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NEW REFINEMENTS OF HÖLDER AND MINKOWSKI INEQUALITIES WITH WEIGHTS

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Abstract. In this paper, we present on new refinements of the discrete Jensen's inequality given in [3] and [4]. Our results are more general than the refinement results given in [5]. Also the parameter dependent results correspond to some new refinements of Hölder's and Minkowski's inequalities.

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1. Introduction and Preliminary Results

The well known discrete Jensen's inequality says: Let U be a convex subset of a real linear space, and let $f : U \to \mathbb{R}$ be a convex function. If $x_i \in U$ $(1 \le i \le n)$ and $p_i \ge 0$ $(1 \le i \le n)$ are such that $\sum\limits_{i=1}^{n} p_i = 1$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds.

Let $I \subset \mathbb{R}$ be an interval, let $h : I \to \mathbb{R}$ be a continuous and strictly monotone function, let $\mathbf{a} = (a_1, \dots, a_n) \in I^n$, and let $\mathbf{p} = (p_1, \dots, p_n)$ be a nonnegative n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$. The quasi-arithmetic h-mean of $\mathbf{a}$ with weights $\mathbf{p}$ is defined by

$$h_n(\mathbf{a}; \mathbf{p}) = h_n(a_i; 1 \le i \le n; \mathbf{p}) = h(\mathbf{a}; \mathbf{p}; n) := h^{-1}\left(\sum_{i=1}^{n} p_i h(a_i)\right).$$

If $p_i = \frac{1}{n}$ $(1 \le i \le n)$, then $\mathbf{p}$ will be ignored from the previous notations.

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The following hypothesis is utilized in [5] to extend Beck's results (see [1]):

(A_1) Let $L_t : I_t \to \mathbb{R}$ $(t = 1, \dots, m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times \cdots \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \geq 2)$ be such that $\mathbf{x}^{(t)} := (x_1^{(t)}, \dots, x_n^{(t)}) \in I_t^n$ for each $t = 1, \dots, m$, and let $\mathbf{p} = (p_1, \dots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

The following extension of Beck's result, given in [5], is a simple consequence of the discrete Jensen's inequality.

Theorem 1.1. *Assume* (A_1). *If N is an increasing function, then the inequality*

$$f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \dots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \geq$$
$$\geq N^{-1}\left(\sum_{i=1}^n p_i N(f(x_i^{(1)}, \dots, x_i^{(m)}))\right), \tag{2}$$

holds for all possible $\mathbf{x}^{(t)}$ $(t = 1, \dots, m)$ and $\mathbf{p}$, if and only if the function H defined on $L_1(I_1) \times \cdots \times L_m(I_m)$ by

$$H(t_1, \dots, t_m) := N\left(f\left(L_1^{-1}(t_1), \dots, L_m^{-1}(t_m)\right)\right)$$

is concave. The inequality in (2) *is reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1, \dots, m)$ and $\mathbf{p}$, if and only if H is convex.*

Beck's original result was the special case of Theorem 1.1, where $m = 2$ and $I_1 = [k_1, k_2]$, $I_2 = [l_1, l_2]$ and $I_N = [n_1, n_2]$ (see [2], p. 249).

In the case $m = 2$ we shall use the following simplified form of (A_1):

(A_2) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \dots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

Then (2) has the form

$$f\big(K_n(\mathbf{a}; \mathbf{p}), L_n(\mathbf{b}; \mathbf{p})\big) \geq N_n(f(\mathbf{a}, \mathbf{b}); \mathbf{p}), \tag{3}$$

where $f(\mathbf{a}, \mathbf{b}) := (f(a_1, b_1), \dots, f(a_n, b_n))$.

The following results (see [5]) are important special cases of Theorem 1.1, and generalize the corresponding results of Beck [5]. The next hypothesis will be used:

(A_3) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x, y) = x + y$ $((x, y) \in I_K \times I_L)$ or $I_K, I_L \subset]0, \infty[$, $I_K \cdot I_L \subset I_N$ and $f(x, y) = xy$ $((x, y) \in I_K \times I_L)$. Assume further that the functions K, L and N are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ be such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

A° means the interior of $A \subset \mathbb{R}$.

Corollary 1.2. *Assume* (A_3) *with* $f(x, y) = x + y$ $((x, y) \in I_K \times I_L)$, *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (3) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x + y), \quad (x, y) \in I_K^\circ \times I_L^\circ.$$

Corollary 1.3. *Assume* (A_3) *with* $f(x, y) = xy$ $((x, y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. Assume further that* K', L', N', A, B *and* C *are all positive. Then* (3) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x, y) \in I_K^\circ \times I_L^\circ.$$

In [3], Mitrinović and Pečarić obtained a new inequality like (3), which is based on the following refinement of the discrete Jensen's inequality (see Pečarić and Volenec [9]):

Lemma A. *Let* f *be a real valued convex function defined on a convex set* U *from a real linear space. If* $x_1, \ldots, x_n \in U$, *and*

$$f_{k,n} = f_{k,n}(x_1, \ldots, x_n) := = \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}\left(x_{i_1} + \cdots + x_{i_k}\right)\right), \quad 1 \leq k \leq n, \tag{4}$$

then

$$f\left(\sum_{i=1}^n \frac{1}{n} x_i\right) = f_{n,n} \leq \cdots \leq f_{k,n} \leq \cdots \leq f_{1,n} = \frac{1}{n} \sum_{i=1}^n f(x_i). \tag{5}$$

Assume (A_2). We denote by α_i^k $(1 \leq i \leq v)$ and β_i^k $(1 \leq i \leq v)$ the k-tuples of $\mathbf{a}$ and $\mathbf{b}$ respectively, where $v = \begin{pmatrix} n \\ k \end{pmatrix}$. Following [7], we introduce the mixed N-K-L means of $\mathbf{a}$ and $\mathbf{b}$:

$$M(N, K, L; k) := N_v\big(f(K_k(\alpha_i^k), L_k(\beta_i^k)); 1 \leq i \leq v\big), \quad 1 < k < n,$$

and

$$\begin{aligned} M(N, K, L; 1) :=& N_n(f(\mathbf{a}, \mathbf{b})), \\ M(N, K, L; n) :=& f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right). \end{aligned}$$

These means are studied in [7] (see also [8] page 195):

Theorem A. *Assume* (A_2). *Let* N *be an increasing* (*decreasing*) *function, and let*

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(s,t) := N\left(f\left(K^{-1}(s), L^{-1}(t)\right)\right)$$

be a convex (*concave*) *function. Then*

$$M(N,K,L;k+1) \le M(N,K,L;k), \quad k = 1,\dots,n-1. \tag{6}$$

If N *is increasing* (*decreasing*) *but* H *is concave* (*convex*) *then the inequalities in* (6) *are reversed.*

In analogy of Corollary 1.2 and Corollary 1.3, the following consequences of Theorem A are given in [5, 7, 8].

Corollary A. *Assume* (A_3) *with* $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$. *Assume further that* K', L', N', K'', L'' *and* N'' *are all positive and* $E(x) + F(y) \le G(x+y)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *where* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$. *Then* (6) *with reverse inequality is valid.*

Corollary B. *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. If* K', L', M', A, B *and* C *are all positive and* $A(x) + B(y) \le C(xy)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, *then* (6) *with reverse inequality is valid.*

The results given in [7] are without weights. By using the refinement of the discrete Jensen's inequality from [6], we gave results in [5] with weights, which cause the improvement of the results in [7]. But in this paper we work on the refinement given in [3] to establish the generalizations of the corresponding results given in [5]. Also we present some parameter dependent refinements of Hölder and Minkowski's inequalities with the help of [4]. First, we give the notations from [3]:

Let X be a set. The power set of X is denoted by $P(X)$. $|X|$ means the number of elements in X. For every nonnegative integer d, let

$$P_d(X) := \left\{Y \subset X \mid |Y| = d\right\}.$$

In the sequel we also need the following hypotheses:

(H_1) Let U be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$.

(H_2) Let $\mathbf{p} := (p_1, \dots, p_n)$ be a positive n-tuple such that $\sum_{i=1}^n p_i = 1$.

(H_3) Let $f : U \to \mathbb{R}$ be a convex function.

(H_4) Let $S_1, \dots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^n S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

(H_5) Suppose $\mathcal{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max \left\{ |A| \mid A \in \mathcal{A} \right\},$$

and let

$$\mathcal{A}_l := \left\{ A \in \mathcal{A} \mid |A| = l \right\}, \quad l = 1, \ldots, k.$$

(We note that $\mathcal{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and of course, $|S| = \sum\limits_{l=1}^{k} l \, |\mathcal{A}_l|$.) The empty sum of numbers or vectors is taken to be zero.

The following refinement of the discrete Jensen's inequality is developed in [3]:

Theorem B. *If* (H_1)–(H_5) *are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j \mathbf{x}_j \right) \le M_k \le M_{k-1} \le \cdots \le M_2 \le M_1 = \sum_{j=1}^{n} p_j f(\mathbf{x}_j),$$

where

$$M_k := \sum_{l=1}^{k} \left(\sum_{A \in \mathcal{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) f \left(\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \right) \right), \tag{7}$$

and for every $1 \le d \le k-1$ *the number* M_{k-d} *is given by*

$$M_{k-d} := \sum_{l=1}^{d} \left(\sum_{A \in \mathcal{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} f(\mathbf{x}_{\tau(s)}) \right) \right) + \sum_{l=d+1}^{k} \left(\frac{d!}{(l-1)\ldots(l-d)} \cdot \right.$$

$$\left. \cdot \sum_{A \in \mathcal{A}_l} \left(\sum_{B \in P_{l-d}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) f \left(\frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}} \right) \right) \right) \right). \tag{8}$$

A parameter dependent refinement of the discrete Jensen's inequality is obtained in [4].

Theorem C. For any real number $\lambda \ge 1$, we suppose (H_1)–(H_3) and consider the sets

$$T_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \,\middle|\, \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N}. \tag{9}$$

Let

$$C_k(\lambda) = C_k(\mathbf{x}_1,\dots,\mathbf{x}_n;p_1,\dots,p_n;\lambda) :=$$
$$= \frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\frac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)f\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j\mathbf{x}_j}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}\right), \quad (10)$$

for any $k\in\mathbb{N}$. Then

$$f\left(\sum_{j=1}^{n}p_j\mathbf{x}_j\right) = C_0(\lambda)\le C_1(\lambda)\le\dots\le C_k(\lambda)\le\dots\le\sum_{j=1}^{n}p_jf(\mathbf{x}_j), \quad k\in\mathbb{N}.$$

2. New Generalizations of Beck's Result

Assume (A_1) with positive n-tuple $\mathbf{p}$, (H_4) and (H_5). Let

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};B) = L_t^{-1}\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}L_t(x^{(t)}_{\tau(s)})}{\sum\limits_{s\in B}c(s)p_{\tau(s)}}\right),$$
$$t=1,\dots,m, \quad B\subset S,$$

and let

$$\mathbf{x}_i := \left(x_i^{(1)},\dots,x_i^{(m)}\right), \quad i=1,\dots,n.$$

Then weighted mixed means corresponding to (7) and (8) are defined in the following ways:

$$M_k^1 := M_k^1(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};c\mathbf{p}) :=$$
$$= N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$
$$\left.\left.\left.\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)};c\mathbf{p};A),\dots,L_m(\mathbf{x}^{(m)};c\mathbf{p};A)\big)\Big)\right)\right)\right),$$

and for $1\le d\le k-1$

$$M_{k-d}^1 := M_{k-d}^1\big(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};c\mathbf{p}\big) :=$$
$$= N^{-1}\left\{\sum_{l=1}^{d}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}N(f(\mathbf{x}_{\tau(s)}))\right)\right)+\right.$$
$$+\sum_{l=d+1}^{k}\left(\frac{d!}{(l-1)\dots(l-d)}\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-d}(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$

$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}; c\mathbf{p}; B), \ldots, L_m(\mathbf{x}^{(m)}; c\mathbf{p}; B)\big)\Big)\Big)\Big)\Big\}.$$

Now, we get an interpolation of (2) by the direct application of Theorem B as follows.

Theorem 2.1. *Assume* (A_1) *with a positive n-tuple* $\mathbf{p}$, (H_4) *and* (H_5). *If N is a strictly increasing (decreasing) function, then the inequalities*

$$f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \leq M_k^1 \leq M_{k-1}^1 \leq \cdots \leq$$
$$\leq M_2^1 \leq M_1^1 = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(\mathbf{x}_i))\right), \tag{11}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$, *if and only if the function H defined in Theorem* 1.1 *is convex (concave). If N is a strictly increasing (decreasing) function, then the inequalities in* (11) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$, *if and only if H is concave (convex).*

Proof. It follows from Theorem B and Theorem 1.1. We apply Theorem B to m-tuples

$$\left(L_1\left(x_i^{(1)}\right), \ldots, L_1\left(x_i^{(m)}\right)\right), \quad i = 1, \ldots, n,$$

and the function H if either H is convex and N is strictly increasing or H is concave and N is strictly decreasing. $-H$ is used if either H is convex and N is strictly decreasing or H is concave and N is strictly increasing. □

The following applications of Theorem 2.1 are based on special cases of Theorem B from [3].

Example 2.2. Let $n \geq 1$ and $k \geq 1$ be fixed integers, and let $I_k \subset \{1, \ldots, n\}^k$ such that

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ means the number of occurrences of i in the sequences $\mathbf{i}_k := (i_1, \ldots, i_k) \in I_k$. For $j = 1, \ldots, n$ we introduce the sets

$$S_j := \Big\{ ((i_1, \ldots, i_k), l) \mid (i_1, \ldots, i_k) \in I_k, \quad 1 \leq l \leq k, \quad i_l = j \Big\}.$$

Let c be a positive function on $S := \bigcup_{j=1}^{n} S_j$ such that

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1, \ldots, i_k), l) = 1, \quad j = 1, \ldots, n.$$

Assume (A_1) with a positive n-tuple $\mathbf{p}$. Then the corresponding weighted mixed means are

$$M_k^1 := N^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k} c\left(\left(i_1,\ldots,i_k\right),l\right)p_{i_l}\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k),\ldots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k)\big)\Big)\right)\right),$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k)=L_t^{-1}\left(\frac{\sum_{l=1}^{k} c\left(\left(i_1,\ldots,i_k\right),l\right)p_{i_l}L_t(x_{i_l}^{(t)})}{\sum_{l=1}^{k} c\left(\left(i_1,\ldots,i_k\right),l\right)p_{i_l}}\right),$$
$$\mathbf{i}_k\in I_k,\ 1\le t\le m,$$

while for $1\le d\le k-1$,

$$M_{k-d}^1 :=N^{-1}\left\{\left(\frac{d!}{(k-1)\ldots(k-d)}\cdot\right.\right.$$
$$\cdot\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{1\le l_1<\ldots<l_{k-d}\le k}\left(\left(\sum_{j=1}^{k-m} c\big((i_1,\ldots,i_k),l_j\big)p_{i_{l_j}}\right)\cdot\right.\right.$$
$$\left.\left.\left.\left.\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}),\ldots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\big)\Big)\right)\right)\right)\right\},$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})=L_t^{-1}\left(\frac{\sum_{j=1}^{k-d} c\big((i_1,\ldots,i_k),l_j\big)p_{i_{l_j}}L_t(x_{i_{l_j}}^{(t)})}{\sum_{j=1}^{k-d} c\big((i_1,\ldots,i_k),l_j\big)p_{i_{l_j}}}\right),$$
$$1\le l_1<\cdots<l_{k-d}\le k,\quad 1\le t\le m.$$

If N is strictly increasing and the function H defined in Theorem 1.1 is convex, then Theorem 2.1 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)\le M_k^1\le M_{k-1}^1\le\cdots\le$$
$$\le M_2^1\le M_1^1=N^{-1}\left(\sum_{i=1}^{n}p_iN(f(x_i^{(1)},\ldots,x_i^{(m)}))\right). \tag{12}$$

Taking

$$c((i_1,\ldots,i_k),l)=\frac{1}{|S_j|}=\frac{1}{\alpha_{I_k,j}},\quad ((i_1,\ldots,i_k),l)\in S_j,$$

in (12) we get Theorem 2.1 of [5].

Example 2.3. Let n, d, r be fixed integers, where $n \geq 3$, $d \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1, 2, \ldots, n$ and for every $l = 0, 1, \ldots, r$ the integer $i+l$ will be identified with the uniquely determined integer j from $\{1, \ldots, n\}$ for which

$$l + i \equiv j \quad (mod\ n). \tag{13}$$

Introducing the notation

$$D := \{1, \ldots, n\} \times \{0, \ldots, r\},$$

let for every $j \in \{1, \ldots, n\}$

$$S_j := \Big\{ (i,l) \in D \mid i + l \equiv j \quad (mod\ n) \Big\} \bigcup \{j\},$$

and let $\mathcal{A} \subset P(S)$ $\big(S := \bigcup\limits_{j=1}^{n} S_j\big)$ contain the following sets:

$$A_i := \big\{(i,l) \in D \mid l = 0, \ldots, r\big\}, \quad i = 1, \ldots, n$$

and

$$A := \{1, \ldots, n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l)\in S_j} c\,(i,l) + c\,(j) = 1, \quad j = 1, \ldots, n.$$

A careful verification shows that the sets $S_1, \ldots, S_n$, the partition $\mathcal{A}$ and the function c defined above satisfy the conditions (H_4) and (H_5),

$$\tau\,(i,l) = i + l, \quad (i,l) \in D,$$

(by the agreement (see (13)), $i + l$ is identified with j)

$$\tau\,(j) = j, \qquad j = 1, \ldots, n,$$
$$|S_j| = r + 2, \quad j = 1, \ldots, n,$$

and

$$|A_i| = r + 1, \quad i = 1, \ldots, n, \quad |A| = n.$$

Assume (A_1) with a positive n-tuple $\mathbf{p}$. If N is increasing and the function H defined in Theorem 1.1 is convex, then from Theorem 2.1 we get

$$f\Big(L_1(\mathbf{x}^{(1)};\mathbf{p};n), \ldots, L_m(\mathbf{x}^{(m)};\mathbf{p};n)\Big) \leq$$
$$\leq N^{-1}\Bigg\{\sum_{i=1}^{n}\Bigg(\sum_{l=0}^{r} c(i,l)p_{i+l}\Bigg)N\Big(f\big(L_1(\mathbf{x}^{(1)},c\mathbf{p};i), \ldots, L_m(\mathbf{x}^{(m)},c\mathbf{p};i)\big)\Big) +$$
$$+\Bigg(\sum_{j=1}^{n} c(j)p_j\Bigg)N\Big(f\big(L_1(\mathbf{x}^{(1)},c\mathbf{p}), \ldots, L_m(\mathbf{x}^{(m)},c\mathbf{p})\big)\Big)\Bigg\} \leq$$

$$\leq N^{-1}\Bigg(\sum_{i=1}^{n} p_i N\Big(f\big(x_i^{(1)},\ldots,x_i^{(m)}\big)\Big)\Bigg),$$

where

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}; i) = L_t^{-1}\left(\frac{\sum\limits_{l=0}^{r} c\,(i,l)\,p_{i+l} L_t(x_{i+l}^{(t)})}{\sum\limits_{l=0}^{r} c\,(i,l)\,p_{i+l}}\right),$$

$$1 \leq i \leq n, \quad 1 \leq t \leq m,$$

and

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}) = L_t^{-1}\left(\frac{\sum\limits_{j=1}^{n} c(j) p_j L_t(x_j^{(t)})}{\sum\limits_{j=1}^{n} c(j) p_j}\right), \quad 1 \leq t \leq m.$$

Example 2.4. Let n and k be fixed positive integers. Let

$$D := \Big\{ (i_1,\ldots,i_n) \in \{1,\ldots,k\}^n \mid i_1+\cdots+i_n = n+k-1 \Big\},$$

and for each $j = 1,\ldots,n$, denote S_j the set

$$S_j := D \times \{j\}\,.$$

For every $\mathbf{i}_n := (i_1,\ldots,i_n) \in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)} := \{((i_1,\ldots,i_n)\,, l) \mid l = 1,\ldots,n\}\,.$$

It is obvious that S_j $(j = 1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n) \in D)$ are decompositions of $S := \bigcup\limits_{j=1}^{n} S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c\,((i_1,\ldots,i_n)\,, j) > 0, \quad ((i_1,\ldots,i_n)\,, j) \in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} c\,((i_1,\ldots,i_n)\,, j) = 1, \quad j = 1,\ldots,n.$$

In summary we have that the conditions (H_5) and (H_6) are valid, and

$$\tau\,((i_1,\ldots,i_n)\,, j) = j, \quad ((i_1,\ldots,i_n)\,, j) \in S.$$

Assume (A_1) with positive n-tuple $\mathbf{p}$. If N is strictly increasing and the function H defined in Theorem 1.1 is convex, then from Theorem 2.1 we get

$$f\Big(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\Big) \leq$$

$$\leq N^{-1}\Bigg(\sum_{(i_1,\ldots,i_n)\in D}\Bigg(\Bigg(\sum_{l=1}^{n} c\big((i_1,\ldots,i_n),l\big)p_l\Bigg)\cdot$$

$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}, c\mathbf{p}; \mathbf{i}_n), \ldots, L_m(\mathbf{x}^{(m)}, c\mathbf{p}; \mathbf{i}_n)\big)\Big)\Bigg) \leq$$

$$\leq N^{-1}\Bigg(\sum_{i=1}^{n} p_i N\big(f(x_i^{(1)}, \ldots, x_i^{(m)})\big)\Bigg),$$

where

$$L_t(\mathbf{x}^{(t)}, c\mathbf{p}; \mathbf{i}_n) = L_t^{-1}\left(\frac{\sum\limits_{l=1}^{n} c\left(\left(i_1, \ldots, i_n\right), l\right) p_l L_t(x_l^{(t)})}{\sum\limits_{l=1}^{n} c\left(\left(i_1, \ldots, i_n\right), l\right) p_l}\right),$$

$$\mathbf{i}_n \in D,\ 1 \leq t \leq m.$$

Now assume (A_1), consider a real number $\lambda \geq 1$, and let S_k be the set defined in (9). Then the mixed means corresponding to (10) are

$$M_k^2(\lambda) := M_k^2\big(L_1, \ldots, L_m; \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; \mathbf{p}; \lambda\big) :=$$

$$= N^{-1}\Bigg(\frac{1}{(n+\lambda-1^k} \sum_{i_1,\ldots,i_n \in S_k} \bigg(\frac{k!}{i_1! \ldots i_n!}\bigg(\sum_{j=1}^{n} \lambda^{i_j} p_j\bigg)\cdot$$

$$\cdot N\Big(f\big(L_1(\mathbf{x}^{(1)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda)\big)\Big)\bigg)\Bigg),$$

where

$$L_t(\mathbf{x}^{(t)}; \mathbf{p}; \mathbf{i}_{n,k}; \lambda) = L_t^{-1}\left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j L_t(x_j^{(t)})}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}\right),$$

$$\mathbf{i}_{n,k} \in S_k, \quad 1 \leq t \leq m.$$

In this case Theorem C gives another interpolation of (2) as follows:

Theorem 2.5. *Assume* (A_1), *let* $\lambda \geq 1$ *be a real number, and let* S_k *be the set defined in* (9). *If* N *is a strictly increasing* (*decreasing*) *function, then the inequalities*

$$f\Big(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), \ldots, L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\Big) = M_0^2(\lambda) \leq M_1^2(\lambda) \leq \cdots \leq$$

$$\leq M_k^2(\lambda) \leq \cdots \leq N^{-1}\Bigg(\sum_{i=1}^{n} p_i N\Big(f\big(x_i^{(1)}, \ldots, x_i^{(m)}\big)\Big)\Bigg), \quad k \in \mathbb{N}, \tag{14}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$, *if and only if the function* H *defined in Theorem* 1.1 *is convex* (*concave*). *If* N *is an increasing* (*decreasing*) *function, then the inequalities in* (14) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$, *if and only if* H *is concave* (*convex*).

Proof. Similar to the proof of Theorem 2.1. □

3. New Generalizations of the Consequences of Beck's Result

Assume (A_2) with positive n-tuple $\mathbf{p}$, (H_4) and (H_5). Then for $m = 2$, the reverse of (11) can be written as

$$f\big(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\big) \geq M_k^1 \geq M_{k-1}^1 \geq \cdots \geq M_1^1 = \\ = N^{-1}\left(\sum_{j=1}^{n} p_j N\big(f(a_j, b_j)\big)\right). \tag{15}$$

Analogous to the results of Corollary A and Corollary B (see [7] and also [8], p. 195), we have immediately from Theorem 2.1 and Corollaries 1.2, 1.3 that

Corollary 3.1. *Assume* (A_3) *with* $f(x,y) = x+y$ ($(x,y) \in I_K \times I_L$) *and with positive n-tuple* $\mathbf{p}$, *assume* (H_4)–(H_5), *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (15) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case

$$M_k^1 := M_k^1(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}) := N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right)\cdot \right.\right.\right. \\ \left.\left.\left. \cdot N\big((K(\mathbf{a}; c\mathbf{p}; A) + L(\mathbf{b}; c\mathbf{p}; A))\big)\right)\right)\right), \tag{16}$$

and for $1 \leq d \leq k-1$

$$M_{k-d}^1 := M_{k-d}^1\big(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}\big) := \\ = N^{-1}\left\{\sum_{l=1}^{d}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}N(a_{\tau(s)} + b_{\tau(s)})\right)\right) + \right. \\ + \sum_{l=d+1}^{k}\left(\frac{d!}{l-1)\ldots(l-d)}\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-d}}(A)\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right)\cdot \right.\right.\right. \\ \left.\left.\left.\left. \cdot N\Big(K(\mathbf{a}; c\mathbf{p}; B) + L(\mathbf{b}; c\mathbf{p}; B)\Big)\right)\right)\right)\right\}. \tag{17}$$

Corollary 3.2. *Assume* (H_4), (H_5) *and consider* (A_3) *with* $f(x,y) = xy$ ($(x,y) \in I_K \times I_L$) *and with positive n-tuple* $\mathbf{p}$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N° *respectively. Assume further that* K', L', M', A,

B and C are all positive. Then (15) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x, y) \in I_K^\circ \times I_L^\circ.$$

In this case

$$M_k^1 := M_k^1\big(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}\big) := N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$
$$\left.\left.\left.\cdot N\big(K(\mathbf{a}; c\mathbf{p}; A)L(\mathbf{b}; c\mathbf{p}; A)\big)\right)\right)\right), \tag{18}$$

and for $1 \leq d \leq k-1$,

$$M_{k-d}^1 : = M_{k-d}^1\big(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}\big) :=$$
$$= N^{-1}\left\{\sum_{l=1}^{d}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}N(a_{\tau(s)}b_{\tau(s)})\right)\right)+\right.$$
$$+\sum_{l=d+1}^{k}\left(\frac{d!}{(l-1)\dots(l-d)}\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-d}(A)}\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right)\cdot\right.\right.\right.$$
$$\left.\left.\left.\left.\cdot\, N\big(K(\mathbf{a}; c\mathbf{p}; B)L(\mathbf{b}; c\mathbf{p}; B)\big)\right)\right)\right)\right\}. \tag{19}$$

Under the considerations of examples in Section 2, we show some special cases of the Corollaries 3.1 and 3.2.

Remark 3.3. Under the settings of Example 2.2, if $f(x_1, x_2) = x_1 + x_2$, then (16) becomes

$$M_k^1 : = M_k^1\big(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}\big) :=$$
$$= N^{-1}\left(\sum_{(i_1,\dots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k} c\big((i_1,\dots,i_k), l\big)p_{i_l}\right)\cdot\right.\right.$$
$$\left.\left.\cdot\ N\big(K(\mathbf{a}; c\mathbf{p}; \mathbf{i}_k) + L(\mathbf{b}; c\mathbf{p}; \mathbf{i}_k)\big)\right)\right),$$

and for $1 \leq d \leq k-1$ (17) becomes

$$M_{k-d}^1 : = M_k^1\big(K, L; \mathbf{a}, \mathbf{b}; c\mathbf{p}\big) :=$$
$$= N^{-1}\left(\left(\frac{d!}{(k-1)\dots(k-d)}\cdot\right.\right.$$
$$\cdot\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{1\leq l_1<\dots<l_{k-d}\leq k}\left(\left(\sum_{j=1}^{k-m} c\big((i_1,\dots,i_k), l_j\big)p_{i_{l_j}}\right)\cdot\right.\right.$$

$$\cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})+L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\Big)\Bigg)\Bigg)\Bigg).$$

Under the conditions of Corollary 3.1, we have

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p})\geq M_k^1\geq M_{k-1}^1\geq\cdots\geq M_1^1= \\ =N^{-1}\left(\sum_{i=1}^n p_iN(a_i+b_i)\right). \tag{20}$$

Similarly, if $f(x_1,x_2)=x_1x_2$, then from (18) we have

$$M_k^1:=M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}):= \\ =N^{-1}\Bigg(\sum_{(i_1,\ldots,i_k)\in I_k}\Bigg(\Bigg(\sum_{l=1}^k c((i_1,\ldots,i_k),l)p_{i_l}\Bigg)\cdot \\ \cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k)L(\mathbf{b};c\mathbf{p};\mathbf{i}_k)\Big)\Bigg)\Bigg),$$

and for $1\leq d\leq k-1$, we have from (19)

$$M_{k-d}^1:=M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}):= \\ =N^{-1}\Bigg(\frac{d!}{(k-1)\ldots(k-d)}\cdot \\ \cdot\sum_{(i_1,\ldots,i_k)\in I_k}\Bigg(\sum_{1\leq l_1<\ldots<l_{k-d}\leq k}\Bigg(\Bigg(\sum_{j=1}^{k-m}c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}\Bigg)\cdot \\ \cdot N\Big(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})\Big)\Bigg)\Bigg).$$

Under the conditions of Corollary 3.2, we have

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p})\geq M_k^1\geq M_{k-1}^1\geq\cdots\geq M_1^1=N^{-1}\left(\sum_{i=1}^n p_iN(a_ib_i)\right). \tag{21}$$

Taking

$$c((i_1,\ldots,i_k),l)=\frac{1}{|S_j|}=\frac{1}{\alpha_{I_k,j}},\quad ((i_1,\ldots,i_k),l)\in S_j,$$

in (20) and (21), we get Corollary 3.1 and Corollary 3.2 of [5], respectively.

Remark 3.4. We consider Example 2.3. If $f(x_1,x_2)=x_1+x_2$, then under the conditions of Corollary 3.1 we have

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p})\geq \\ \geq N^{-1}\Bigg\{\sum_{i=1}^n\left(\sum_{l=0}^r c(i,l)p_{i+l}\right)N\Big(K_r(\mathbf{a},c\mathbf{p};i)+L_r(\mathbf{b},c\mathbf{p};i)\Big)+$$

$$+\left(\sum_{j=1}^{n} c(j)p_j\right)N\big(K_n(\mathbf{a};c\mathbf{p})+L_n(\mathbf{b};c\mathbf{p})\big)\Bigg\} \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Similarly, if $f(x_1, x_2) = x_1 x_2$, then under the conditions of Corollary 3.2 we have

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq$$
$$\geq N^{-1}\Bigg\{\sum_{i=1}^{n}\left(\sum_{l=0}^{r} c(i,l)p_{i+l}\right)N\big(K_r(\mathbf{a};c\mathbf{p};i)L_r(\mathbf{b};c\mathbf{p};i)\big)+$$
$$+\left(\sum_{j=1}^{n} c(j)p_j\right)N\big(K_n(\mathbf{a};c\mathbf{p})L_n(\mathbf{b};c\mathbf{p})\big)\Bigg\} \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Remark 3.5. We now consider Example 2.4. If $f(x_1, x_2) = x_1 + x_2$, then under the conditions of Corollary 3.1 we have

$$K_n(\mathbf{a};\mathbf{p}) + L_n(\mathbf{b};\mathbf{p}) \geq$$
$$\geq N^{-1}\left(\sum_{(i_1,\dots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c((i_1,\dots,i_n),l)p_l\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(K_n(\mathbf{a};c\mathbf{p},\mathbf{i}_n + L_n(\mathbf{b};c\mathbf{p},\mathbf{i}_n))\right)\right) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i + b_i)\right).$$

Similarly, if $f(x_1, x_2) = x_1 x_2$, then under the conditions of Corollary 3.2 we have

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq$$
$$\geq N^{-1}\left(\sum_{(i_1,\dots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c\big((i_1,\dots,i_n),l)p_l\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(K_n(\mathbf{a},c\mathbf{p},\mathbf{i}_n)L_n(\mathbf{b},c\mathbf{p},\mathbf{i}_n)\big)\right)\right) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i\right).$$

Next, assume (A_2), let $\lambda \geq 1$, and let T_k be the set defined in (9). Then for $m = 2$, the reverse of (14) becomes

$$f\big(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\big) = M_0^2(\lambda) \geq M_1^2(\lambda) \geq \cdots \geq M_k^2(\lambda) \geq \cdots \geq$$
$$\geq N^{-1}\left(\sum_{i=1}^{n} p_i N\big(f(a_i, b_i)\big)\right), \quad k \in \mathbb{N}, \tag{22}$$

where

$$M_k^2(\lambda) := M_k^2(K, L; \mathbf{a}, \mathbf{b}; \mathbf{p}; \lambda) :=$$

$$= N^{-1}\left(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\left(\frac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(f(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda), L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda))\big)\right)\right).$$

By using Theorem 2.5 (for $m=2$) and Corollaries 1.2, 1.3, we get parameter dependent generalizations of Beck's results.

Corollary 3.6. *Assume* (A_3) *with* $f(x,y)=x+y$ $((x,y)\in I_K\times I_L)$, *let* $\lambda\geq 1$, *and let* T_k *be the set defined in* (9). *Assume further that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E:=\frac{K'}{K''}$, $F:=\frac{L'}{L''}$, $G:=\frac{N'}{N''}$, (22) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x)+F(y)\leq G(x+y),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

In this case for $k\in\mathbb{N}$, *we have*

$$M_k^2(\lambda):=M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda):=$$
$$= N^{-1}\left(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\left(\frac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)\cdot\right.\right.$$
$$\left.\left.\cdot N\big(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda)+L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda)\big)\right)\right).$$

Corollary 3.7. *Assume* (A_3) *with* $f(x,y)=xy$ $((x,y)\in I_K\times I_L)$, *let* $\lambda\geq 1$, *and let* T_k *be the set defined in* (9). *Suppose the functions* $A(x):=\frac{K'(x)}{K'(x)+xK''(x)}$, $B(x):=\frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x):=\frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N° *respectively. Assume further that* K', L', M', A, B *and* C *are all positive. Then* (22) *holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x)+B(y)\leq C(xy),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

In this case for $k\in\mathbb{N}$, *we have*

$$M_k^2(\lambda):=M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda):=$$
$$= N^{-1}\left(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\left(\frac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)\cdot\right.\right.$$
$$\left.\cdot N\big(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda)L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda))\big)\right).$$

4. Generalization of Minkowski's Inequality

We need the following hypothesis:

(A_4) Let I be an interval in $\mathbb{R}$, and let $M:I\to\mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i\in I^m$ $(i=1,\dots,n)$, let $\mathbf{p}=(p_1,\dots,p_n)$

be a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$, and let $\mathbf{w} = (w_1, \ldots, w_m)$ be a nonnegative m-tuple such that $\sum_{i=1}^{m} w_i = 1$.

We give a generalization of the Minkowski's inequality by using Theorem B.

Theorem 4.1. *Assume* (A_4), (H_4) *and* (H_5). *Further, assume that the quasi-arithmetic mean function*

$$\mathbf{x} \to M_m(\mathbf{x}; \mathbf{w}), \quad \mathbf{x} \in I^m \tag{23}$$

is convex. Then

$$M_m\left(\sum_{r=1}^{n} p_r \mathbf{x}_r; \mathbf{w}\right) \le A_k \le A_{k-1} \le \cdots \le A_2 \le A_1 = \sum_{r=1}^{n} p_r M_m(\mathbf{x}_r; \mathbf{w}),$$

where

$$A_k := \sum_{l=1}^{k} \left(\sum_{A \in \mathcal{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) M_m \left(\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}}; \mathbf{w} \right) \right) \right), \tag{24}$$

and for $1 \le d \le k-1$

$$\begin{aligned} A_{k-d} : = &\sum_{l=1}^{d} \left(\sum_{A \in \mathcal{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} M_m(\mathbf{x}_{\tau(s)}; \mathbf{w}) \right) \right) + \\ &+ \sum_{l=d+1}^{k} \left(\frac{d!}{(l-1)\ldots(l-d)} \cdot \sum_{A \in \mathcal{A}_l} \left(\sum_{B \in P_{l-d}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) \cdot \right.\right.\right. \\ &\left.\left.\left. \cdot M_m \left(\frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}; \mathbf{w}} \right) \right) \right) \right). \end{aligned} \tag{25}$$

Proof. We apply Theorem B to the convex function $M_m(\cdot; \mathbf{w})$ and the vectors $\mathbf{x}_i$ $(i = 1, \ldots, n)$. We get A_d $(k \ge d \ge 1)$ in (24) and (25) from (7) and (8) respectively. □

Similarly, by using Theorem C we get

Theorem 4.2. *Let* $\lambda \ge 1$ *be a real number, assume* (A_4) *and suppose* T_k $(k \in \mathbb{N})$ *is the set given in* (9). *If the quasi-arithmetic mean function* 23 *is convex, then*

$$\begin{aligned} M_m\left(\sum_{r=1}^{n} p_r \mathbf{x}_r; \mathbf{w}\right) = C_0(\lambda) \le C_1(\lambda) \le \cdots \le C_k(\lambda) \le \cdots \le \\ \le \sum_{r=1}^{n} p_r M_m(\mathbf{x}_r; \mathbf{w}), \quad k \in \mathbb{N}, \end{aligned}$$

where

$$C_k(\lambda) = C_k(\mathbf{x}_1, \ldots, \mathbf{x}_n; p_1, \ldots, p_n; \lambda) :=$$
$$= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) \cdot$$
$$\cdot M_m \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j \mathbf{x}_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}; \mathbf{w} \right), \quad k \in \mathbb{N}.$$

The following result gives a necessary and sufficient condition for the quasi-arithmetic mean function to be convex (see [8], p. 197):

Theorem D. *If $M : [m_1, m_2] \to \mathbb{R}$ has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function $M_m(\cdot; w)$ is convex if and only if M'/M'' is a concave function.*

(A_5) Let $M :]0, \infty[\to]0, \infty[$ be a continuous and strictly monotone function such that $\lim\limits_{x\to 0} M(x) = \infty$ or $\lim\limits_{x\to\infty} M(x) = \infty$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be positive m-tuples such that $w_i \geq 1$ $(i = 1, \ldots, m)$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$.

Then we define

$$\widetilde{M}_m(\mathbf{x}; \mathbf{w}) = M^{-1} \left(\sum_{i=1}^{m} w_i M(x_i) \right). \tag{26}$$

The following result is also given in ([8], page 197):

Theorem E. *If $M :]0, \infty[\to]0, \infty[$ has continuous derivatives of second order and it is strictly increasing and strictly convex, then $\widetilde{M}_m(\cdot; w)$ is a convex function if M/M' is a convex function.*

By using (26) we have

Theorem 4.3. *Assume* (A_5) *and let*

$$\mathbf{x} \to \widetilde{M}_m(\mathbf{x}; \mathbf{w}), \quad \mathbf{x} \in]0, \infty[^m$$

be a convex function.

(a) *Consider* (H_4) *and* (H_5). *Then Theorem* 4.1 *remains valid for* $\widetilde{M}_m(\mathbf{x}; \mathbf{w})$ *instead of* $M_m(\mathbf{x}; \mathbf{w})$.

(b) *Consider $\lambda \in \mathbb{R}$ such that $\lambda \geq 1$ and suppose T_k $(k \in \mathbb{N})$ is the set defined in* (9). *Then Theorem* 4.2 *also remains valid for* $\widetilde{M}_m(\mathbf{x}; \mathbf{w})$ *instead of* $M_m(\mathbf{x}; \mathbf{w})$.

Remark 4.4. All special cases (as given in Section 2) can also be considered for Theorem 4.1, Theorem 4.2 and Theorem 4.3.

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