Partitioning the bases of the union of matroids

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Abstract: Let $B = \cup_{i=1}^{n} B_i$ be a partition of base B in the union (or sum) of n matroids into independent sets B_i of M_i. We prove that every other base B' has such a partition where B_i and B'_i span the same set in M_i for $i = 1, 2, \ldots, n$.

Keywords: matroid theory, union of matroids

1 Introduction

For the definitions and notations in matroid theory the reader is referred to [5] or [6]. In particular, let E denote the common underlying set of every matroid and let $r_1, r_2, \ldots, r_n$ denote the rank functions of the matroids $M_1, M_2, \ldots, M_n$, respectively. Throughout M will denote the union (or sum) $\vee_{i=1}^{n} M_i$ of these matroids, and R will denote the rank function of M. A subset $X \subseteq E$ is independent in M if and only if it arises as $X = \bigcup_{i=1}^{n} X_i$ with X_i independent in M_i for each i. Recall that

$$R(X) = \min_{Y \subseteq X} \left[\sum_{i=1}^{n} r_i(Y) + |X - Y| \right]$$

by the fundamental results of [1] and [4].
An element of the underlying set E of a matroid is a *loop* if it is dependent as a single element subset, and it is a *coloop* if it is contained in every base. We shall need the following observation ([3], independently rediscovered in [2]):

Proposition 1 *If M has no coloops, then $R(E) = \sum_{i=1}^{n} r_i(E)$.*

The *weak map* relation is defined as follows: the matroid B is *freer* than A (denoted by $A \preceq B$) if every independent set of A is independent in B as well. Clearly $M_j \preceq \vee_{i=1}^{n} M_i$ for every $j = 1, 2, \ldots, n$ and $A \preceq B$ implies $A \vee C \preceq B \vee C$ for every C.
Let $\sigma_i(X)$ denote the *closure* of a set $X \subseteq E$ in M_i, that is, $\sigma_i(X) = \{e | r_i(X \cup \{e\}) =$

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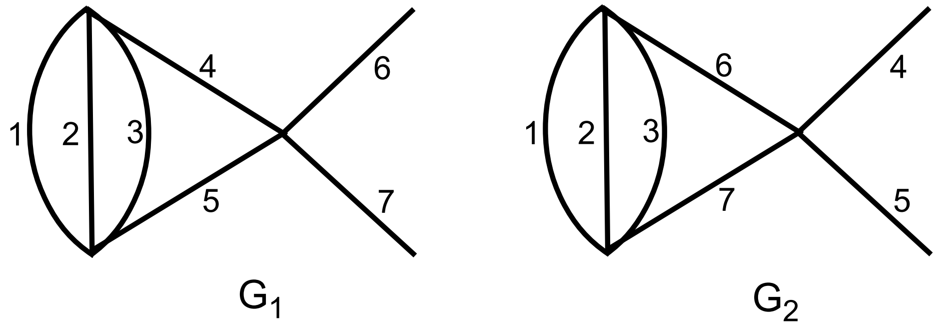

Fig. 1.

$r_i(X)\}$. Let $\sigma(X)$ denote the closure of X in M. A set $X \subseteq E$ is *closed* if $\sigma(X) = X$. The closed sets are also called *flats*. In particular, the set of loops, that is $\sigma(\emptyset)$ is the smallest and E is the largest flat. We shall need the following easy property of the closure function:

Proposition 2 *Let $S_1, S_2 \subseteq E$ be independent subsets with $\sigma(S_1) = \sigma(S_2) = S$. Let, furthermore, $S_0 \subseteq E$ so that $S \cap S_0 = \emptyset$ and $S_1 \cup S_0$ is independent. Then $S_2 \cup S_0$ is also independent.*

PROOF: Observe that $|S_1| = |S_2|$ since both are independent and span the same subset S. Indirectly suppose that $r(S_2 \cup S_0) < |S_2| + |S_0| = |S_1| + |S_0| = |S_1 \cup S_0|$. Since $S_1 \cup S_0$ is independent, there exists an element $x \in S_1 - S_2$ so that $r(S_2 \cup S_0 \cup \{x\}) > r(S_2 \cup S_0)$. However, $x \in S_1 \subseteq S = \sigma(S_2)$ implies that $r(S_2 \cup \{x\}) = r(S_2)$, a contradiction. $\square$

2 Partitioning the bases

Let B be a base of M. The partition $B_1, B_2, \dots, B_n$ of B is a *good partition* if B_i is independent in M_i for $i = 1, 2, \dots, n$.
Let $F_i = \sigma_i(B_i)$ for every i. This collection of flats $F_1, F_2, \dots, F_n$ depends on the actual good partition of B, as illustrated by the following example.

Example 3 *If M_1 and M_2 are the cycle matroids of the graphs G_1 and G_2 of Figure 1, respectively, then M will be the cycle matroid of the graph of Figure 2. The base $B = \{1, 2, 4, 5, 6, 7\}$ of M has 54 good partitions, see the first two columns of Table 1, where each row represents six good partitions (put $a, b \in \{1, 2, 3\}$, $a \neq b$ in every possible way). These good partitions lead to 9 different collections of flats, see columns 3 and 4 of Table 1.*

Table 1

	B_1	B_2	F_1	F_2
1	$\{a,4,6,7\}$	$\{b,5\}$	E	$\{1,2,3,5\}$
2	$\{a,5,6,7\}$	$\{b,4\}$	E	$\{1,2,3,4\}$
3	$\{a,4,6\}$	$\{b,5,7\}$	$E-\{7\}$	$E-\{4\}$
4	$\{a,4,7\}$	$\{b,5,6\}$	$E-\{6\}$	$E-\{4\}$
5	$\{a,5,6\}$	$\{b,4,7\}$	$E-\{7\}$	$E-\{5\}$
6	$\{a,5,7\}$	$\{b,4,6\}$	$E-\{6\}$	$E-\{5\}$
7	$\{a,6,7\}$	$\{b,4,5\}$	$E-\{4,5\}$	$E-\{6,7\}$
8	$\{a,6\}$	$\{b,4,5,7\}$	$\{1,2,3,6\}$	E
9	$\{a,7\}$	$\{b,4,5,6\}$	$\{1,2,3,7\}$	E

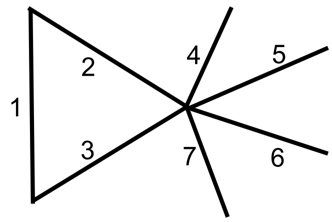

Fig. 2.

Surprisingly if we consider any other base of the union, the list of the possible collections of flats will always be the same.

Theorem 4 *Let $M_1, M_2, \ldots, M_n$ be matroids and let M be their union. Let B be a base of M with a good partition $B_1, B_2, \ldots, B_n$. For any base B' of M there is a good partition $\cup_{i=1}^n B_i'$ so that $\sigma_i(B_i) = \sigma_i(B_i')$ for $i = 1, 2, \ldots, n$.*

Proof: Suppose that B' is a base of the union with a good partition $X_1, X_2, \ldots, X_n$. Let A denote the set of the non-coloop elements of the union. B' is independent in the union so $|B' \cap A| = R(B' \cap A)$. Clearly $R(B' \cap A) = R(A)$ since B' is a base in the union, and $\sigma(A) = A$. According to Proposition 1 $\Sigma_{i=1}^n r_i(A) = R(A)$. Now $r_i(A) \geq r_i(X_i \cap A)$ since $X_i \cap A \subseteq A$, and $r_i(X_i \cap A) = |X_i \cap A|$ since X_i is independent in M_i. These together give the following:

$$|B' \cap A| = R(B' \cap A) = R(A) = \Sigma_{i=1}^n r_i(A) \geq \Sigma_{i=1}^n r_i(X_i \cap A) = |B' \cap A|$$

Since the two sides are equal, the inequality must be satisfied as equality, so $r_i(A) =$

$r_i(X_i \cap A)$. This means that every good partition $X_1, X_2, \ldots, X_n$ of a base B' of the union will satisfy $\sigma_i(A \cap X_i) = A$, that is, $X_i \cap A$ spans A in M_i for $i = 1, 2, \ldots, n$.
These results are true for B, too, so $B_i \cap A$ spans A in M_i for $i = 1, 2, \ldots, n$. All the coloops of M are in $B \cap B'$, this way we can get a good partition of B', namely $B'_i = (X_i \cap A) \cup (B_i \setminus A)$ according to Proposition 2. This partition satisfies the requirements of Theorem 4. $\square$

3 Weak maps with the same union

Let B be an arbitrary base of M with an arbitrary good partition $\cup_{i=1}^n B_i$. Let $F_i = \sigma_i(B_i)$ for every i and let M'_i be obtained from M_i by replacing all the elements of $E - F_i$ by loops. (That is, M'_i has ground set E and $X \subseteq E$ is independent in M'_i if and only if $X \subseteq F_i$ and X is independent in M_i.)

Proposition 5 *If $M' = \vee_{i=1}^n M'_i$ then $M' = M$.*

Proof: Clearly $M'_i \preceq M_i$, and therefore $M' = \vee_{i=1}^n M'_i \preceq \vee_{i=1}^n M_i = M$.
On the other hand we have to prove that any independent set X of M is independent in M' as well.
Let B' be a base of M, containing X. By Theorem 4, there exists a good partition $\cup_{i=1}^n B'_i$ of B' so that $\sigma_i(B'_i) = F_i$ for every i. Since B'_i is independent in M'_i, so is $B'_i \cap X$. Hence $X = \cup_{i=1}^n (B'_i \cap X)$ is independent in M', as requested. $\square$

Example 6 illustrates Proposition 5.

Example 6 *Let M_1 and M_2 be the cycle matroids of the graphs G_1 and G_2 of Figure 1, as in Example 3. Consider the pair of flats $E, \{1, 2, 3, 5\}$ as in the first row of Table 1. The corresponding restricted matroids M'_1, M'_2 are represented by the graphs of the first row of Figure 3. One can easily see that $M'_1 \vee M'_2$ is still the cycle matroid of the graph of Figure 2. Similarly, the pairs of flats, given by rows 3 and 9 of Table 1 lead to the second and third rows of Figure 3, respectively.*

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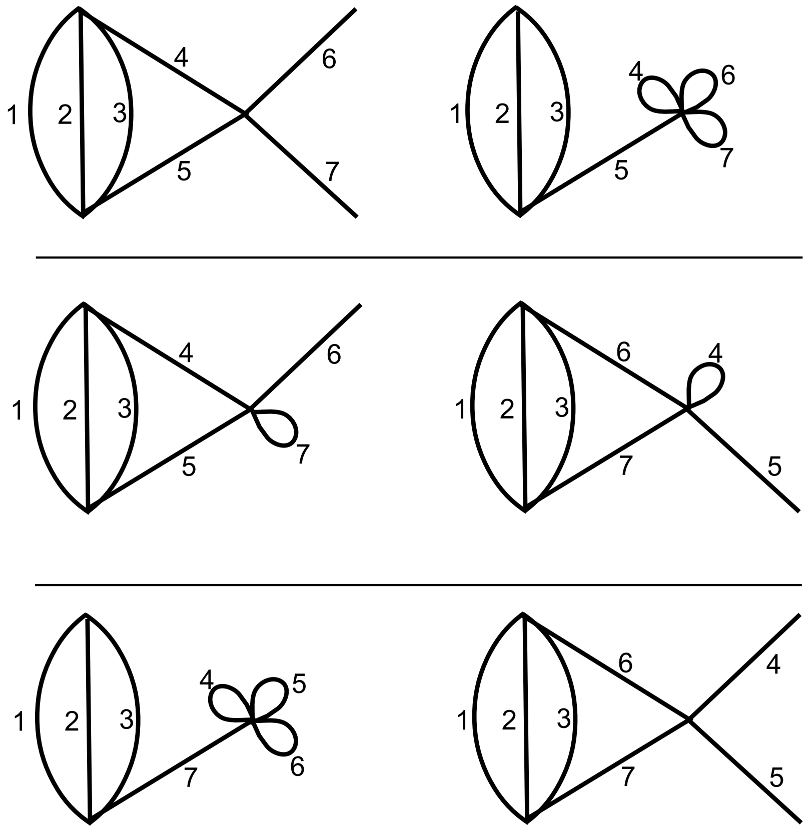

Fig. 3.

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