, the square of a path with p vertices connecting v_1 and v_2 from which at most two short diagonals are missing.*

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[1]For background, this variant and other variants of the Regularity Lemma see [20].

We emphasize that Lemma 10 is true for any value of p between 4 and $3m$, not just for the ones that are divisible by 3. The price we pay is that two short diagonals might be missing which is allowed in our application. Note also that an easier approximate version of this lemma would suffice as well, but for simplicity we use this lemma.

Proof. We think of G as having the orientation (V_1, V_2, V_3). Because of the Blow-up Lemma it is sufficient to check the statement for the complete tripartite graph (using $r = 3$ and $\Delta = 4$ in the Blow-up Lemma). We distinguish three cases depending on p. If $p = 3k$ for some integer $2 \leq k \leq m$, then we just go around (V_1, V_2, V_3) k times starting with v_1 and ending with v_2, so in this case actually no short diagonal is missing. If $p = 3k + 1$ with $k \geq 1$, then we go around (V_1, V_2, V_3) $(k - 1)$ times starting with v_1, but then in the last round we "turn around", i.e. we finish with vertices u_1, u_2, u_3, v_2 chosen from V_1, V_2, V_1, V_3, respectively. Then the only short diagonal missing is between u_1 and u_3. Finally, if $p = 3k + 2$ with $k \geq 1$, then we go around (V_1, V_2, V_3) $(k - 1)$ times starting with v_1, but then in the last round we "double up", i.e. we finish with vertices u_1, u_2, u_3, u_4, v_2 chosen from V_1, V_2, V_1, V_2, V_3, respectively. Then the only two short diagonals missing are between u_1 and u_3 and between u_2 and u_4. $\square$

With these preparations now we are ready to prove Theorem 3 from Theorem 7. Let

$$\varepsilon \ll \eta \ll 1, \tag{1}$$

m_0 sufficiently large compared to $1/\varepsilon$ and M_0 obtained from Lemma 8. Suppose we have a 2-coloring of a complete graph with vertex set V, $|V| = (1 + \eta)7n/3$ (for simplicity assume that this is a sufficiently large integer). We apply Lemma 8. We obtain a partition of V, that is $V = \cup_{0 \leq i \leq \ell} V_i$. We define the following *reduced graph* G^R: The vertices of G^R are $p_1, \ldots, p_\ell$, and there is an edge between vertices p_i and p_j if the pair (V_i, V_j) is ε-regular in both colors. The edge $p_i p_j$ is colored with the majority color in $K(V_i, V_j)$. Thus G^R is a $(1 - \varepsilon)$-dense 2-colored graph on ℓ vertices. Then we "trim" G^R in the standard way: there is a subgraph H^R on at least $(1 - \sqrt{\varepsilon})\ell$ vertices where the maximum degree of the complement is less than $\sqrt{\varepsilon}\ell$ (see for example Lemma 9 in [13]). In other words the third color class does not contain a star $K_{1,t}$ with $t = \sqrt{\varepsilon}\ell$, as we need in Theorem 7.

Applying Theorem 7 to H^R with $t = \sqrt{\varepsilon}\ell$, we can get a large monochromatic (say red) connected triangle matching in H^R (and thus in G^R). For a triangle $T_i, 1 \leq i \leq \ell_1$ in this connected triangle matching denote the corresponding clusters by (V_1^i, V_2^i, V_3^i). Thus (using (1)) we may assume that the number of vertices in the union of these clusters is between $(1 + \frac{\eta}{2})n$ and $(1 + \eta)n$. Next, first using the fact that this is a connected triangle matching we find red connecting paths P_i^R in G^R between T_i and $T_{i+1}, 1 \leq i \leq \ell_1$ (where $T_{\ell_1+1} = T_1$) and then from these connecting paths P_i^R we can find vertex disjoint red connecting paths P_i in the original graph between a vertex of

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V_3^i to a vertex of V_1^{i+1}. This procedure is rather standard by now, see for example [13] for the details.

These connecting paths P_i will be part of the final monochromatic $C_n^{2;c}$ we are constructing, so on these segments there will not be any short diagonals guaranteed. However, since the paths P_i^R are in G^R, their total length is indeed a constant depending on ε only. We remove the internal vertices of the paths P_i from our graph; let us denote their total number by c_1. Thus on the remaining segments we need exactly $n - c_1$ vertices.

Furthermore, we remove some more vertices from each $(V_1^i, V_2^i, V_3^i), 1 \leq i \leq l_1$ to achieve super-regularity in red in all of the three pairs. Finally we remove some more vertices to get a balanced super-regular tripartite graph. The number of remaining vertices in the union of the clusters in the triangles is still between n and $(1 + \eta)n$ using (1). For simplicity we still denote the clusters by V_j^i.

Finally we will lift the triangles back to almost square-paths in the original graph using Lemma 10. Let us denote by $(1 - \eta')$ the ratio of n and the number of remaining vertices in the union of the clusters in the triangles, so $0 \leq \eta' \ll 1$. Let us use Lemma 10 in each balanced super-regular tripartite graph $(V_1^i, V_2^i, V_3^i), 2 \leq i \leq l_1$ with $p_i = \lfloor (1 - \eta')3|V_1^i| \rfloor$ to connect the two endpoints of the connecting paths P_{i-1} and P_i with an almost square-path of length p_i. Finally we use Lemma 10 one more time in the balanced super-regular tripartite graph (V_1^1, V_2^1, V_3^1) with a p_1 value that makes the total length exactly n, to connect the two endpoints of the connecting paths P_{ℓ_1} and P_1 with an almost square-path of length p_1. This is possible since this p value is less than $\lfloor (1 - \eta')3|V_1^1| \rfloor$ only by a constant. Putting together the almost square-paths within the triangles with the connecting paths we get the red almost square-cycle of length n with only a constant number of short diagonals missing. $\square$

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