Ramsey number of paths and connected matchings in Ore-type host graphs

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Abstract

It is well-known (as a special case of the path-path Ramsey number) that in every 2-coloring of the edges of K_{3n-1}, the complete graph on $3n-1$ vertices, there is a monochromatic P_{2n}, a path on $2n$ vertices. Schelp conjectured that this statement remains true if K_{3n-1} is replaced by any host graph on $3n-1$ vertices with minimum degree at least $\frac{3(3n-1)}{4}$. Here we propose the following stronger conjecture, allowing host graphs with the corresponding Ore-type condition: If G is a graph on $3n-1$ vertices such that for any two non-adjacent vertices u and v, $d_G(u)+d_G(v) \geq \frac{3}{2}(3n-1)$, then in any 2-coloring of the edges of G there is a monochromatic path on $2n$ vertices. Our main result proves the conjecture in a weaker form, replacing P_{2n} by a *connected matching* of size n. Here a monochromatic, say red, matching in a 2-coloring of the edges of a graph is connected if its edges are all in the same connected component of the graph defined by the red edges. Applying the standard technique of converting connected matchings to paths with the Regularity Lemma, we use this result to get an asymptotic version of our conjecture for paths.

1 Background, summary of results.

The path-path Ramsey number was determined in [10], and its diagonal case (stated for convenience for even paths) is that $R(P_{2n}, P_{2n}) = 3n-1$, i.e. in every 2-coloring of the edges of K_{3n-1}, the complete graph on $3n-1$ vertices, there is a monochromatic P_{2n}, a path on $2n$ vertices. It is a natural question whether a similar conclusion is true if K_{3n-1} is replaced by some other host graph G. The first result in this direction was obtained in [13] where it was proved that in every 2-coloring of the edges of the complete 3-partite graph $K_{n,n,n}$ there is a monochromatic $P_{(1-o(1))2n}$. We focus in this paper on an other example, a conjecture of Schelp [21], stating that K_{3n-1} can be replaced by any host graph G of order $3n-1$ with large minimum degree $\delta(G)$.

Conjecture 1 (Schelp [21]). *Suppose that n is large enough and G is a graph on $3n-1$ vertices with $\delta(G) \geq \frac{3(3n-1)}{4}$. Then in every 2-coloring of the edges of G there is a monochromatic P_{2n}.*

Asymptotic versions of Schelp's conjecture were proved independently in [3] and [15]. In this paper we go one step further and consider graphs satisfying an Ore-type degree condition replacing the minimum degree condition. Here we call a degree condition Ore-type if it gives a lower bound on the degree sum for any two non-adjacent vertices. There has been a lot of efforts in trying to extend results from minimum degree conditions to Ore-type conditions. The first result of this type was proved by Ore [20]: If for any two non-adjacent vertices x and y of G, we have

$d_G(x)+d_G(y) \geq n$, then G is Hamiltonian. Some other results of this type include for example [7] (Ore-type conditions for k-ordered Hamiltonian graphs), [16] (Ore-type results on equitable colorings), [17] (Ore-type versions of Brooks' theorem), [8] (Ore-Type Conditions for H-Linked Graphs) or [2] (Ore-type conditions for partitioning into two monochromatic cycles).

Generalizing Conjecture 1 for graphs satisfying an Ore-type condition here we pose

Conjecture 2. *Suppose that n is large enough and G is a graph on $3n-1$ vertices such that for any two non-adjacent vertices u and v of G, we have $d_G(u)+d_G(v) \geq 3(3n-1)/2$. Then in every 2-coloring of the edges of G there is a monochromatic P_{2n}.*

The condition "n is large enough" seems to be a kind of safety belt in Conjecture 1, so we kept it also in Conjecture 2, although as far as we know, both can be true for all n. It is also worth mentioning that the condition $\delta(G) \geq \frac{3(3n-1)}{4}$ (or the sum of degrees of nonadjacent vertices is at least $\frac{3(3n-1)}{2}$ in Conjecture 2) is close to best possible in these conjectures as the following example ([15], [21]) shows.

Suppose that $3n-1=4m$ for some m and consider a graph whose vertex set is partitioned into four parts A_1, A_2, A_3, A_4 with $|A_i|=m$. Assume there are no edges from A_1 to A_2 and from A_3 to A_4 and all other pairs are edges. Edges in the complete bipartite graphs $[A_1, A_3], [A_2, A_4]$ ($[A_1, A_4], [A_2, A_3]$) are colored red (blue). Edges inside the A_i-s can be colored arbitrarily. In this coloring the longest monochromatic path has $\frac{3n-1}{2}$ vertices, much smaller than $2n$, while the minimum degree is $3m-1=\frac{3(3n-1)}{4}-1$ and the sum of degrees of nonadjacent pairs is $6m-2=\frac{3(3n-1)}{2}-2$. Thus, a small increase in the minimum degree (or in the sum of degrees of nonadjacent pairs) results a dramatic increase of the length of the longest monochromatic path.

To state our main result, Theorem 1, we need a definition. A matching in a graph is called a *connected matching* if its edges belong to the same connected component of the graph. When the edges are colored, a monochromatic, say red connected matching is a matching with red edges in a connected component of the graph defined by the red edges.

Theorem 1. *Let G be a graph with $3n-1$ vertices such that for any two non-adjacent vertices u and v of G, we have $d_G(u)+d_G(v) \geq 3(3n-1)/2$. Then in any 2-coloring of the edges of G there exists a monochromatic connected matching of size n.*

Although Theorem 1 is weaker than Conjecture 2 since it proves the existence of a *connected matching* of the right size instead of a *path*, it is valid for every n. The special case of Theorem 1 with minimum degree condition $\frac{3}{4}(3n-1)$ was proved in [15].

Theorem 1 can be used as a stepping stone to prove Theorem 2, an asymptotic form of Conjecture 2.

Theorem 2. *For every $\eta > 0$, there is an $n_0 = n_0(\eta)$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices such that for any two non-adjacent vertices x and y of G, we have $d_G(x) + d_G(y) \geq (\frac{3}{2} + \eta)n$. Then in every 2-coloring of the edges of G there is a monochromatic path with at least $(\frac{2}{3} - \eta)n$ vertices.*

Our proof technique is based on a method of Łuczak established in [19] and used successfully in many results of this area, see e.g. [4], [9], [11], [12], [13], [14]. The crucial idea of this method is that "paths" in a statement to be proved are replaced by "connected matchings". We will apply Theorem 1 to the cluster graph of a regular partition of the target graph of Theorem 2 obtained from the Regularity Lemma. Through several technical details, the regularity of the partition is used to "lift back" the connected matching of the cluster graph to a path in the original graph. This became a rather standard method by now, we give an outline in Sections 5 and 6.

The proof of Theorem 1 (Section 4) relies on two other results that may be interesting on their own. One of them is a lemma on matchings in multipartite graphs satisfying an Ore-type condition (proof is in Section 2).

Lemma 1. *Let H be a multipartite graph with classes $C_0, C_1, \ldots, C_m$ such that $|C_0| \geq |C_1| \geq \ldots \geq |C_m|$. If the following three conditions hold, then there is a matching of H with n edges:*
(1) $|V(H)| \geq 2n$,
(2) $d_H(u) + d_H(v) \geq 2n$ for every $uv \notin E(H)$ with $u \in C_i, v \in C_j$ and $i \neq j$,
(3) $|V(H - C_0)| \geq n$.

In case of $|C_0| = \cdots = |C_m| = 1$ Lemma 1 yields (an extension of) a folklore remark (Erdős and Pósa in [6] gave credit to Dirac): if $|V(H)| \geq 2n$ and $d_H(v) \geq n$ for every $v \in V(H)$ then there is a matching in H with n edges.

The other result we need (proof is in Section 3) is Theorem 3, an extension of a result about the 3-color Ramsey number $R(n_1K_2, n_2K_2, S_t)$, where n_iK_2 is a matching with n_i edges and S_t is a star with t edges. It was proved in [15] that for $n_1 \geq n_2 \geq 1$, $t \geq 1$,

$$R(n_1K_2, n_2K_2, S_t) = f(n_1, n_2, t) := \begin{cases} 2n_1 + n_2 - 1 & \text{if } t \leq n_1 \\ n_1 + n_2 - 1 + t & \text{if } t \geq n_1. \end{cases}$$

A 2-colored host graph G of order n with $\delta(G) \geq n - t$ can be considered as a 3-coloring of a K_n such that there is no star S_t in the third color. To handle a 2-colored host graph with an Ore-type condition, we need a more general result as follows.

Theorem 3. *Assume that $n_1 \geq n_2 \geq 1$, $t \geq 1$ and let G be a graph on $f(n_1, n_2, t)$ vertices such that for each pair of non-adjacent vertices, the sum of the number of their non-neighbors is at most $2(t-1)$. Then in any 2-coloring of the edges of G there exists either a matching of size n_1 in the first color or a matching of size n_2 in the second color.*

2 Matchings in multipartite graphs with Ore-type condition

In this section we prove Lemma 1. Let M be a maximum matching of H. Suppose to the contrary that $|M| < n$, and let $U \subset V(H)$ be the set of all vertices unsaturated by M. Then, by condition (1), $|U| \geq 2$, and if $u \in U$ and $uv \in E(H)$, then v is saturated by M.

Case 1: there are $u \in U \cap C_i$, $v \in U \cap C_j$, with $i \neq j$, and $uv \notin E(H)$.

By condition (2), the pair $\{u, v\}$ has at least $2n$ neighbors which are saturated by M. By the pigeon-hole principle, there is an edge $xy \in M$ incident with three edges from $\{u, v\}$. Then we have two independent edges, say $ux, vy \in E(G)$, and (u, x, y, v) is a path augmenting M, a contradiction.

Case 2: $U \subseteq C_i$, for some $i \neq 0$.

Since $|C_0| \geq |C_i|$, there is an edge $xy \in M$ such that $x \in C_0$ and $y \in C_j$, for some $j \notin \{0, i\}$. We claim that all neighbors of y are saturated by M. If this is not the case, then let $uy \in E(H)$, for some $u \in U$, and let $v \in U \setminus \{u\}$. Now $vx \notin E(H)$, since otherwise (v, x, y, u) is a path augmenting M. Then $M' = (M \setminus \{xy\}) \cup \{uy\}$ is a maximum matching which does not saturate $x \in C_0$ and $v \in C_i$, thus Case 1 applies. In a similar way, we obtain that all neighbors of x are saturated by M, in particular, $vx \notin E(H)$.

Now by (2), $d_H(u) + d_H(y) \geq 2n$, thus by the pigeon-hole principle there is an edge $x'y' \in M$ such that (x, y, x', y', u) is a path. Then $(M \setminus \{xy, x'y'\}) \cup \{yx', y'u\}$ is a maximum matching which does not saturate $x \in C_0$ and $v \in C_i$. Since $vx \notin E(H)$, Case 1 applies.

Case 3: $U \subseteq C_0$.

Assume that M saturates the maximum number of vertices of C_0 among all maximum matchings of H. Let $M_0 \subseteq M$ be the set of all edges of M with one end vertex in C_0. By the definition of M, every neighbor of $u \in U$ must be saturated by M_0. Let X be the set of all vertices $x \in V(H - C_0)$ such that, $ux \in E(H)$, for some $u \in U$, and let $Y = \{y \in C_0 \mid yx \in M_0, \text{ for some } x \in X\}$. Set $|X| = |Y| = n - t$ $(0 < t < n)$.

Observe that by (3), $M_0 \neq M$, let $vw \in M \setminus M_0$. If there is an edge $xy \in M_0$ and $u \in U$ such that $ux, vy \in E(H)$, then the set $M' = (M \setminus \{xy, vw\}) \cup \{ux, vy\}$ is a maximum matching which saturates the additional vertex $u \in C_0$, a contradiction.

Thus we obtain that v has all neighbors in $D = V(H) \setminus (U \cup Y)$. Since $d_H(u) \leq |X| = n - t$, by condition (2), we obtain $d_H(v) \geq n + t$. This implies $|D \setminus X| \geq d_H(v) - |X| \geq (n+t) - (n-t) = 2t$. Then the perfect matching of $D \setminus X$ which has at least t edges can be added to the $n - t$ edges of the perfect matching on $X \cup Y$ to obtain a matching of order n in H, a contradiction.

3 2-color Ramsey numbers of matchings in graphs with an Ore-type condition

In this section we prove Theorem 3. Let G be a 2-colored graph on $f(n_1, n_2, t)$ vertices such that for each pair of non-adjacent vertices, the sum of the number of their non-neighbors is at most $2(t-1)$. We shall prove that G contains either a matching of size n_1 in the first color or a matching of size n_2 in the second color.

Consider an arbitrary red-blue coloring of the edges of G. Notice that the case $t < n_1$ obviously follows from the case $t = n_1$, so we will assume that $|V(G)| = n_1 + n_2 - 1 + t$ and $t \geq n_1 \geq n_2$. We use induction on n_1; for $n_1 = 1$ (thus $n_2 = 1$), the statement is obvious, for every t.

In the induction step we reduce the triple (n_1, n_2, t) to $(n_1 - 1, n_2, t)$ if $n_1 > n_2$ and to $(n_1 - 1, n_1 - 1, t)$ if $n_1 = n_2$. Depending on which case we have, either there is a red matching of size $n_1 - 1$ or there is a blue matching of size n_2 or a blue matching of size $n_1 - 1$. If there is a blue matching of size n_2 there is nothing to prove. Otherwise, by switching colors if necessary, we may assume that there is a red matching of size $n_1 - 1$ and our goal is to find a blue matching of size n_2.

We will use the Berge-Tutte formula [5] several times in the paper. Let $G_r \subset G$ be the subgraph of all red edges of G. Defining $def(G_r) = |V(G_r)| - 2\nu(G_r)$, the deficiency of G_r, a well-known (e.g. see in [23]) form of the formula states that there is a cutset $X \subset V(G_r)$ such that $V(G_r) \setminus X$ is partitioned into $def(G_r) + |X|$ odd connected components. Then

$$def(G_r) = |V(G_r)| - 2\nu(G_r) = (n_1 + n_2 - 1) + t - 2(n_1 - 1) = t - n_1 + n_2 + 1,$$

and the number of odd components of $V(G_r) \setminus X$ in G_r is $t - n_1 + n_2 + 1 + |X|$. Label these components as $C_0, C_1, \ldots C_m$ so that the sizes are in decreasing order. Note that $m = t - n_1 + n_2 + |X| \geq 1$.

Let $H \subset G$ be the graph with vertex set $V(G) \setminus X$ and with all those edges of G which connect different C_i-s. Obviously all edges of H are blue. We shall prove that

H has a (blue) matching of size n_2. For this purpose we will apply Lemma 1 with H and n_2. It remains to check the three conditions of the lemma.

For (1) notice that the set X together with one vertex from each C_i, $i = 0, \ldots, m$, is included in $V(G)$, thus $|X|+(t-n_1+n_2+1+|X|) \leq |V(G)| = n_1+n_2-1+t$. Hence $|X| \leq n_1-1$, which implies $|V(H)| = |V(G)|-|X| \geq (n_1+n_2-1+t)-(n_1-1) \geq 2n_2$.

Secondly we have to consider non-adjacent vertices u and v in H such that $u \in C_i$ and $v \in C_j$, where $i \neq j$, and show that $d_H(u) + d_H(v) \geq 2n_2$. Assume to the contrary that $2n_2 > d_H(u) + d_H(v)$. The (co-)degree condition on G translates into $(|V(G)| - 1 - d_G(u)) + (|V(G)| - 1 - d_G(v)) \leq 2(t-1)$, implying $d_G(u) + d_G(v) \geq 2(n_1 + n_2 - 2 + t) - 2(t-1)$. This leads to

$$2n_2 > d_H(u) + d_H(v) \geq 2(n_1 + n_2 - 1) - 2|X| - (|C_i| - 1) - (|C_j| - 1),$$

where we subtract from $d_G(u)+d_G(v)$ the potential edges going from u and from v to X and to the vertices' own components, C_i and C_j. From here rearrangement gives $|C_i| + |C_j| > 2(n_1 - |X|)$. Now for the total number of vertices we have the following estimate:

$$\begin{aligned} |V(G)| &= n_1 + n_2 - 1 + t \geq |X| + |C_i| + |C_j| + m - 1 \\ &> |X| + 2(n_1 - |X|) + (t - n_1 + n_2 + |X| - 1) = n_1 + n_2 - 1 + t = |V(G)|, \end{aligned}$$

a contradiction.

Finally we have to verify $|V(H - C_0)| \geq n_2$. Indeed, by taking one vertex from each C_i different from C_0, and using $t \geq n_1$, we obtain

$$|V(H - C_0)| \geq t - n_1 + n_2 + |X| \geq n_2,$$

as desired.

4 2-color Ramsey numbers of connected matchings in graphs with an Ore-type condition

In this section we prove Theorem 1. Let G be a 2-edge colored graph with $3n-1$ vertices such that $d_G(u)+d_G(v) \geq \frac{3}{2}(3n-1)$, for any pair u, v of non-adjacent vertices. We shall prove that G has a monochromatic connected matching of size n.

Let O_1 be the vertex set of a largest monochromatic component of G, say red.

Case 1: $|O_1| < |V(G)|$.

Set $D = V(G) \setminus O_1$, and let A be the set of those vertices in O_1 which are adjacent to D by a blue edge.

Claim: $A \cup D$ is a connected blue component. Assume that $A \cup D$ has a cut $(A_1 \cup D_1, A_2 \cup D_2)$, w.l.o.g. $|A_1 \cup D_2| \geq |A_2 \cup D_1|$. By the definition of D and the cut, there is no edge between the non-empty sets $O_1 \setminus A_2$ and D_2. Thus $d_G(u) + d_G(v) \geq \frac{3}{2}(3n-1)$ for $u \in O_1 \setminus A_2, v \in D_2$. On the other hand,

$$d_G(u)+d_G(v) \leq (3n-2-|D_2|)+(3n-2-|O_1 \setminus A_2|) = 6n-4-(|D_2|+|O_1 \setminus A|+|A_1|)$$

$$< 6n-2-\frac{1}{2}(3n-1) = \frac{3}{2}(3n-1),$$

a contradiction proving the claim (in the last step we used $|A_1 \cup D_2| \geq |A_2 \cup D_1|$).

Let O_2 be the vertex set of the blue component covering D. Let $|O_1 \setminus O_2| = p$ and $|O_2 \setminus O_1| = q$. Since $u' \in O_1 \setminus O_2$ and $v' \in O_2 \setminus O_1$ are non-adjacent, $d_G(u')+d_G(v') \geq \frac{3}{2}(3n-1)$. If $d_G(u') < \frac{3}{4}(3n-1)$, for some $u' \in O_1 \setminus O_2$, then $d_G(v') \geq \frac{3}{4}(3n-1)$, for every $v' \in O_2 \setminus O_1$. By symmetry, we may assume $d_G(v) \geq \frac{3}{4}(3n-1)$ for all $v \in O_2 \setminus O_1$. This implies $p < (3n-1)/4$.

Case 1.1: $n/2 \leq p < (3n-1)/4$.

Let $p = \frac{3n-1}{4} - x$, for some $0 < x \leq n/4$. We first show that $d(u, O_1 \cap O_2) \geq 2(n-p)+p$, for each $u \in O_1 \setminus O_2$ (where $d(u, O_1 \cap O_2)$ is the number of neighbors of u in $O_1 \cap O_2$). Since $d_G(v) \leq (3n-1)-p$, the Ore-condition implies

$$d_G(u) \geq \frac{3}{2}(3n-1) - d_G(v) \geq \frac{3}{2}(3n-1) - (\frac{3}{4}(3n-1)+x) = \frac{3}{4}(3n-1) - x.$$

Therefore, $d(u, O_1 \cap O_2) \geq \frac{3}{4}(3n-1) - x - (p-1) = (3n+1)/2 \geq 2(n-p)+p$, since $p \geq n/2$.

We apply Theorem 3 to the subgraph $G[O_1 \cap O_2]$ with parameters $t = \left\lceil \frac{3n-1}{4} \right\rceil, n_1 = n-q, n_2 = n-p$ $(n_1 \geq n_2)$. We claim that with these choices of the parameters t, n_1, n_2 we have $|O_1 \cap O_2| = 3n-1-p-q \geq f(n_1, n_2, t)$. Indeed, for $t \leq n_1$ we have to check that $3n-1-p-q \geq 2(n-q)+(n-p)-1$ which reduces to $q \geq 0$. For $t > n_1$ we have to check $3n-1-p-q \geq (n-p)+(n-q)-1+t$ which reduces to $n \geq t$, obviously true for our choice of t. Thus by Theorem 3 (switching colors) we have either a red matching M of size $n-p$ or a blue matching M' of size $n-q$. In the former case, we can extend M to a connected matching of size n by including p additional edges, since any vertex $u \in O_1 \setminus O_2$ has at least p neighbors in $(O_1 \cap O_2) \setminus V(M)$. In the latter case, we observe that $d(v, O_1 \cap O_2) \geq \frac{3}{4}(3n-1)-(q-1) \geq 2(n-q)+q$, for any $v \in O_2 \setminus O_1$. Therefore we can extend M' by including q additional edges to obtain a connected blue matching of size n.

Case 1.2: $n/2 > p$.

By the previous paragraph we may assume that $G[O_1 \cap O_2]$ does not contain a blue matching of size $n-q$. We apply the Berge-Tutte formula for the subgraph $G_b \subset G[O_1 \cap O_2]$ formed by the blue edges of $G[O_1 \cap O_2]$. If $def(G_b)$ is the deficiency of G_b, then there exists a cutset $X \subset V(G_b)$ such that $V(G_b) \setminus X$ is the union of $|X| + def(G_b)$ odd components. Thus for the number of odd components we have

$$|X| + def(G_b) \geq |X| + (3n-1-p-q) - 2(n-q-1) = |X| + n - p + q + 1.$$

We include the set $O_1 \setminus O_2$ to the odd components and label them as $C_0, C_1, \dots, C_{m+1}$, where the sizes are in decreasing order and $m \geq |X| + n - p + q$.

Let us define a multipartite graph H with classes $C_0, C_1, \dots, C_{m+1}$ and with all red edges of G going between these classes (there are no blue edges between them). Since $V(H) \subset O_1$, a matching of H is a red connected matching. We claim that H satisfies the three conditions of Lemma 1.

First we deduce an upper bound on $|X|$. The sum of the size of X, plus at least 1 for each odd component, and $p+q$ is at most the total number of vertices. Thus $|X| + (|X| + n - p + q + 1) + p + q \leq 3n - 1$, and we have

$$|X| \leq n - q - 1.$$

This implies

$$|V(H)| = (|V(G)| - |O_2 \setminus O_1|) - |X| = (3n-1-q) - |X| \geq 3n-1-q-(n-q-1)-q = 2n,$$

which is condition (1) in Lemma 1.

Secondly we show that $d_H(u) + d_H(v) \geq 2n$, for non-adjacent vertices $u \in C_i$ and $v \in C_j$, where $i \neq j$. We will distinguish two subcases.

<u>Subcase a:</u> Neither C_i nor C_j is $O_1 \setminus O_2$.

Assume to the contrary that $2n > d_H(u) + d_H(v)$. We use the Ore-condition in G to get $d_H(u) + d_H(v) \geq \frac{3}{2}(3n-1) - 2q - 2|X| - (|C_i| - 1) - (|C_j| - 1)$, where we subtract the potential edges going from u and v to $(O_2 \setminus O_1)$, to X, and to their own components. Rearrangement gives

$$2|X| + |C_i| + |C_j| \geq 2.5n - 2q.$$

We observe $|O_1 \cap O_2| \geq |X| + |C_i| + |C_j| + (|X| + n - p + q - 1)$, by counting 1 vertex in each odd component different from C_i, C_j, and $O_1 \setminus O_2$. Using this bound on $|O_1 \cap O_2|$, for the total number of vertices we obtain the following estimation

$$\begin{aligned} |V(G)| &= p + q + |O_1 \cap O_2| \geq p + q + (2|X| + |C_i| + |C_j| + n - p + q - 1) \\ &= p + q + (2.5n - 2q) + (n - p + q - 1) = 3.5n - 1 > 3n - 1, \end{aligned}$$

a contradiction.

<u>Subcase b:</u> $C_i = O_1 \setminus O_2$.

Repeating the previous argument leads to a slightly different estimate:

$$2n > d_H(u) + d_H(v) \geq \frac{3}{2}(3n-1) - q - 2|X| - (|C_i| - 1) - (|C_j| - 1),$$

since now u has no neighbor in $(O_2 \setminus O_1)$. This implies $2|X| + |C_i| + |C_j| \geq 2.5n - q$. Then $|V(G)| \geq q + (2.5n - q) + (n - p + q) = 3.5n - p > 3n - 1$, a contradiction since $n/2 > p$.

Thirdly we have to control the size of the largest partition class C_0. If $C_0 \subseteq O_1 \cap O_2$, then $|V(H) \setminus C_0| \geq |O_1 \setminus O_2| + (|X| + n - p + q) \geq n$, since $p = |O_1 \setminus O_2|$. If $C_0 = O_1 \setminus O_2$, then using $|X| \leq n - q - 1$ we get $|V(H) \setminus C_0| = 3n - 1 - q - |C_0| - |X| \geq 3n - 1 - q - p - (n - q - 1) = 2n - p > n$, since $n/2 > p$, hereby finishing Case 1.

Case 2: $O_1 = V(G)$ (i.e. $q = 0$).

We suppose there is no red matching of size greater than $n - 1$. Apply again the Berge-Tutte formula on the red graph G_r by considering all vertices of G, but only the red edges. Then there exists a cutset $X \subset V(G_r)$ such that $V(G_r - X)$ is the union of $|X| + def(G_r)$ odd components, where the deficiency of G_r satisfies $def(G_r) \geq (3n - 1) - 2(n - 1) = n + 1$

Let us label the components again as $C_0, C_1, \ldots, C_m$, where the sizes are in decreasing order and $m \geq |X| + n$. We will apply Lemma 1 on the graph H that consists of $C_0, C_1, \ldots, C_m$ and the blue edges between these sets (there are no red edges between them). We have to verify the three premises of Lemma 1.

Since each odd component contains at least one vertex, we obtain $2|X| + n + 1 \leq |X| + (|X| + def(G_r)) \leq |V(G_r)| = 3n - 1$. Therefore $|X| < n$, and $|VH)| = |V(G) \setminus X| \geq 2n$ follows.

Secondly let $u \in C_i$ and $v \in C_j$, for $i \neq j$, two non-adjacent vertices of H. Observe that u and v are non-adjacent in G, by the definition of the (red) components C_i and C_j. Therefore $d_G(u) + d_G(v) \geq \frac{3}{2}(3n - 1)$. Assume to the contrary that $2n > d_H(u) + d_H(v)$. Since $d_G(u) \leq d_H(u) + |X| + (|C_i| - 1)$ and $d_G(v) \leq d_H(v) + |X| + (|C_j| - 1)$, we deduce $d_H(u) + d_H(v) \geq \frac{3}{2}(3n - 1) - 2|X| - |C_i| - |C_j| + 2$. That is, $2|X| + |C_i| + |C_j| \geq 2.5n$. Using again that each odd component contains at least one vertex we obtain:

$$\begin{aligned}
|V(G)| &\geq |X| + |C_i| + |C_j| + (m - 2) = |X| + |C_i| + |C_j| + (|X| + def(G_r) - 2) \\
&\geq 2.5n + (n + 1) - 2 = 3.5n - 1 > 3n - 1,
\end{aligned}$$

a contradiction.

Thirdly, we have to show $|V(H) \setminus C_0| \geq n$. Suppose to the contrary $|V(H) \setminus C_0| < n$. It yields $|C_0| + |X| = |V(G)| - |V(H) \setminus C_0| > 3n - 1 - n = 2n - 1$. Now again we use that each odd component contains at least one vertex: $|V(G)| \geq |X| + |C_0| + m - 1 =$

$(|X| + |C_0|) + (|X| + def(G_r)) - 1 \geq 2n + |X| + n \geq 3n$, a contradiction. That is, condition (3) holds.

Thus Lemma 1 yields a blue matching M of size n in H. Now this matching may not necessarily be connected. We finish the proof by showing that this M is indeed a connected matching in blue.

Claim: M is a connected blue matching.

The claim is certainly true if H is connected. Suppose to the contrary that H is disconnected. Let A be a connected component of H, which intersects the smallest component C_m and let $B = V(H) \setminus A$. First we observe that $|C_m| \leq 2$. Indeed, if $|C_m| \geq 3$, then each component has at least 3 vertices. Therefore $|V(G)| \geq 3(|X| + n + 1) > 3n - 1 = |V(G)|$, a contradiction.

We will pick a vertex $u \in C_m \cap A$ and an appropriate vertex $v \in B$. Assume first that there is a vertex $v \in C_i \cap B$, $i \notin \{0, m\}$. Since u and v are non-adjacent in G, we have $d_G(u) + d_G(v) \geq \frac{3}{2}(3n-1)$. Furthermore, observe $d_G(v) \leq (3n-2) - |A| + |C_i| - 1$, since v cannot be adjacent to a vertex of A except the ones in C_i by a possible red edge. Similarly $d_G(u) \leq (3n-2) - |B| + |C_m| - 1$.

Combining the inequalities above, we obtain

$$\begin{aligned} \frac{3}{2}(3n-1) &\leq d(u) + d(v) \leq 2(3n-2) - (|A| + |B|) + |C_i| - 1 + |C_m| - 1 \\ &= (3n-1) + (3n - 1 - |A| - |B|) + |C_i| + |C_m| - 4. \end{aligned}$$

Since $(3n-1) - |A| - |B| = |X|$, and using that $|C_m| \leq 2$ the previous inequality implies $(3n-1)/2 \leq |X| + |C_i| - 2$. This leads to the contradiction

$$\begin{aligned} 3n - 1 &\leq 2|X| + 2|C_i| - 4 \leq 2|X| + |C_i| + |C_0| - 4 \\ &\leq |X| + (|X| + n - 1) + |C_i| + |C_0| - 4 \leq |V(G)| - 4 < 3n - 1, \end{aligned}$$

where we use that $C_i \neq C_m$ and the number of the remaining odd components is at least $|X| + n - 1$. Thus if we could pick an appropriate vertex $v \in C_i \cap B$, $i \notin \{0, m\}$, then we would be done.

Observe first that there must be an edge $e \in M$ disjoint from A, since otherwise M is connected. If $|C_m| = 1$, then $e \cap C_m = \emptyset$, and $v \in e \setminus (C_0 \cup C_m)$ is an appropriate choice for v leading to a contradiction.

Assume now that $|C_m| = 2$. If we cannot pick a vertex v as before, then e goes between C_0 and $C_m \cap B$ (the other vertex in C_m). Then let v be the vertex of e in C_0. A computation identical to the above yields $(3n-1)/2 \leq |X| + |C_0| - 2$. Using this inequality and the fact that $|C_i| \geq |C_m| = 2$, for each $1 \leq i \leq m$, we obtain the contradiction

$$3n - 1 = |V(G)| \geq |X| + |C_0| + 2(|X| + n) \geq (3n-1)/2 + 2 + 2n > 3n - 1.$$

We conclude that H is a connected graph and the claim follows.

5 Applying the Regularity lemma; perturbations

As in many applications of the Regularity Lemma, one has to handle irregular pairs, that translates to exceptional edges in the reduced graph. A graph G on n vertices is *ε-perturbed* if at most $\varepsilon\binom{n}{2}$ of its edges are marked as exceptional (or perturbed). For a perturbed graph G, let G^- denote the graph obtained by removing all perturbed edges. We are not allowed to use the exceptional edges for our connected matching. Thus first we need a perturbed version of Theorem 1.

Theorem 4. *For every $\eta > 0$, there exist $n_0 = n_0(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)(\ll \eta)$ such that the following holds. Suppose that $\varepsilon \leq \varepsilon_0$ and G is a 2-edge-colored ε-perturbed graph on $n \geq n_0$ vertices and G satisfies the following Ore-type condition: for any two non-adjacent vertices x and y of G, we have $d_G(x)+d_G(y) \geq (3/2+\eta)n$. Then there exists a monochromatic connected matching in G^- spanning at least $(\frac{2}{3} - (\varepsilon)^{1/3})n$ vertices.*

These perturbation arguments are fairly standard modifications of the original argument, for example in [2] we presented all the details in a similar situation. Here we are not going to present all the details, we just present the perturbed version of Lemma 1 and its proof for demonstrative purposes. The other details are left to the interested reader.

Lemma 2. *For every $\eta > 0$, there exist $n_0 = n_0(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)(\ll \eta)$ such that the following holds for every $n \geq n_0$. Suppose that $\varepsilon \leq \varepsilon_0$ and let H be a multipartite graph with at most εn^2 exceptional edges and with classes $C_0, C_1, \ldots, C_m$ such that $|C_0| \geq |C_1| \geq \ldots \geq |C_m|$. If the following three conditions hold, then there is a matching in H^- with n edges:*
(1) $|V(H)| \geq (2 + 3\sqrt{\varepsilon})n$,
(2) $d_H(u) + d_H(v) \geq (2 + \eta)n$ *for every* $uv \notin E(H)$ *with* $u \in C_i, v \in C_j$ *and* $i \neq j$,
(3) $|V(H - C_0)| \geq (1 + \sqrt{\varepsilon})n$.

Proof of Lemma 2. We may assume that n is sufficiently large and $\varepsilon \ll \eta$. Let us start by the standard "trimming" of the graph, i.e. by deleting those vertices of H that are adjacent to at least $\sqrt{\varepsilon}n$ exceptional edges. There are less than $\sqrt{\varepsilon}n$ such vertices. This way we get a slightly smaller graph H_ε, with $|V(H_\varepsilon)| \geq (2 + 2\sqrt{\varepsilon})n$. By renaming we may assume that C_0 is still the largest class, from condition (3) we still have $|V(H_\varepsilon - C_0)| \geq n$. Secondly we delete the remaining exceptional edges to form the graph H_ε^-. We will find a matching of size n in H_ε^-. We will denote the complement of a class of vertices in H_ε by $\overline{C_i} = \cup\{C_j \mid j \neq i\}$.

Let M be a maximum matching of H_ε^-. Suppose to the contrary that $|M| < n$, and let $U \subset V(H_\varepsilon)$ be the set of all vertices of H_ε unsaturated by M. Now $|U| > 2\sqrt{\varepsilon}n$, and if $u \in U$ and $uv \in E(H_\varepsilon^-)$, then v must be saturated by M.

Case 1: there exists a vertex $u \in U \cap C_i$ such that $|U \cap \overline{C_i}| \geq \sqrt{\varepsilon} n$.

In this case we may pick a vertex v in $U \cap \overline{C_i}$ such that u and v are non-adjacent in H (since u has fewer than $\sqrt{\varepsilon} n$ exceptional neighbors). By condition (2), the pair $\{u, v\}$ has at least $(2 + \eta - 2\sqrt{\varepsilon})n > 2n$ non-exceptional neighbors and they are saturated by M. By the pigeon-hole principle, there is an edge $xy \in M$ incident to three non-exceptional edges coming from $\{u, v\}$. Therefore, there are two independent non-exceptional edges, say $ux, vy \in E(G)$, and (u, x, y, v) is a path augmenting M, a contradiction.

Note that again if Case 1 does not hold, we must have $U \subseteq C_i$, for some i, since $|U| > 2\sqrt{\varepsilon} n$. Indeed, consider a C_i such that $|U \cap C_i| > 0$. We have two possibilities: either $|U \cap C_i| \geq \sqrt{\varepsilon} n$ or $|U \cap \overline{C_i}| \geq \sqrt{\varepsilon} n$. The latter is in Case 1, and the former is also in Case 1 if $|U \cap \overline{C_i}| > 0$. Thus otherwise in fact $U \subseteq C_i$.

Case 2: $U \subseteq C_i$, for some i, where $i \neq 0$.

Since $|C_0| \geq |C_i|$, there is an edge $xy \in M$ such that $x \in C_0$ and $y \in C_j$ for some $j \notin \{0, i\}$. We claim that all non-exceptional neighbors of y are saturated by M. If this is not the case, then let $uy \in E(H_\varepsilon^-)$ for some $u \in U$. Let $v \in U \setminus \{u\}$ be a vertex such that vx is not an exceptional edge. Then $vx \notin E(H_\varepsilon^-)$, since otherwise (v, x, y, u) is a path augmenting M. Thus v and x are non-adjacent in H. Now $M' = (M \setminus \{xy\}) \cup \{uy\}$ is a maximum matching which does not saturate $x \in C_0$ and $v \in C_i$, which are non-adjacent in H, and thus we can proceed as in Case 1. In a similar way, we obtain that all non-exceptional neighbors of x are saturated by M. Thus if $v \in U$ is such that vx is not an exceptional edge, then $vx \notin E(H_\varepsilon^-)$. Let u be a vertex in U different from v so that uy is not an exceptional edge and thus u and y are non-adjacent in H. Now by condition (2), the pair $\{u, y\}$ has at least $(2 + \eta - 2\sqrt{\varepsilon})n > 2n$ non-exceptional neighbors and they are saturated by M. By the pigeon-hole principle there is an edge $x'y' \in M$ such that (x, y, x', y', u) is a path (vertices x' and y' may be reversed). Therefore, $(M \setminus \{xy, x'y'\}) \cup \{yx', y'u\}$ is a maximum matching which does not saturate $x \in C_0$ and $v \in C_i$. Since $vx \notin E(H)$, we can proceed as in Case 1.

Case 3: $U \subseteq C_0$.

Assume that M saturates the maximum number of vertices of C_0 among all maximum matchings of H_ε^-. Let $M_0 \subseteq M$ be the set of all edges of M with one end vertex in C_0.

By the definition of M, every non-exceptional neighbor of $u \in U$ must be saturated by M_0. Let X be the set of all vertices $x \in V(H_\varepsilon - C_0)$ such that $ux \in E(H_\varepsilon^-)$, for some $u \in U$, and let $Y = \{y \in C_0 \mid yx \in M_0, \text{ for some } x \in X\}$. Set $|X| = |Y| = n - t$ $(0 < t < n)$.

Observe that by $|V(H_\varepsilon - C_0)| \geq n$ we have $M_0 \neq M$, and thus let $vw \in M \setminus M_0$. If there is an edge $xy \in M_0$ and $u \in U$ such that $ux, vy \in E(H_\varepsilon^-)$, then the set $M' = (M \setminus \{xy, vw\}) \cup \{ux, vy\}$ is a maximum matching which saturates the additional vertex $u \in C_0$, a contradiction.

Thus we obtain that v has all non-exceptional neighbors in $D = V(H_\varepsilon) \setminus (U \cup Y)$. Pick a vertex $u \in U$ such that uv is not an exceptional edge (using $|U| > 2\sqrt{\varepsilon}n$), thus u and v are non-adjacent in H. Then since $d_{H_\varepsilon^-}(u) \leq |X| = n - t$, by condition (2), we obtain $d_{H_\varepsilon^-}(v) \geq n+t+\eta n-2\sqrt{\varepsilon}n \geq n+t$. This implies $|D \setminus X| \geq d_{H_\varepsilon^-}(v)-|X| \geq (n+t)-(n-t) = 2t$. Now M induces a perfect matching on $D \setminus X$, that has at least t edges. We can add these edges to the $n-t$ edges of the perfect matching on $X \cup Y$ to obtain a matching of order n in H_ε^-, a contradiction. $\square$

6 Building paths from connected matchings

Next we show how to prove Theorem 2 from Theorem 4 and the Regularity Lemma [22]. The material of this section is again fairly standard by now (see e.g. [1, 11, 12, 13, 14, 15]) so we omit some of the details. The discussion closely follows the treatment in [2] where also an Ore-type condition was transferred to the reduced graph.

We use a 2-edge-colored version of the Regularity Lemma.[1]

Lemma 3. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \dots, V_\ell$ such that*

- *$m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \cdots = |V_\ell|$, $|V_0| < \varepsilon n$,*
- *apart from at most $\varepsilon\binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are ε-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 2$.*

Proof of Theorem 2: Fixing an $\eta \ll 1$, let $\varepsilon \ll \rho \ll \eta$, and let m_0 be sufficiently large compared to $1/\varepsilon$ (so we will be able to apply Theorem 4 in the reduced graph). Lemma 3 with parameters ε, m_0 defines M_0. Let G be a graph on $n \geq M_0$ vertices such that for any two non-adjacent vertices x and y of G, we have $d_G(x)+d_G(y) \geq (\frac{3}{2}+\eta)n$. Consider a 2-edge-coloring of G, that is $G = G_1 \cup G_2$. Let $V = \cup_{0 \leq i \leq \ell} V_i$ be the partition ensured by Lemma 3, set $|V_i| = L$ for $1 \leq i \leq \ell$.

We define the *reduced graph* G^R as follows. The vertices $p_1, \dots, p_\ell$ of G^R correspond to the clusters. There is an *exceptional edge* between vertices p_i and p_j if the pair (V_i, V_j) is ε-irregular in G_1 or in G_2. If the pair (V_i, V_j) is ε-regular in both G_1 and G_2 with density in G exceeding ρ, then $p_i p_j$ is a (non-exceptional) edge of G^R.

Note that G^R is an ε-perturbed graph where a non-edge corresponds to a regular pair with density is at most ρ. Any edge $p_i p_j$ is colored by the color which is used on

[1]For background, this variant and other variants of the Regularity Lemma see [18].

most edges of $G[V_i, V_j]$ (the bipartite subgraph of G with edges between V_i and V_j). If the edge is non-exceptional, the density of this majority color is still at least $\rho/2$ in $G[V_i, V_j]$. This defines a 2-edge-coloring $G^R = G_1^R \cup G_2^R$.

We claim that G^R inherits a similar Ore-type condition from G: for any two non-adjacent vertices p_i and p_j of G^R, we have $d_{G^R}(p_i) + d_{G^R}(p_j) \geq (\frac{3}{2} + \frac{\eta}{2})\ell$. Indeed, let p_i and p_j be non-adjacent in G^R and consider the corresponding clusters V_i and V_j. Set

$$S = \sum_{u \in V_i} \sum_{v \in V_j} (d_G(u) + d_G(v)).$$

By definition, the number of non-edges in $G[V_i, V_j]$ is at least $(1-\rho)|V_i||V_j| = (1-\rho)L^2$. For each of these non-edges we can use the Ore-condition in G so we get the following lower bound for S:

$$S \geq (1-\rho)L^2 \left(\frac{3}{2} + \eta\right) n.$$

On the other hand we can get the following upper bound for S:

$$S \leq (d_{G^R}(p_i) + d_{G^R}(p_j))L^3 + 2\rho n L^2 + 2\varepsilon n L^2 + 2L^3$$

where the main term estimates the degrees to clusters corresponding to neighbors of p_i, p_j; the first error term is an upper bound for the number of edges to clusters corresponding to non-neighbors of p_i, p_j (where the density is at most ρ); the second error term stands for the number of edges of G from $V_i \cup V_j$ to V_0 and finally the third error term is an upper bound for the number of edges within V_i and V_j. Comparing the bounds of S and using that $\frac{n}{L} \geq \ell$, we get

$$d_{G^R}(p_i) + d_{G^R}(p_j) \geq \left(\left(\frac{3}{2} + \eta\right)(1-\rho) - 2\rho - 2\varepsilon\right)\frac{n}{L} - 2 \geq \left(\frac{3}{2} + \frac{\eta}{2}\right)\ell,$$

as desired, because ε, ρ are small compared to η and ℓ is large enough in terms of $\frac{1}{\varepsilon}$, more precisely, we need

$$\left(\frac{\eta}{2} - \rho\left(\frac{3}{2} + \eta\right) - 2\rho - 2\varepsilon\right)\ell \geq 2.$$

Applying Theorem 4 to the 2-colored, ε-perturbed and Ore-type G^R, we get a monochromatic connected matching, say in $(G_1^R)^-$, that spans at least a $(\frac{2}{3} - (\varepsilon)^{1/3})$-fraction of G^R. Finally, we lift this connected matching back to a path in the original graph using the following lemma[2] in our context.

[2] As in [12, 13, 14, 15].

Lemma 4. *Assume that there is a monochromatic connected matching M (say in $(G_1^R)^-$) saturating at least $c|V(G^R)|$ vertices of G^R, for some positive constant c. Then in the original G there is a monochromatic path in G_1 covering at least $c(1-3\varepsilon)n$ vertices.*

Using our choice of $\varepsilon \ll \eta$ we obtain that G has a monochromatic path with at least $(\frac{2}{3}-\eta)n$ vertices thus concluding the proof of Theorem 2. $\square$

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